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ACKERBERG-O'MALLEY RESONANCE REVISITED

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1. **Introduction.** In a 1970 article in Studies in Applied Mathematics Ackerberg and O'Malley [1] described a curious phenomenon, which they called resonance, associated with the singular perturbation boundary value problem

$$(1)\qquad \begin{aligned} &\text{(a)}\quad \epsilon y'' - p(x)y' + q(x)y = 0, \quad (0 < \epsilon \ll 1) \\ &\text{(b)}\quad y(a) = \alpha, \quad y(b) = \beta \end{aligned}$$

where the function $p(x)$ changes sign within the interval $[a, b]$. Assume in fact that $p'(x) > 0$ and that p increases from negative to positive, so that $p(x_c) = 0$ at some "turning point" (or "critical point") $x_c \in (a, b)$. Then normally the B.V.P. is uniquely solved by a function $y(x)$ which is exponentially small throughout the body of the interval, except in two thin boundary layers (of width $O(\epsilon)$), where it adjusts rapidly to match the prescribed boundary values. However in special circumstances, depending delicately on $p(x)$ and $q(x)$, a kind of resonance occurs: $y(x)$ is no longer asymptotically zero within (a, b), but is instead asymptotically close to a nontrivial solution of the *reduced* differential equation

$$(2)\qquad p(x)y_0' - q(x)y_0 = 0$$

obtained by setting $\epsilon = 0$ in (1). In general, at least one boundary layer correction is needed in order that both boundary conditions may be satisfied, and depending on circumstances this correction may appear at the left or right endpoint. Under exceptional conditions there may even be boundary layer corrections at both ends.

Ackerberg and O'Malley gave a simple necessary condition for resonance to occur: it is that the ratio $\ell = q(x_c)/p'(x_c)$ must be a nonnegative integer. Later W. D. Lakin [7] and P. Cook and W. Eckhaus [3] independently discovered that this necessary condition is only the first of an infinite sequence of such conditions involving p and q. In fact the sequence of conditions can be shown to be sufficient. The conditions after the first become exceedingly burdensome to compute or even to write down, and the problem of finding effectively applicable criteria for resonance remains open. (But see Kreiss and Parter [6] and B. Matkowsky [14].)

In this article we shall examine several phenomena which while quite diverse in appearance seem to have a common origin related to resonance. For much of the asymptotic analysis we shall rely on two early papers of R. McKelvey [15] and [16] which utilize the methods of R. E. Langer.

In paragraph 2 we review the basic facts about the "classical" Ackerberg-O'Malley resonance phenomenon. In paragraph 3 we examine the "interior flaring" phenomenon which occurs when the slope of the coefficient function $p(x)$ in (1a) is reversed. An explicitly solvable example has already been treated by R. O'Malley [18].

In paragraph 4 we examine the asymptotic form of solutions of the D.E.

$$-iy'' = \lambda[p(x)y' - q(x)y] \text{ as } \lambda \to +\infty$$

where $p(a) < 0 < p(b)$. The particular solution which to one side (left or right) of the turning point is asymptotic to a solution of the reduced equation, breaks up on passing across the turning point into a pattern of rapid oscillations. Exceptionally, there is a resonant state in which the slowly varying regular behavior persists across the entire interval.

In paragraph 5 we examine the stability properties of the differential equation

$$\epsilon y'' - [U(x) - c](y' + \sigma y) + n(\epsilon)U'(x)y = 0 \tag{3}$$

where $U(x)$ is to be interpreted as a velocity, ϵ as an inverse Reynolds number and σ and c as frequencies in space and time. When the equation is uniformly in resonance (via conditions on $n(\epsilon)$) its stability properties are analogues to those encountered in viscous shear flow, as described by the fourth order Orr-Sommerfield equation of hydrodynamics. Equation (3) is therefore a kind of second order "model" of O.S. The O.S. equation itself exhibits certain residual characteristics of "resonance," and we discuss this briefly.

