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Ackermann's Method: Revisited, Extended, and Generalized to Uncontrollabe Systems

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ABSTRACT The celebrated method of Ackermann for eigenvalue assignment of single-input controllable systems is revisited in this paper, contributing an elegant proof. The new proof facilitates a compact formula which consequently permits an extension of the method to what we call incomplete assignment of eigenvalues. The inability of Ackermann's formula to deal with uncontrollable systems is considered a weakness inherent in the method. The notion of incomplete assignment leads to a straightforward generalization of our method to eigenvalue assignment of uncontrollable systems, thus mitigating such a drawback of a popular method with minor modifications. Further results concerning the incomplete assignment are stated, verified, and commented. Such results reveal the trace of the state matrix A a worthy feature pertinent to an open loop system . Finally, four numerical examples are worked out to demonstrate cases of incomplete and uncontrollable eigenvalues assignment. The examples consider a case where the structure of the feedback matrix can be easily simplified. The paper ends with a commentary brief concerning some commonly used MATLAB commands for eigenvalue assignment.

INDEX TERMS - Ackermann's method, eigenvalue assignment, incomplete assignment, MATLAB, uncontrollability.

I. INTRODUCTION

The eigenvalue assignment problem is well established in system theory and continues to attract further research over the years. Since the study concentrates on a particular method namely Ackermann's method [1-2], attempting to review such myriad methods is beyond the scope of this study.

Nonetheless, in a brief, the problem of eigenvalue assignment has been tackled through many approaches ranging from an algebraic approach [3-4], a geometric approach [5], conjectural and explicit determination [6-7], recursive methods [8], eigenstructure methods [9-10], closed loop robustness [11], Minimization of certain condition numbers [12], algorithmic, numerical, and computer-aided design approaches [13-17]. Additional approaches have been through the theory of variable structure systems [18], partial assignment [19-21], mechanical structures [22], and derivative and acceleration feedback [23-24], to mention but a few.

Among the numerous methods for eigenvalue assignment is the well- known method of Ackermann [1-2], [25-26]. The method is simple in concept, and of clear-cut nature. Besides, it doesn't require knowledge of the open loop characteristic polynomial nor the requirement of similarity transformations. However, the method applies to controllable systems only. This is evident by its vivid reliance on the inverse of the controllability matrix.

We revisit the method in this paper, shedding more light on its nature, and provide an elegant proof. Such alternative proof results in a compact form of the state feedback matrix and facilitates an extension of the method to incomplete assignment of eigenvalues (in the sense of assigning a limited number of eigenvalues). In certain cases, demonstrated later in the examples, incomplete assignment becomes the usual partial eigenvalue assignment.

Since Ackermann's method deals with controllable systems only, extending the method to uncontrollable systems should be a significant improvement. In fact, it turns out to be a special case within the incomplete assignment approach, and is shown to be possible with minor modification. Beside the use of incomplete assignment as a mechanism to extend the method to the uncontrollable case, it can be justified on its own right in some cases, leading to a simpler controller of lower dimensionality.

II. NATURE OF THE ACKERMANN'S FORMULA

The method is well known, simple in concept, of explicit nature, and requires no particular transformation. However, it suffers from the drawback of only applying to controllable single-input systems [1-2], [25-26]. It is implemented by MATLAB [35], through the built-in *acker* function command.

Many control theorists commented on the derivation of the Ackermann's formula as being nontrivial. P.C. Chau [25]

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commented: "it is not trivial to prove Ackermann's formula and most introductory texts do not derive it". Nonetheless, some authors approach the derivation of the method by restricting the proof to the case of a three state system as a compromise, as that presented in [25-26]. To the author's knowledge , other proofs are carried out also for the case of $n = 3$, [27-29].

In this section, we present a general derivation in a straightforward manner irrespective of the order of the system.

Consider a linear time invariant system given by,

$$\dot{x} = Ax + Bu \quad ; \quad x \in R^n, u \in R^1 \tag{1}$$

Following the same lines as that of Ackermann's , since matrix B has full rank 1, it is routinely replaced by b ; the state feedback controller used is $u = -K\,x$.

where K is a $1 \times n$ matrix resulting in a closed loop system given by.

$$\dot{x} = (A - bK)x \tag{2}$$

Given that γ_i ; $i = 1, 2, \cdots, n$ are the coefficients of the closed loop characteristic equation (polynomial), Ackermann's formula calculates K as in [1-2], and [25-26] as,

$$K = [0 \quad 0 \quad \cdots \quad 1]([b \quad Ab \quad \cdots \quad A^{n-1}b]^{-1}$$
$$\times [A^n + \gamma_1 A^{n-1} + \gamma_2 A^{n-2} + ... + \gamma_n I_n]) \tag{3}$$

Due to the reliance of K on the inverse of the controllability matrix, such description of the feedback matrix as in (3) necessarily assumes and requires the system to be controllable, thus limiting the applicability of the method to controllable systems only.

