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Acoustic cloaking transformations from attainable material properties

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Abstract. We propose a general methodology and a set of practical recipes for the construction of ultra-broadband acoustic cloaks—structures that can render themselves and a concealed object undetectable by means of acoustic scattering. The acoustic cloaks presented here are designed and function analogously to electromagnetic cloaks. However, acoustic cloaks in a fluid medium do not suffer the bandwidth limitations imposed on their electromagnetic counterparts by the finite speed of light in vacuum. In the absence of specific metamaterials having arbitrary combinations of quasi-static speed of sound and mass density, we explore the flexibility of continuum transformations that produce approximate cloaking solutions. We show that an imperfect, eikonal acoustic cloak (that is, one which is not impedance matched but is valid in the geometrical optics regime) with negligible dispersion can be designed using a simple layered geometry. Since a practical cloaking device will probably be composed of combinations of solid materials rather than fluids, it is necessary to consider the full elastic properties of such media, which support shear waves in addition to the compression waves associated with the acoustic regime. We perform a systematic theoretical and numerical investigation of the role of shear waves in elastic cloaking devices. We find that for elastic metamaterials with Poisson's ratio $\nu > 0.49$, shear waves do not alter the cloaking effect. Such metamaterials can be built from nearly incompressible rubbers (with $\nu \approx 0.499$) and fluids. We expect this finding to have applications in other acoustic devices based on the form-invariance of the scalar acoustic wave equation.

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Contents

1. Introduction	2
2. Construction of the cloaking transformation from achievable ranges of material properties	6
3. Implementation using the bi-layer metamaterial model	9
4. Tri-layer metamaterial	12
5. The effect of shear modulus	14
6. Conclusions	19
Acknowledgments	20
References	20

1. Introduction

Propagating waves are among the most significant tools for the remote sensing of material objects. Although modern sensors vastly expand the frequency ranges that humans can perceive and have become increasingly sophisticated, they nevertheless rely on the same underlying principle: detecting waves that propagate a sizable distance in the medium separating an object and its observer. Only two types of easily detectable waves can propagate in bulk media, such as air or water, without significant damping: electromagnetic and acoustic waves. Regardless of the progress in other areas of physics, detection of these two types of waves will probably remain the most practical means for probing material objects, especially those unreachable with ‘near-field’ instruments such as our tactile senses or other, more sophisticated probes (electron and atomic force microscopy, scanning tunneling microscopy and so forth).

The uniqueness theorem for the scattering problem, which can be derived for the wave equation, suggests that every scattering object gives rise to a unique field pattern [1, 2]. In other words, field patterns allow one, in principle, to distinguish different objects from one other, as well as from vacuum. However, this theorem does not apply to the scattering problems involving anisotropic media. For example, in the consideration of electrical impedance tomography measurements on physiological systems, it was found that an anisotropic conductivity could produce identical voltage and current measurements over the boundary, as would a homogeneous, isotropic conductivity [3, 4]. In the electromagnetic case, further advances made in the metamaterials field [5] over the past decade have confirmed that an observer equipped only with far-field instrumentation may fail to sense an object that is normally considered detectable [6]–[8]. We refer to devices based on anisotropic media that render a material object undetectable as cloaking devices or cloaks [5, 6].

The fundamental principle behind the cloaking devices is quite simple (figure 1): they compress the true size of the object a to an apparent size $a' \ll \lambda$, rendering a' much smaller than the wavelength λ of propagating waves in the ambient medium. In the long-wavelength (Rayleigh scattering) limit, the total (all-angle) scattering cross-section (TSCS) of a spherical or cylindrical object scales as a high power of its radius a ; e.g. $\sigma_{sc} \propto k^4 a^6$ for electromagnetic waves scattering from a perfectly conducting sphere [9], as well as for acoustic waves scattering from a rigid sphere [10]. In the case of infinitely long circular cylinders, the TSCS per unit length scales as $\sigma_{sc}/l \propto k^3 a^4$ for acoustic waves scattering from a rigid cylinder [10],