In paragraph 6 we examine a "resonance" phenomenon associated with a class of differential equations which are singular at one end of the interval of interest: The D.E. is

$$\epsilon y'' - \varphi(x,\epsilon)y' + \left[\frac{m^2-1}{4x^2} + \frac{\psi(x,\epsilon)}{x}\right] y = 0, \ (\varphi(x) \neq 0)$$

with B.C.:

$$-iy'' = 0(x^{m+1/2}) \text{ at } x = 0; \quad y(1) = 1.$$

For brevity we confine our description to the prototypical case where φ and ψ are constant; the general class can be handled using the analysis of [16]. The behavior here is very similar to the original Ackerberg-O'Malley situation.

The phenomena considered in this paper all have in common two things: First an order-reducing limiting process whereby certain solutions of the full D.E. are represented asymptotically on subintervals by solutions of a reduced equation. Second the coalescence, at certain critical values of parameters in the D.E., of solutions which are of asymptotically distinguished types. It is this latter occurrence which we call resonance.

Extensions of results announced here, as well as precise statements and detailed computations, will be found in R. Bohac's University of Montana dissertation.

We wish to thank Seymour Parter and Robert O'Malley for calling our attention to Ackerberg-O'Malley resonance, and for bringing us up to date on the research dealing with it.

2. **Classical Ackerberg-O'Malley resonance.** The occurrence of resonance can be explained from several points of view (H. O. Kreiss and S. Parter [6]; P. P. N de Groen [4]). The simplest, it seems to us, is in the original framework of asymptotic turning point analysis. Let

$$\xi_x = \frac{1}{\epsilon} \int_{x_c}^{x} p(t)\, dt.$$

Thus, for each fixed $\epsilon > 0$, ξ_x vanishes at the turning point x_c and increases monotonically as one moves away from x_c, either to the left or to the right. By the W.K.B. method (E. Zauderer [23]) or otherwise, one finds on either side of the turning point a pair of asymptotic solutions, respectively,

$$(4) \qquad \begin{aligned} &(a) \quad y_0(x) + \epsilon\, y_1(x) + \epsilon^2 y_2(x) + \cdots, \\ &(b) \quad \exp(\xi_x)[u_0(x) + \epsilon\, u_1(x) + \epsilon^2 u_2(x) + \cdots]. \end{aligned}$$

The first of these has the structure of a regular perturbation or "balanced" solution: formal substitution in the differential equation (1 a) shows that the leading term $y_0(x)$ must satisfy the reduced equation (2). The second is of asymptotically "dominant" type, becoming infinite at $x \neq x_c$ as $\epsilon \to 0$. Alternatively, this second solution may be modified to a boundary layer form by inserting an exponential damping factor independent of the variable x:

$$\exp[-(\xi_b - \xi_x)] \cdot [u_0(x) + \epsilon\, u_1(x) + \epsilon^2 u_2(x) + \cdots], \text{ for } x > x_c$$

or

$$\exp[-(\xi_a - \xi_x)]\ [u_0(x) + \epsilon\, u_1(x) + \epsilon^2 u_2(x) + \cdots] \text{ for } x < x_c.$$

In general the asymptotic forms (4) cannot be continued across the turning point, e.g., the solution represented by (4 a) to the left of the turning point normally will be linearly independent of that solution which is represented by (4 a) to the right. Indeed resonance occurs precisely in the exceptional circumstance that these two solutions happen to become dependent!

Briefly, *out* of resonance the solution which is balanced to left of the turning point connects to one which is dominant to the right. Modifying the overall solution by a factor $c \cdot \exp(-\xi_b)$ yields a solution which is exponentially small everywhere in $[a, b)$ outside of a narrow boundary layer at b and, with adjustment of the constant c, takes on the boundary value β at b. Similarly one finds a solution which takes on the boundary value α at a, and is exponentially small in $(a, b]$ outside of a narrow boundary layer at a. Superimposing these two solutions one solves the B.V.P.

On the other hand, *in* resonance the balanced solutions are connected — indeed, with no jump in the value of $y_0(x)$ on crossing the critical point. A second solution is exponentially large at both ends. When $\xi_a \geqq \xi_b$, that solution should be modified by the factor $\exp(-\xi_a)$ to one which is exponentially small in the interval and rapidly changing within a boundary layer at a. In the special case where $\xi_a = \xi_b$, i.e., where

$$\int_a^b p(t)\,dt = 0,$$

this modified solution also grows in a boundary layer at b. The B.V.P. can once again be satisfied by a linear combination of the two solutions, but now the combination will be asymptotically non-zero and, outside of the boundary layers and a shrinking critical layer — within which it remains bounded $O(\epsilon^{1/2})$ — is asymptotically close to a non-zero solution of the reduced equation.