We shall now consider another approach to the derivation of K , and thus provide an alternative general proof to the above expression given in (3) , ending with a compact form for K . Such depiction facilitates incomplete eigenvalue assignment and enables the method to be extended to uncontrollable systems.

III. AN ELEGANT PROOF

An alternative proof of Ackermann's formula is now worked out. As a consequent, we gain further insight into the method yielding what we call incomplete eigenvalue assignment. Such novel treatment of the classical Ackermann's method culminates in a generalization of the method to eigenvalue assignment of uncontrollable systems.

Assuming the closed loop system matrix $(A - bK$) has $\lambda_1, \lambda_2, \cdots, \lambda_n$ as closed loop eigenvalues. The associated closed loop characteristic equation $P_{cl}(\lambda)$ is therefore,

$$P_{cl}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \tag{4}$$

Expanding we get,

$$P_{cl}(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \gamma_2 \lambda^{n-2} + \cdots + \gamma_{n-1}\lambda + \gamma_n \tag{5}$$

Where γ_i are obtained using the theory of elementary symmetric polynomials [30].

Applying Cayley-Hamilton theorem to (5), i.e. setting $P_{cl}(A - bK) = 0$ gives.

$$(A - bK)^n + \gamma_1(A - bK)^{n-1} + \gamma_2(A - bK)^{n-2} + \cdots$$
$$+ \gamma_{n-1}(A - bK) + \gamma_n I_n = 0 \tag{6}$$

Let the controllability matrix T be given by,

$$T = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-2}b & A^{n-1}b \end{bmatrix} \tag{7}$$

With the following partitioning of the inverse of T as,

$$T^{-1} = \begin{bmatrix} b^g \\ (Ab)^g \\ \vdots \\ (A^{n-2}b)^g \\ (A^{n-1}b)^g \end{bmatrix} \tag{8}$$

The fact that $T^{-1}T = I_n$ necessitates,

$$(A^i b)^g (A^j b) = 0 \text{ for } i \neq j \ ; \ i, j = 0, 1, 2, \cdots, n-1$$
$$= 1 \quad i = j \tag{9}$$

while $TT^{-1} = I_n$ necessitates,

$$\sum_{i=0}^{n-1} (A^i b)(A^i b)^g = I_n \tag{10}$$

Each row matrix with the superscript g is a unique generalized inverse of a column of the matrix T ,[31-34]. The generalized inverses are unique in our case since they satisfy the additional conditions given in (9), and (10).

Pre-multiplying (6) by $(A^{n-1}b)^g$ where $(A^{n-1}b)^g$ is the last row of T^{-1} , and , observing the conditions in (9) , we simplify each term as follows.

$$(A^{n-1}b)^g (A - bK) = (A^{n-1}b)^g A$$

$$(A^{n-1}b)^g (A - bK)^2 = (A^{n-1}b)^g (A - bK)(A - bK)$$
$$= (A^{n-1}b)^g A(A - bK)$$
$$= (A^{n-1}b)^g A^2$$

$$(A^{n-1}b)^g (A - bK)^3 = (A^{n-1}b)^g (A - bK)^2 (A - bK)$$
$$= (A^{n-1}b)^g A^2(A - bK)$$
$$= (A^{n-1}b)^g A^3 \tag{11}$$

Continuing this process, the last term ends up as,

$$(A^{n-1}b)^g (A - bK)^n = (A^{n-1}b)^g (A - bK)^{n-1}(A - bK)$$
$$= (A^{n-1}b)^g A^{n-1}(A - bK)$$
$$= (A^{n-1}b)^g A^n - K \tag{12}$$

Hence, (6) is now compactly represented as,

$$-K + (A^{n-1}b)^g [A^n + \gamma_1 A^{n-1} + \cdots + \gamma_{n-1}A + \gamma_n I_n] = 0 \tag{13}$$

Consequently,

$$K = (A^{n-1}b)^g [A^n + \gamma_1 A^{n-1} + \cdots + \gamma_{n-1}A + \gamma_n I_n] \tag{14}$$

It's not hard to recognize that the product of the first two terms of (3) is effectively $(A^{n-1}b)^g$, i.e.