Sometimes the matching across x_c of the balanced solutions can be detected by direct examination of the series (4 a), should the expected singularities at x_c fail to occur in the coefficients $y_j(x)$ (see Matkowsky [14]). But this regularity of coefficients at x_c is in no way necessary for resonance; in general there is a critical layer (of width $O(\epsilon^{1/2})$) about x_c within which the W.K.B. expressions (4) simply are invalid. Then one must match up solutions (4) across x_c with the help of explicit linear connection formulas involving so-called Stokes' multipliers. These may be derived by utilizing "local turning point solutions" or, as we prefer, by use of comparison equations which are solvable across the whole interval, in terms of special functions of known structure at the turning point.

This latter method was developed, in the first instance, by R. E. Langer [9], [10], and then by several others (McKelvey [15], Hanson and Russell [5], Roy Lee [11], F. W. J. Olver [17]). If the D.E. (1 a) is normalized by the transformation $y = v \exp(1/2\xi_x)$ to

$$v'' = \left[\frac{1}{4\epsilon^2} p^2 - \frac{1}{2\epsilon} (p' + 2q) \right] v \tag{5}$$

then one sees that the turning point is of order 2: i.e., the term $(1/4\epsilon^2)p^2$, which is dominant away from the turning point, has a second order zero at the turning point. Hence a comparison equation, in order to capture the asymptotic properties of solutions, must faithfully duplicate this given equation *at least* to $1/\epsilon$ terms in the vicinity of the turning point. An algorithm for doing this was given by McKelvey [15], providing exact solutions to a sequence of equations

$$(6)\; z'' = \left[\frac{1}{4\epsilon^2} p^2 - \frac{1}{2\epsilon} (p' + 2q) + 0(\epsilon^{-N}) \right] z \quad (N = 0, 1, 2, \cdots)$$

by

$$z = \eta(\xi) \sum_{n=0}^{N} \epsilon^n A_n(x) + \frac{d\eta(\xi)}{d\xi} \sum_{n=0}^{N} B_n(x) \tag{7}$$

where $\eta(\xi)$ is essentially a Whittaker function (= confluent hypergeometric function)

$$\eta(\xi) = \xi^{-1/4}\, W_{1/4(1+2\ell),1/4}, \text{ with } \ell(\epsilon) = \sum_{n=0}^{N} \epsilon^n \ell_n$$

and $A_n(x)$, $B_n(x)$, and ℓ_n are determined recursively. The role of the adjustable constants ℓ_n is crucial, for without them the formal recursion formulas for the B_n's would contain non-integrable singularities at x_c. The adjustment dictates the values of the ℓ_n's; in particular $\ell_0 = q(x_c)/p'(x_c)$, the number which occurs in the Ackerberg-O'Malley resonance condition.

The relevant connection formula between solutions of (6) is an immediate reflection of the connection formula for the (multiple-valued) Whittaker functions

$$W_{k,1/4}(\xi e^{-2\pi i}) = \frac{-2\pi i \cdot \exp(-k\pi i)}{\Gamma(-\ell/2)\Gamma((1-\ell)/2)} W_{-k,1/4}(\xi e^{-\pi i}) + e^{-k\pi i} W_{k,1/4}(\xi), \tag{8}$$

where $k = 1/4(1 + 2\ell)$. The condition for resonance for (6) is the linear dependence of $W_{k,1/4}(\xi)$, which is exponentially small to the right of the turning point, and $W_{k,1/4}(\xi e^{-2\pi i})$ which is exponentially small to the left. A glance at the gamma functions in (8) shows that resonance occurs in (6) whenever $\ell(\epsilon)$ is a non-negative integer, i.e., when $\ell_0 = q(x_c)/p'(x_c)$ is a non-negative integer and $\ell_1 = \ell_2 = \cdots = \ell_N = 0$. The latter are equivalent to the successive resonance conditions of Cook-Eckhaus and Lakin.

Resonance for (5) and hence for (1) requires resonance in (6) for *every* $N = 1, 2, \cdots$ and hence that $\ell_n = 0$ for all positive integers. The sufficiency of the conditions is an easy consequence of the uniform simplification theorem, first proved for a second order turning point by Roy Lee [11] and described in the monograph of Sibuya [21]. More precisely, by Lee's result the conditions imply resonance for a certain differential equation which cannot be distinguished asymptotically from the given equation. Within the context of asymptotic analysis, that is the most that can be said.

3. **The interior flaring phenomenon.** Significantly different behavior occurs for the B.V.P.

$$(9)\quad \begin{aligned} &(a)\quad \epsilon y'' - p(x)y' + q(x)y = 0 \qquad (p'(x) < 0, p(x_c) = 0) \\ &(b)\quad y(a) = \alpha, \quad y(b) = \beta, \end{aligned}$$

which superficially differs from B.V.P.(1) only in the sign of the coefficient of y'. Defining ξ_x now to be,

$$\xi_x = -(1/\epsilon)\int_{x_c}^{x} p(t)\,dt,$$

the transformation $y = v \cdot \exp(-1/2\xi_x)$ normalizes the differential equation (9 a) to (5), so that once again there is a second order turning point and the algorithm of [15] can be applied. As before there are two asymptotically simple solution forms on either side of the turning point:

$$(10)\quad \begin{aligned} &(a)\quad y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots \\ &(b)\quad \exp(-\xi)[u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots], \end{aligned}$$

where now the solutions (10 b) are asymptotically *recessive*, i.e., $\to 0$ as $\epsilon \to 0$ for x away from the turning point. It is these asymptotically recessive solutions which arise via the transformation $y = v \exp(-1/2\xi_x)$ from the Whittaker functions $W_{k,1/4}(\xi)$ and $W_{k,1/4}(\xi e^{-2\pi i})$, respectively to right and left of the turning point. As we shall demonstrate there is again a kind of resonance phenomenon when these two (recessive) Whittaker functions become dependent. Because here $k = -1/4(1 + 2\ell)$, the conditions for resonance are:

$$\ell_0 = \textit{negative} \text{ integer}; \quad \ell_1 = \ell_2 = \cdots = 0$$

The solution of the boundary value problem can now be sketched: *Out* of resonance, the solution which is balanced to the left of the turning point connects to a recessive solution to the right, and hence is asymptotically zero at b. This solution, multiplied by a constant, will satisfy the boundary condition at a. A second solution displays the image behavior, i.e., is balanced to the right satisfying the B.C. at b, and damps exponentially to zero to the left of the turning point. Superimposing the two solutions yields a solution to the full boundary value problem. This full solution is asymptotic to a solution y_0^- of the reduced equation on $[a, x_c)$ and to another y_0^+ on $(x_c, b]$. In general y_0^+ and y_0^- *cannot* be expected to match at x_c since boundary values at a and b are assigned independently. Hence a finite adjustment must occur across a narrow critical layer at x_c. Its detailed structure in the layer reflects that of the Whittaker functions $W_{\pm k,1/4}(\xi)$ for bounded ξ.

In resonance one solution is exponentially small on both sides of the turning point (and the lead coefficient $\eta_0(x)$ suffers no jump on crossing the critical layer.) A second solution is balanced on both sides of x_c (but with a definite jump in its lead coefficient $y_0(x)$ — the size of the jump may be calculated by the use of the connection formulas). Suppose $\xi_a > \xi_b$, and define a point a' by $a' < x_c$, $\xi_{a'} = \xi_b$. Now modify the recessive solution (b) by the expansion factor $\exp(\xi_b)$ to

$$c \exp(\xi_b - \xi_x)[u_0(x) + \epsilon u_1(x) + \cdots]. \tag{11}$$

By adjusting c, one gives this solution any desired value at $x = b$; it will be exponentially asymptotic to zero for $x \in [a, a')$. One now solves the B.V.P. by matching the B.C. at $x = a$ by a multiple of the balanced solution and correcting the discrepancy at $x = b$ by adding in a solution of form (11).