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$$[0\ 0\ \cdots\ 0\ 1][\ b\ \ Ab\ \cdots A^{n-2}\ \ A^{n-1}b\ ]^{-1}$$

$$=[0\ 0\ \cdots\ 0\ 1]\begin{bmatrix} b^g \\ (Ab)^g \\ \vdots \\ (A^{n-1}b)^g \end{bmatrix}$$

$$=(A^{n-1}b)^g \qquad (15)$$

Hence, K as in (3) is elegantly proved for the general case of n states as compared to special case proofs [25-26]. Furthermore, the new form,

$$K = (A^{n-1}b)^g [A^n + \gamma_1 A^{n-1} + \cdots + \gamma_{n-1}A + \gamma_n I_n]$$
$$= (A^{n-1}b)^g (A - \lambda_1 I_n)(A - \lambda_2 I_n)\cdots(A - \lambda I_n)$$
$$= (A^{n-1}b)^g P_{cl}(A) \qquad (16)$$

stands as a compact alternative to the original classical form presented in (3).

Such depiction proves to be convenient in extending the method to incomplete eigenvalue assignment and to the assignment of uncontrollable systems as will be shown in sec.V, and sec.VI.

IV. FURTHER INSIGHT

The original depiction of the Ackermann's formula artificially accentuates the fixed positional order of the number 1 in the first row and the fixed positional order of the columns $b, Ab, \cdots, A^{n-1}b$ in the controllability matrix. The new depiction shows that such order can be dispensed with. The importance is in fact to the singling of the $(A^{n-1}b)^g$ row out of T^{-1}. In other words, the controllability matrix needs not be in its common layout. A possible alternative is,

$$K = \begin{bmatrix} 1 & 0 & 0 \cdots 0 \end{bmatrix}\begin{bmatrix} A^{n-1}b & Ab & b & A^2b & \cdots \end{bmatrix}^{-1}$$
$$\times [A^n + \gamma_1 A^{n-1} + \ldots + \gamma_{n-1}A + \gamma_n I_n] \quad (17)$$

Another reordering may be,

$$K = \begin{bmatrix} 0 & 0 & 1 \cdots 0 \end{bmatrix}\begin{bmatrix} A^2b & b & A^{n-1}b & Ab & \cdots \end{bmatrix}^{-1}$$
$$\times [A^n + \gamma_1 A^{n-1} + \ldots + \gamma_{n-1}A + \gamma_n I_n] \quad (18)$$

In fact, any ordered arrangement of the columns of the controllability matrix is eligible as long as the 1 in the first row matrix has the same column position as that of $A^{n-1}b$ in the controllability matrix. Such observation couldn't have been naturally spotted when considering the original classical proof of the method. In other words, expressing $\left((A^{n-1}b)^g \right)$ as

$$\left(A^{n-1}b \right)^g = \begin{bmatrix} 0 & 0 & 0 \cdots 1 \end{bmatrix}\begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix}^{-1} \quad (19)$$

And consequently K as,

$$K = \left(A^{n-1}b\right)^g [A^n + \gamma_1 A^{n-1} + \ldots + \gamma_{n-1}A + \gamma_n I_n]$$
$$= \left(A^{n-1}b\right)^g P_{cl}(A) \qquad (20)$$

further enhances compactness, and renders the method a closed-form formula which is analytically more appealing.

V. INCOMPLETE EIGENVALUES ASSIGNMENT

Partial assignment of eigenvalues is well-known in control theory analysis and applications. It is generally needed whenever the mainstream of the system eigenvalues are satisfactory while the remaining ones are not. In which case, a subset of the spectrum of the system matrix A is reassigned leaving the rest of the spectrum invariant. In certain cases, this is favorable and usually results in a simplified feedback controller in terms of structure, dimension, and robustness [19-21]. In our study we will not pursue partial assignment any further. The reader is warned not to mix it with what we shall call incomplete assignment described next.

Our method has an analogous feature to partial assignment, but of different nature. It will be called incomplete eigenvalue assignment. It enjoys the liberty of assigning a number of eigenvalues less than n, say q if so desired without any concern of the remaining $n-q$ eigenvalues. In other words, the remaining assigned eigenvalues are enforced and are not known in advance. They depend on the structure of the system and the particular newly defined T which may differ from the usual controllability matrix especially if the system is uncontrollable.

One may argue the point behind that; in which case the answer is that such an approach enables an extension of the method to uncontrollable systems. Besides, as shall be shown in sec.VII, certain preliminary information can be obtained beforehand regarding these eigenvalues and in certain cases can simplify the feedback K matrix. In the worst case such incomplete assignment can be looked upon as a characteristic feature pertinent to the newly shaped Ackermann's method.

In the light of the new derivation and depiction of the state feedback matrix, the case of incomplete assignment of eigenvalues is now shown to be easily carried out.

Assume we are only interested in the closed loop assignment of the q eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_q$, where $q < n$ with no interest in the remaining $n-q$ closed loop eigenvalues. Let these remaining enforced closed loop eigenvalues be the roots of an $(n-q)th$ polynomial say, $P(\lambda)$. Consequently, the characteristic equation of our closed loop feedback system is,

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_q)\ P(\lambda) = 0 \qquad (21)$$

Or equivalently,

$$(\lambda^q + v_1 \lambda^{q-1} + \cdots + v_{q-1}\lambda + v_q)\ P(\lambda) = 0 \qquad (22)$$

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Where $v_1, v_2, \cdots, v_q$ are obtained by expanding the first q terms of (21). Alternatively, they are compactly expressed using the theory of elementary symmetric polynomials [30]. The first parenthesis accounts for the q eigenvalues to be assigned while $P(\lambda)$ is in fact irrelevant to the forthcoming derivation.

Due to the uncontrollability of the system we now introduce an alternative T matrix modified to have the following form.

$$T = \begin{bmatrix} b & Ab & \cdots & A^{q-1}b & \vdots & N \end{bmatrix} \qquad (23)$$

Where N is any $n \times n - q$ matrix ensuring the nonsingularity of T. Accordingly, the following conditions are to be observed and subsequently will be referred to quite frequently.

$$T = \begin{bmatrix} b & Ab & \cdots & A^{q-1}b & \vdots & N \end{bmatrix}$$

$$\Rightarrow \quad T^{-1} = \begin{bmatrix} b^g \\ (Ab)^g \\ \vdots \\ (A^{q-1}b)^g \\ \cdots \\ N^g \end{bmatrix} \qquad (24)$$

$T^{-1}T = I_n$ necessitate,

$$(A^i b)^g (A^j b) = 0 \text{ for } i \neq j$$
$$= 1 \text{ for } i = j$$
$$i, j = 0, 1, 2, \cdots, q-1$$
$$(A^i b)^g N = 0_{1 \times n-q} \text{ for } i = 0, 1, \cdots, q-1$$
$$N^g (A^i b) = 0_{n-q \times 1} \text{ for } i = 0, 1, \cdots, q-1$$
$$N^g N = I_{n-q \times n-q} \qquad (25)$$

$T T^{-1} = I_n$ necessitate,

$$\left[\sum_{i=0}^{q-1} (A^i b)(A^i b)^g \right] + N N^g = I_n \qquad (26)$$

Applying Cayley-Hamilton theorem to (22) in a similar manner as has been done to (5) one gets.

$$((A-bK)^q + v_1(A-bK)^{q-1} + v_2(A-bK)^{q-2}$$
$$+ \cdots + v_{q-1}(A-bK) + v_q I_n) \, P(A-bK) = 0 \qquad (27)$$

Note that the terms in the first parenthesis are analogous to those in (6) with q replacing n. Repeating the same methodology as in sec. III, we get,

$$((A^{q-1}b)^g A^q - K + (A^{q-1}b)^g [v_1 A^{q-1} + v_2 A^{q-2}$$
$$+ \cdots + v_{q-1}A + v_q I_n]) \, P(A-bK) = 0 \qquad (28)$$

for (28) to hold, it is sufficient that the terms within the overall first parenthesis sum to zero, in which case,

$$(A^{q-1}b)^g A^q - K + (A^{q-1}b)^g [v_1 A^{q-1} + v_2 A^{q-2}$$
$$+ \cdots + v_{q-1}A + v_q I_n] = 0 \qquad (29)$$

Consequently,

$$K = (A^{q-1}b)^g [A^q + v_1 A^{q-1} + \cdots + v_{q-1}A + v_q I_n] \qquad (30)$$

Note that the number of terms involved are now reduced : q instead of n and the highest power of A is q which is necessarily less than n. Both facts add to the numerical attractiveness of the incomplete assignment approach.

This value of K guarantees incomplete assignment of only q eigenvalues . The remaining enforced eigenvalues are dependent upon the structure of the system and the selection of a particular N as will be shown in sec.VII.

VI. UNCONTROLLABLE EIGENVALUE ASSIGNMENT

The new depiction of Ackermann's method superficially disguises the dependence of the method on the controllability matrix, thus helping in justifying an extension of the method to uncontrollable systems. We accomplish this by modifying the T matrix to an alternative nonsingular one as given by (23).