A similar approach, with endpoints interchanged handles the case $\xi_a < \xi_b$. If $\xi_a = \xi_b$ solution (11) has non-zero values at both endpoints, but a linear combination with the balanced solution will still solve B.V.P. (9).

The presence of a component of form (11) in the resonant solution means that in the interior subinterval (a', b) where $\xi_x < \xi_b$, the resonant solution becomes exponentially large as $\epsilon \to 0$. In particular the resonant solution in $[a', b]$ is *not* asymptotically close to a solution of the reduced equation. Despite these differences from Ackerberg-O'Malley, perhaps the term "resonance" remains descriptive of the dramatic consequences of a minor adjustment in the parameter ℓ.

4. **An oscillatory system.** As is well-known in classical analysis, a proper understanding of the special functions of mathematical physics is gained only by examining their behavior in the complex domain. In the case of the Whittaker functions such a treatment suggests defining

$$W_\nu(\xi) = W_{\delta k, 1/4}(\xi e^{-\nu\pi i}),$$

where ν is any integer and $\delta(\nu) = (-1)^\nu$. For $|\xi|$ large, $W_\nu(\xi)$ has the asymptotic representation

$$W_\nu(\xi) = e^{-1/2\delta\xi}(\xi e^{-\nu\pi i})^{\delta k}[1 + O(1/\xi)], \tag{12}$$

valid in the sector

$$\textstyle\sum_\nu : (\nu - 3/2\pi) + \epsilon \leqq \arg \xi \leqq (\nu + 3/2\pi) - \epsilon.$$

In the middle third of this sector, namely in

$$S_\nu : \nu - 1/2\pi < \arg \xi < \nu + 1/2\pi,$$

the function $W_\nu(\xi)$ is exponentially recessive in character; in the outer thirds it is exponentially dominant, and on the dividing lines (anti-Stokes lines) where $\operatorname{Re} \xi = 0$, $W_\nu(\xi)$ is purely oscillatory. To represent W_ν outside of $\sum_\nu$ one must have recourse to lateral connection formulas, e.g., formula (8) expresses W_2 as a linear combination of W_0 and W_1.

In our applications,

$$\xi = 2\lambda \int_{x_c}^{x} p(t)\, dt, \tag{13}$$

where up to now the parameter λ has been large and positive. The asymptotic sectors S_ν in ξ are imaged in asymptotic sectors $S_\nu{}'$ in the x-plane, sectors with central angle 90°. When λ is positive and $p'(t) > 0$, the x-sectors are bounded by the 45° lines $\operatorname{Im} x = \pm \operatorname{Re} x$, and so when x is *real* e^ξ has always the dominant and $e^{-\xi}$ the recessive character.

To obtain *oscillatory* behavior for real x, one must take λ to be pure imaginary; this is the basic ingredient of our next example.

Accordingly, we consider the D.E.

$$-iu'' = \lambda[p(x)u' - q(x)u], \quad \lambda \to \infty, \tag{14}$$

where p and q are real, $p'(x) > 0$ and $p(x_c) = 0$ for some $x_c \in (a, b)$.

With ξ defined by (13), the sectors $S_\nu{}'$ in the x-plane are ordinary cartesian quadrants centered at x_c. The positive axis $x > x_c$ is the boundary between $S_0{}'$ (below) and $S_1{}'$ (above); the negative axis is the boundary between $S_2{}'$ (above) and $S_3{}'$ (below). Consequently, to the right of the critical point W_0 and W_1 are given directly by asymptotic formulas (12), while to the left W_2 and W_3 have this direct representation. No solution can be continued across the critical point without use of a connection formula.

Since Whittaker functions enter into the solution of (14) only after a normalization $u = v \exp(-1/2\xi)$, one concludes that the *balanced* solutions are $v_0(x, \lambda)$ and $v_2(x, \lambda)$, i.e., those obtained as transformations from $W_0(\xi)$ and $W_2(\xi)$. Thus *resonance* corresponds to the linear dependence of these solutions, i.e., that ℓ be a non-negative integer in the connection formula (8). Working through the algorithm for a solution of the form (7) one may in principal determine recursively the infinite set of conditions for resonance; the first is that $q(x_c)/p'(x_c) = \ell$.