Such approach to the problem of uncontrollable eigenvalues through incomplete assignment effectively extends the method to uncontrollable systems; evading the impossible task of dealing with a would be singular controllability matrix.

An established fact in control theory is that state feedback cannot change the uncontrollable eigenvalues and their associated left eigenvectors. So, any K chosen, leaves these eigenvalues invariant. In other words any K which re-assigns the controllable eigenvalues automatically reassigns the remaining uncontrollable eigenvalues their original open loop values.

The extension of Ackermann's method to the uncontrollable case tackles the problem from incomplete assignment point of view. A state feedback matrix K is calculated to only re-assign the controllable eigenvalues. This automatically leaves the uncontrollable eigenvalues unchanged. This is considered an advantage as knowledge of the uncontrollable eigenvalues actual values is not required.

Therefore, for eigenvalue assignment of uncontrollable systems , the feedback matrix used is K_u ,

$$K_u = (A^{\kappa-1}b)^g [A^\kappa + \sigma_1 A^{\kappa-1} + \cdots + \sigma_{\kappa-1}A + \sigma_\kappa I_n] \qquad (31)$$

Where κ is the number of controllable eigenvalues. The remaining $n - \kappa$ uncontrollable eigenvalues are thus inescapably re-assigned. Bear in mind that $(A^{\kappa-1}b)^g$ has to satisfy the conditions in (25), and (26).

In essence, the method has a safe-guard property when it comes to uncontrollability. Since the method is based on the controllability matrix ,any further incorporation of columns representing the controllable subspaces greater than κ will result in a singular T matrix affirming uncontrollability, consequently, forcing a compulsory need for (23).

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A careful rethinking of (30) when assigning a number of q eigenvalues reveals that K in the case $q < n - \upsilon$ where υ is the number of uncontrollable eigenvalues only ensures the assignment of q eigenvalues, and υ uncontrollable eigenvalues. The υ uncontrollable eigenvalues are automatically reassigned irrespective of the T chosen while the remaining $n - q - \upsilon$ controllable eigenvalues are dependent upon the particular choice of N. As mentioned before, N is needed to ensure the non-singularity of T. Further results concerning these enforcedly assigned eigenvalues are developed below.

VII. THE ROLE OF THE TRACE OF A

In tackling the problem of incomplete eigenvalue assignment of completely controllable systems the usual controllability matrix T can be used. However, a more general approach which allows for systems to be uncontrollable is more favorable and we subsequently pursue it further.

It is now essential to choose T as in (23), where the inclusion of the $n \times n - q$ matrix N is obligatory to accommodate for cases where the system is uncontrollable. Later on, it is shown that choosing T as in (23) can be relaxed in the case of controllable systems. Such relaxation yields a simple bound on the sum of the remaining enforced eigenvalues mainly equaling the trace of A as proved later.

To Accomplish that, let $\sum_{i=1}^{q} \lambda_i$ be the sum of the desired q eigenvalues to be assigned using K as given by (30), and let $\sum_{j=q+1}^{n} \lambda_j$ be the sum of the remaining enforced $n - q$ eigenvalues.

In addition, let $tr(A)$ stand for the trace of a matrix. Using the fact ,[31] that for any two matrices L, and R; $tr(LR) = tr(RL)$, and that the sum of all closed loop eigenvalues $\sum_{l=1}^{n} \lambda_l$ is given by $tr(A - bK)$ then,

$$tr(A - bK) = \sum_{l=1}^{n} \lambda_l = \sum_{i=1}^{q} \lambda_i + \sum_{j=q+1}^{n} \lambda_j$$
$$= tr(A) - tr(bK)$$
$$= tr(A) - tr(Kb) \qquad (32)$$

In order to determine $tr(Kb)$, following the elementary symmetric polynomials notation[30], K in (30) can be conveniently expressed as

$$K = \left(A^{q-1}b\right)^g \left(A^q - \sum_{i=1}^{q} \lambda_i A^{q-1} + \sum_{1 \le i < j}^{q} \lambda_i \lambda_j A^{q-2}\right.$$
$$- \sum_{1 \le i < j < p}^{q} \lambda_i \lambda_j \lambda_p A^{q-3} + \cdots$$
$$\left. + (-1)^q \sum_{1 \le l_1 < l_2 < \cdots < l_q}^{q} \lambda_{l_1} \lambda_{l_2} \cdots \lambda_{l_q} I_n\right) \qquad (33)$$