The behavior of solutions may now be simply described: *Out* of resonance one finds one solution which is balanced to the left of x_c, i.e., is asymptotic there to a solution of the reduced equation. On crossing to the right of the critical point that solution becomes the superposition of a balanced solution and a rapidly oscillating part of a comparable magnitude. A second solution has a structure which is mirror image to the first.

In resonance these two solutions coalesce into one, which is balanced on both sides of x_c with no jump through the critical layer except in smaller order terms. A second solution is purely oscillatory to the right of x_c and a superposition of both forms to the left. A third solution is oscillatory to the left and a superposition to the right.

The change in behavior of resonance is clearly shown when one imposes B.C. such as $pu' = qu$ at $x = a, b$, since these are automatically satisfied by a solution which is balanced.

5. **A uniformly resonant model.** The Orr-Somerfeld B.V.P. has the form

$$
\begin{aligned}
&(D^2 - \sigma^2)^2\varphi - i\,R[U(x) - c](D^2 - \sigma^2)\varphi \\
&\qquad + U''(x)\varphi = 0, (D = d/dx) \qquad (15)\\
&\varphi = \varphi' = 0 \text{ at } x = a \text{ and } x = b,
\end{aligned}
$$

where $U(x)$ is an unperturbed velocity profile, $R \gg 1$ is Reynolds' number and $\varphi(x)\exp[i\sigma(y - ct)]$ is the stream function for a velocity perturbation wave. Unstable waves correspond to $\operatorname{Im} c > 0$; the profile is stable for given R when there are no unstable waves for *any* real σ. The "stable boundary" consists of waves, specified by (c, σ, R), for which c is *real*.

A second order B.V.P. which at least superficially resembles (15) is

$$
(16) \quad \begin{aligned} &(a) \quad \epsilon D^2\varphi - [U(x) - c](D + \sigma)\varphi \\ &\qquad + \eta(\epsilon)U'(x)y = 0, (\epsilon = ((b - a)/R) \\ &(b) \quad \varphi(a) = 0, \quad \varphi(b) = 0. \end{aligned}
$$

Whenever $U'(x) > 0$ on $[a, b]$, this is for real σ and small positive ϵ a B.V.P. of the type (1) studied by Ackerberg and O'Malley [1]. If c is real and in $[U(a), U(b)]$ there is a turning point at $x_c \in [a, b]$, i.e., for $U(x_c) = c$.

For Orr-Sommerfeld, the "outer" solution, away from boundary layers and critical layer, is asymptotic to a non-zero solution of the reduced equation. As we know, to achieve the same for (16), this second order equation must be in resonance for all σ and all c. The special structure of (16a) makes this possible; in particular, the Ackerberg-O'Malley necessary condition for resonance is simply that the leading term of $\eta(\epsilon) = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots$ shall be a non-negative integer ℓ. By examining the details of the algorithm in [15], it becomes evident that $\eta_1, \eta_2, \cdots$ may be determined recursively so that (16) will satisfy *all* Lakin-Cook-Eckhaus conditions — unfortunately subsequent η_j's must vary with c and σ. We *assume* that the adjustments have been made and therefore that B.V.P. (16) is in resonance, uniformly in σ and c.

It happens, in studying the neutral curve for Orr-Sommerfeld, that the entire lower branch of this curve may correspond to values of x_c which are near to a boundary point $x = a$ or b, causing *the boundary layer and critical layer to coalesce.* This state of affairs turns out to be important for B.V.P. (16) as well and brings into play a new connection formula, namely

$$
(17) \quad W_{k,1/4}((\xi) = \frac{\Gamma(-1/2)}{\Gamma(-\ell/2)} M_{k,1/4}(\xi) + \frac{\Gamma(1/2)}{\Gamma((1-\ell)/2)} M_{k,-1/4}(\xi).
$$

Here $M_{k,\pm 1/4}(\xi)$ are those Whittaker functions which are real for real ξ and are distinguished by their functional form at the branch point $\xi = 0$. At resonance (where ℓ is a non-negative integer) $W_{k,1/4}(\xi)$ becomes a multiple of one of the M's, which itself reduces to $\xi^{-3/4}e^{\xi/2}$ times a Laguerre polynomial. Thus both functions $M_{\pm k,1/4}$ have only finitely many oscillations on the axis. The complex-valued function $W_{-k,1/4}(\xi e^{-\pi i})$ is given by a connection formula similar to (17), from which as ξ traverses the real axis the function values wind around the origin finitely many times.