$$tr(Kb) = tr((A^{q-1}b)^g A^q b) - \sum_{i=1}^{q} \lambda_i tr((A^{q-1}b)^g A^{q-1} b)$$
$$+ \sum_{1 \le i < j}^{q} \lambda_i \lambda_j tr((A^{q-1}b)^g A^{q-2} b)$$
$$- \sum_{1 \le i < j < p}^{q} \lambda_i \lambda_j \lambda_p tr((A^{q-1}b)^g A^{q-3} b) + \cdots$$
$$+ (-1)^q \sum_{1 \le l_1 < l_2 < \cdots < l_q}^{q} \lambda_{l_1} \lambda_{l_2} \cdots \lambda_{l_q} tr((A^{q-1}b)^g I_n b) \quad (34)$$

Using (25),

$$tr(Kb) = tr((A^{q-1}b)^g A^q b) - \sum_{i=1}^{q} \lambda_i \times tr(1)$$
$$+ \sum_{1 \le i < j}^{q} \lambda_i \lambda_j \times tr(0) - \sum_{1 \le i < j < p}^{q} \lambda_i \lambda_j \lambda_p tr(0) + \cdots$$
$$+ (-1)^q \sum_{1 \le l_1 < l_2 < \cdots < l_q}^{q} \lambda_{l_1} \lambda_{l_2} \cdots \lambda_{l_q} tr(0) \qquad (35)$$

Ending with,

$$tr(Kb) = tr((A^{q-1}b)^g A^q b) - \sum_{i=1}^{q} \lambda_i$$
$$= tr((A^{q-1}b)^g A A^{q-1} b) - \sum_{i=1}^{q} \lambda_i$$
$$= tr((A^{q-1}b)(A^{q-1}b)^g A) - \sum_{i=1}^{q} \lambda_i \qquad (36)$$

Invoking (26), and using the trace of a product of matrices property to get traces of zero matrices we arrive at the final answer.

$$tr(Kb) = tr(A) - tr(N^g A N) - \sum_{i=1}^{q} \lambda_i \qquad (37)$$

Invoking (32)

$$\sum_{i=1}^{q} \lambda_i + \sum_{j=q+1}^{n} \lambda_j = tr(A) - tr(A) + tr(N^g A N) + \sum_{i=1}^{q} \lambda_i$$

$$\sum_{j=q+1}^{n} \lambda_j = tr(N^g A N) \qquad (38)$$

i.e. the sum of the $n - q$ enforced eigenvalues is given by $tr(N^g A N)$ which depends on A and the specific choice of N. A necessary condition for stability when using incomplete assignment is a negative trace for $N^g A N$, which may be the case for some systems and certain N.
However, if the system dynamics are not satisfactory and the uncontrollable eigenvalues have negative real parts, one

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can always resort to a full specification of the whole κ controllable eigenvalues ,i.e. letting $q = \kappa$ and use (31).

For the special case of completely controllable systems, T can involve the whole controllable subspace as in (7) , yielding

$$N = [A^q b \ A^{q+1} b \ \cdots \ A^{n-2} b \ A^{n-1} b] \qquad (39)$$

In which case, it is now shown that the sum of the remaining enforced eigenvalues is equal to $tr(A)$.

First, $tr(N^g A N)$ is shown equal to $(A^{n-1} b)^g A^n b$ which is generally not zero. Then we show it equal to $tr(A)$ as demonstrated below.

$$N = [A^q b \ A^{q+1} b \ \cdots \ A^{n-2} b \ A^{n-1} b]$$

$$N^g = \begin{bmatrix} (A^q b)^g \\ (A^{q+1} b)^g \\ \vdots \\ (A^{n-2} b)^g \\ (A^{n-1} b)^g \end{bmatrix} \qquad (40)$$

Hence, using (9) for $j = q, q+1, \cdots, n-2, n-1$.

$$N^g A N = \begin{bmatrix} 0 & 0 & \cdots & 0 & (A^q b)^g A^n b \\ 1 & 0 & \cdots & 0 & (A^{q+1} b)^g A^n b \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & (A^{n-2} b)^g A^n b \\ 0 & 0 & \cdots & 1 & (A^{n-1} b)^g A^n b \end{bmatrix} \qquad (41)$$

$$\therefore \qquad tr(N^g A N) = (A^{n-1} b)^g A^n b \qquad (42)$$

Furthermore, after proper manipulation using trace of a product of product of matrices and using (10) it can be shown that

$$tr(N^g A N) = tr(A) \qquad (43)$$

i.e. the sum of the enforced eigenvalues are now given by $tr(A)$ and therefore their sum is known beforehand. For a single enforced eigenvalue case , this is fine as stability is secured once $tr(A)$ is negative . Sufficient conditions for instability, however, are non-negative $tr(A)$.