Returning to the B.V.P. (16), let $\bar{U}$ be the average value of $U(x) = U_x$ on the interval $[a, b]$:

$$\bar{U} = \int_a^b U_t \, dt/(b - a).$$

For values of c away from the endpoint velocities U_a and U_b, the characteristic equation for B.V.P. (16) is

$$-e^{R(\bar{U}-c)} = \left(\frac{U_b - c}{c - U_a} \right)^{2\ell+1} e^{-2\sigma(b-a)}. \tag{18}$$

On the other hand when $x_c = b - O(R^{-1/2})$ the characteristic equation becomes

$$\begin{aligned} &-e^{+k\pi i} R^{-(\ell+1/2)} e^{R(c-\bar{U})} \\ &\quad = \underline{\bar{K}} e^{2\sigma(b-a)} e^{-\xi_b} [W_{-k,1/4}(\xi_b e^{-\pi i})/W_{k,1/4}(\xi_b)] \end{aligned} \tag{19}$$

where $\xi_b = R(U_b - c)^2/(2U_b'(b - a))$ and $\underline{\bar{K}} = (U_b')^{\ell-1/2}(U_b - U_a)^{2\ell+1}/[2(b - a)]^{\ell+1/2}$ is a real constant.

In order to satisfy the characteristic equation it is necessary to allow complex values of σ; the simplest case is where

$$2\sigma(b - a) = \gamma\pi i, \quad (\gamma \text{ real}). \tag{20}$$

With assumption (20) one obtains a neutral curve, most conveniently graphed by plotting $\gamma = \gamma(c)$ and $R = R(c)$ against $c \in [U_a, U_b]$ as independent variable.

Referring to (19) the factor $e^{\gamma\pi i}$ must wind around the unit circle a finite number of times to compensate for the variations in $W_{-k,1/4}(\xi e^{-\pi i})$. Thus $\gamma(c)$ is virtually constant for c away from the end points, and near $c = U_b$ varies motonically according to

$$\gamma(\xi_b) = (\ell/2 + 5/4) - \frac{\arg W_{-k,1/4}(\xi_b e^{-\pi i})}{\pi}.$$

The graph $R = R(c)$ displays vertical asymptotes, from equation (18) at $(U_a + U_b)/2$ and from equation (19) at the finitely many zeroes of $W_{k,1/4}(\xi)$. Under the symmetry assumption that $\bar{U} = (U_a + U_b)/2$, all components of the graph are U-shaped, opening toward $R = +\infty$.

Other conditions than (20) on σ yield other kinds of behavior. One interesting case is to allow σ to vary over any radial line in the complex plane and to search out eigenvalues c for any large but fixed value of R. There will be a finite number of such eigenvalues on $[U_a, U_b]$, interlacing the zeroes of $M_{k,1/4}$ and tending to the endpoints as $R \to \infty$. Here $[U_a, U_b]$ is a line of continuous spectrum for any reduced B.V.P.

We comment briefly on the role of resonance for the O.S. equation (15). Formal asymptotic solution of O.S. by Lin and Rabenstein [13] requires recursive determination of two functions $\alpha(\epsilon) = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \cdots$ and $\beta(\epsilon)$, which play the same role as $k(\epsilon)$ in our theory, in suppressing non-integrable singularities which arise in the formal algorithm. (See [13]).

Now $\alpha(\epsilon)$ figures in a much-used connection formula, namely

$$A_1 + A_2 + A_3 = B_0 + [1 - e^{-2\pi i\alpha(\epsilon)}]\, B_2. \tag{21}$$

Here A_1, A_2, A_3 are solutions of dominant-recessive type, B_0 is a wholly balanced solution and B_2 is balanced in one sector and dominant in its complement.