VIII. EXAMPLES
Example 1
An unstable single input controllable system has the following system matrices,

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

It is required to assign the two eigenvalues $-4 \pm 5i$, hence,

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} * \begin{bmatrix} b & Ab \end{bmatrix}^{-1} * (A - (-4 + 5i) I_2)(A - (-4 - 5i) I_2)$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} * \begin{bmatrix} b^g \\ (Ab)^g \end{bmatrix} * (A + 8A + 41 I_2)$$

$$K = (Ab)^g * (A + 8A + 41 I_2) = \begin{bmatrix} \tfrac{4}{3} & \tfrac{28}{3} \end{bmatrix} = \begin{bmatrix} 1.3333 & 9.3333 \end{bmatrix}$$

The closed loop system matrix is

$$AK = A - bK = \begin{bmatrix} -\tfrac{2}{3} & -\tfrac{65}{3} \\ \tfrac{5}{3} & -\tfrac{22}{3} \end{bmatrix} = \begin{bmatrix} -0.6667 & -21.6667 \\ 1.6667 & -7.3333 \end{bmatrix}$$

Which can be checked to have the two eigenvalues $-4 \pm 5i$

Example 2
A third-order uncontrollable system has the following system matrices,

$$A = \begin{bmatrix} -8 & 3 & 3 \\ -6 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The uncontrollable eigenvalue is -2, which cannot be changed by state feedback. Since it has negative real part, we have no problem with stability. The remaining two eigenvalues have to have negative real parts, say $-4 \pm 5i$. Since the system is uncontrollable we have no choice but to choose a modified controllability matrix as in (23), i.e.

$$K = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} b & Ab & N \end{bmatrix}^{-1} * (A + (4 - 5i) I_3)(A + (4 + 5i) I_3)$$

Where an $N = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ ensures an invertible modified controllability matrix, hence,

$$K = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} b^g \\ (Ab)^g \\ N^g \end{bmatrix} * (A + 8A + 41 I_3)$$

Resulting in

$$K = (Ab)^g * (A + 8A + 41 I_3) = \begin{bmatrix} -6 & 5 & 30 \end{bmatrix}$$

The closed loop system matrix is

$$AK = A - bK = \begin{bmatrix} -2 & -2 & -27 \\ 0 & -5 & -26 \\ 0 & 1 & -3 \end{bmatrix}$$

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By inspection, -2 is an eigenvalue, while the two remaining eigenvalues are those of the matrix M given by,

$$M = \begin{bmatrix} -5 & -26 \\ 1 & -3 \end{bmatrix}$$

Example 3

Consider assigning repeated eigenvalues for the following controllable system

$$A = \begin{bmatrix} -5 & 3 & 3 & 0 \\ -6 & 3 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It is required to assign $-5, -5$, and $-2 \pm j$ as eigenvalues. Since $tr(A) = -5$, and -5 is an eigenvalue to be assigned, then in order to reduce the number of terms, the theory of incomplete assignment can be used. In this case , according to (38), and (43), N has to be $A^3 b$, and the K used is

$$K = (A^2 b)^g (A + 5I_4)(A + (2-j)I_4)(A + (2+j)I_4) \quad (44)$$

Where $(A^2 b)^g$ is given by the third row of T^{-1},

$$T^{-1} = \begin{bmatrix} b & Ab & A^2 b & A^3 b \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 2 & -8 & 1 & 13 \\ -16 & -7 & -8 & 16 \\ 17 & -8 & -14 & -17 \\ 19 & -17 & -13 & -19 \end{bmatrix} / 15 = \begin{bmatrix} b^g \\ (Ab)^g \\ (A^2 b)^g \\ (A^3 b)^g \end{bmatrix}$$

i.e.
$$(A^2 b)^g = \begin{bmatrix} 17 & -8 & -14 & -17 \end{bmatrix} / 15$$

Using (30), K is exactly, and uniquely,
$$K = \begin{bmatrix} 16.4 & -16.6 & -19.8 & -7.4 \end{bmatrix} \quad (45)$$

The *acker* command has been used to validate the unique K in (51), and the *jordan(A-b*K)* command has been used to validate the assignment of $-5, -5$, and $-2 \pm j$ as eigenvalues .