Clearly a coalesence of solutions occurs when $\alpha(\epsilon)$ is exactly an integer. In the case of the Orr-Sommerfeld equation, $\alpha_0 = 0$, so resonance may be suspected. However, as calculated by Lakin and Reid [8], $\alpha_1 \neq 0$ so the equation is *not* in resonance. On the other hand, Wasow's first approximation to O.S. [22] uses $\alpha \equiv 0$ and so *is* in resonance. The use of a resonant approximant to a non-resonant equation would seem to require some caution, but the impact of resonance in (21) is presently more obscure than for the second order equation.

The state-of-affairs for O.S. suggests that in some respects a more revealing second order model may be furnished by the interior flaring phenomenon: one again uses (16), but now with $U' < 0$ and with $\eta(\epsilon)$ *exactly* equal to a negative integer. We do *not* adjust subsequent coefficients $\eta_1, \eta_2, \cdots$ into resonance. Details will be given in R. Bohac [2].

6. **Resonance in a singular B.V.P.** We shall, in this report, confine attention to B.V.P.'s associated with the special D.E.

$$\epsilon y'' - y' + (k/x - \epsilon\tau/x^2)y = 0 \text{ on } [0, b]. \tag{22}$$

In fact, everything goes over, with little extra effort and some complication of notation, to the class of D.E.'s.

$$\epsilon y'' - \varphi(x,\epsilon)y' + (\psi(x,\epsilon)/x - \epsilon\tau(\epsilon)/x^2)y = 0,$$

where $\varphi(x,\epsilon) \neq 0$ on $[0, b]$: the asymptotic analysis in McKelvey [16] establishes that (22) is entirely typical in this class.

Allowing $\epsilon = 0$ one obtains the reduced equation

$$xy' - ky = 0, \tag{23}$$

with the explicit solution $y = x^k$.

The transformation

$$\xi = x/\epsilon,\ y = e^{\xi/2}W,\ m = (1 + 4\tau)^{1/2}\ (\operatorname{Re} m \geqq 0)$$

carries (22) into Whittaker's differential equation

$$\frac{d^2W}{d\xi^2} = \left(\frac{1}{4} - \frac{k}{\xi} - \frac{1 - m^2}{4\xi^2}\right)W.$$

Hereafter, we confine attention to the case $\tau \geqq -1/4$, for which m is real. We may immediately draw the relevant facts about the solutions of (22):

1. Away from the singular point $x = 0$, the solutions of distinguished asymptotic form are:

$$y_+ = e^{\xi/2}W_{k,m/2}(\xi) = c_+x^k + O(\epsilon),$$
$$y_- = e^{\xi/2}W_{-k,m/2}(\xi e^{-\pi i}) = e^{x/\epsilon}[c_-x^{-k} + O(\epsilon)].$$

2. At the singular point $x = 0$ the solution of highest index is $y_0 = e^{\xi/2}M_{k,m/2}(\xi) = O(\xi^{m+1/2})$ as $\xi \to 0$. All other solutions have index $\xi^{-m+1/2}$(or $\xi^{1/2} \mid n\xi$ when $m = 0$).

3. The condition for *resonance*, i.e., for the dependence of the balance solution y_+ and the high index solution y_0 is that

$$\ell \equiv k - (1 + m)/2$$

be a non-negative integer.

Boundary conditions natural to (22) on $[0, b]$ are of the form

$$\lim_{x \downarrow 0} x^{-m+1/2}y(x) = \alpha, \quad y(b) = \beta,$$

Of particular interest to us will be the case $\alpha = 0, \beta = 1$, or

(24) $$y(x) = O(x^{m+1/2}) \text{ as } x \to 0, \quad y(b) = 1.$$

In resonance it is satisfied by cy_+, non-zero balanced solution which, because of resonance, is $O(x^{m+1/2})$ at the origin. *Out* of resonance the candidate is y_0, the only solution which is $O(x^{m+1/2})$. This solution behaves like $e^{x/\epsilon}x^{-k}$ at $x = b$; by modifying it to $ce^{b/\epsilon}y_0(x)$ one converts it into boundary layer form. Thus the modified solution, which does indeed satisfy B.C. (24) is exponentially small away from the endpoints.

Note that, had we allowed $\tau < -1/4$, m would have become pure imaginary, and the solution behavior at the singular point would have been decidedly different.

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