If the fact $tr(A) = -5$ was ignored, the assignment problem becomes that of the classical Ackermann's method . In which case and since the system is controllable, we should get the same numerical value for K using (3) or (16), though K is now structurally different from that given in (44).

$$K = (A^3 b)^g (A + 5I_4)^2 (A + (2-j)I_4)(A + (2+j)I_4)$$

It still results in a numerical value for K as given by (45). Such answer can be checked using the *acker* command of MATLAB[35], but cannot be checked using the *place* command as it involves assignment of two identical eigenvalues where their number exceeds the rank of b .

Example 4

A process has the following system matrices, [7]

$$A = \begin{bmatrix} -5 & 3 & 3 & 0 \\ -6 & 3 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

It is required to assign the eigenvalues $-3, -4, -5$ and of course to reassign any uncontrollable eigenvalue (in fact its value need not be known beforehand and even if it was known it will not be needed in the calculations). At least the sign of the real part has to be negative. The uncontrollable eigenvalue is this example is -2.

The system is unstable and also uncontrollable, but we need not test for that in advance. The classical method detects uncontrollability due to the incalculability of T^{-1} and therefore the classical method cannot be used. However, the incomplete method is well suited to mitigate such defect through any choice for N .

The number of columns in N , should equal the number of uncontrollable eigenvalues, having any values ensuring a nonsingular T . Since the system has a single uncontrollable eigenvalue, let N be as simple as $N = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. The T matrix used is therefore.

$$T = \begin{bmatrix} b & Ab & A^2 b & N \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -3 & 11 & 0 \\ 0 & 2 & -6 & 0 \\ 1 & -3 & 9 & 1 \end{bmatrix}$$

Which has the following inverse

$$T^{-1} = \begin{bmatrix} 2 & -1 & 0.5 & 0 \\ -3 & 3 & 2 & 0 \\ -1 & 1 & 0.5 & 0 \\ -2 & 1 & 1 & 1 \end{bmatrix}$$

$(A^2 b)^g$ is now extracted and is given by the third row of T^{-1}, i.e.

$$(A^2 b)^g = \begin{bmatrix} -1 & 1 & 0.5 & 0 \end{bmatrix}$$

Using (31) where $\kappa = 3$,
$$K = (A^2 b)^g (A + 3I_4)(A + 4I_4)(A + 5I_4) \quad (46)$$

Yields,
$$K = \begin{bmatrix} -90 & 81 & 69 & 18 \end{bmatrix} \quad (47)$$

Note that the -2 uncontrollable eigenvalue is not involved in (46). Either *eig(A-b*K)* or *jordan(A-b*K)* can now be used to validate the assignment of $-2, -3, -4, and -5$.

Commenting:

For example 4 and due to uncontrollability, the above answer for K cannot be checked using the *place* command of MATLAB as *place* will give a second alternative yet valid answer. This is due to the fact that K for uncontrollable systems is not unique even for single input systems.

Note also that the *acker* function of MATLAB doesn't work in this case since the classical Ackermann's method only deals with controllable systems. In other words, the *acker* command of MATLAB can be perfected by our outlined procedure for it to deal with uncontrollable systems.

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Using MATLAB, a general check method for eigenvalue assignment is eig$(A-b*K)$. For the case of repeated eigenvalues one better use the numerically more expensive command *jordan(A-b*K)* especially when the eigenvectors are required.

To sum things up: the *place* command cannot be used with example 3 as it cannot assign eigenvalues with multiplicity greater than $rank(B)$, but the *acker* command can. The *acker* command cannot be used with example 4 due to uncontrollability, but the *place* command can though it gives a different answer for K.

In conclusion, our modified and improved Ackermann's method easily works in both cases. It offers liberty of choice as in example 3, and flexibility resulting in simplicity as in example 4.

CONCLUSION

The study presents an elegant general proof of the method of Ackermann, resulting in an alternative concise depiction. The new depiction results in a more compact formula, and facilitates a method of incomplete assignment of eigenvalues if so desired. An advantage of incomplete assignment is that it can help in simplifying the structure for the feedback matrix in certain cases especially when the system is uncontrollable. A principal contribution of our study is the extension of the method to deal with uncontrollable systems at no extra cost. This has been accomplished through treating uncontrollable systems as a special case of the incomplete assignment approach.

Future work may tackle the problem of adaptation of the Ackermann's method to the eigenvalue assignment of controllable and uncontrollable multi-input systems. Moreover, exploration studies are needed concerning particular selections for N to further pin-point the values of the remaining enforced eigenvalues. Further investigations may delve into multi-input eigenstructure assignment concerning the assignment of eigenvectors in addition to that of eigenvalues.

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