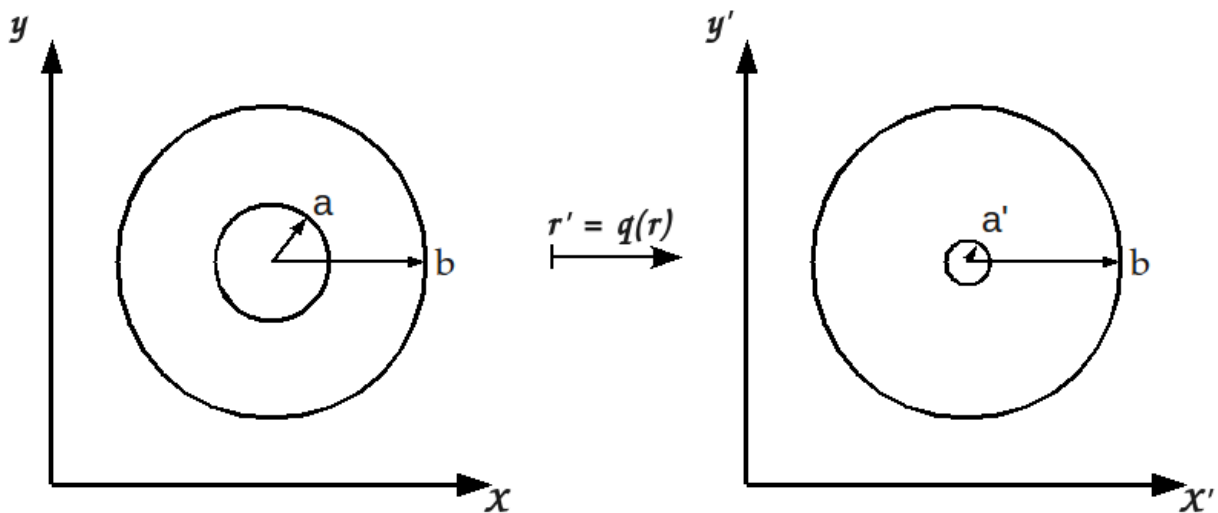


Figure 1. Schematic view of the cloaking transformation: an isomorphism between a circular annulus of inner radius a and outer radius b in physical coordinates (x, y) (left), and another annulus of inner radius $a' < a$ and the same outer radius b in some auxiliary coordinates (x', y') (right). Perfect cloaking is achieved in the limit $a' \rightarrow 0$. Transformations studied in this paper are rotationally invariant: in cylindrical coordinates $r' = q(r)$, $\theta' = \theta$.

as well as for TM-polarized electromagnetic waves scattering from a perfectly conducting cylinder [9].

The compression of the effective electromagnetic or acoustic width of an object can be achieved using appropriately chosen distributions of material properties. This is possible thanks to the form-invariance of Maxwell's equations [6]; the technique has been termed *transformation optics* in electromagnetism [11]–[13]. Form-invariance of the scalar Helmholtz equation enables analogous approaches in acoustics, which collectively can be referred to as *transformation acoustics*.

The ‘compressed’ object is hard to detect—at least at the wavelength of measurement λ . Knowing this, a clever observer might try to switch its sensor to operate at a shorter wavelength, thus improving the spatial resolution of the obtained image. To avoid detection by such a countermeasure, a perfect cloaking device therefore needs to be sufficiently broadband; that is, it needs to cover the entire range $\lambda_{\min} \leq \lambda \lesssim a$, where λ_{\min} is the shortest wavelength the observer could use given the limitations of the far-field sensor and the dispersive properties of the lossy medium separating the object and detector. An ultra-broadband cloak that provides $a' \ll \lambda_{\min}$ for all $\lambda \geq \lambda_{\min}$ accomplishes complete invisibility of the object to that observer.

Since electromagnetic waves in any physical medium cannot transfer signals faster than they do in vacuum, special relativity places stringent constraints on the performance of ultra-broadband electromagnetic cloaks in empty space. In particular, it is generally believed that a passive, lossless electromagnetic metamaterial cannot be constructed to enable cloaking at all wavelengths $\Lambda \ll \lambda_{\text{vac}} < \infty$, where $\Lambda = \max(L_{\text{cell}}, \lambda_{\text{res}}^{\max})$ is the largest of the two scales: the diameter L_{cell} of the metamaterial unit cell (defined as the maximum of its three dimensions) and the largest resonant wavelength $\lambda_{\text{res}}^{\max}$. In this regime, we may exclude all finite-frequency,

resonant phenomena from consideration, as well as the quasi-static resonances of the surface plasmon type, which require mixtures of positive and negative dielectric permittivities [14]. Cloaking devices based on resonant phenomena cannot be extremely broadband; in particular, their operation band cannot extend into the static regime. Quasi-static resonances based on negative permittivity can be made arbitrarily sub-wavelength ($\lambda_{\text{res}}^{\text{max}}/L_{\text{cell}} \gg 1$); however, the prospects for building acoustic metamaterials with the analogous property of negative effective density $\rho_{\text{eff}}(\omega) < 0$ that remains negative in the limit $\omega \ll 2\pi c_0/L_{\text{cell}}$ are unclear [15, 16].

In what follows, we consider only the long-wavelength regime $\lambda_{\text{vac}} \gg \Lambda$, as defined above. It should be emphasized that cloaks operating in this regime can still hide objects that are larger than or comparable to the wavelength $\Lambda \ll \lambda \lesssim a$; the remaining part of the long-wavelength regime, $\lambda > a$, is less interesting, since the object is barely detectable in the far field without any cloaks. We can see that the metamaterial scale Λ must be chosen not to exceed λ_{min} ; in other words, the observer should not be able to see scattering from individual metamaterial unit cells. Although trivial, this inequality places a fundamental constraint on the maximum granularity of metamaterial cloaks when a practical application requires a specific λ_{min} .

In order to support propagating waves in the limit $\lambda \rightarrow \infty$, a periodic metamaterial, electromagnetic or acoustic, must have a dispersion branch $\omega(k)$ that starts at $\omega(k=0) = 0$. Such branches of dispersion are known as *acoustic branches* in the theory of photonic and phononic crystals; their dispersion $\omega(k)$ is a linear function in the limit $\omega \rightarrow 0$. Therefore, in the regime $\lambda_{\text{vac}} \gg \Lambda$, the phase velocity $v_{\text{ph}} = \omega/k$ in a periodic metamaterial coincides with the group velocity $v_{\text{g}} = \partial\omega/\partial k$. For electromagnetic metamaterials with negligible loss, the group velocity cannot be superluminal, as this would violate special relativistic causality. Since cloaking devices require phase velocities that exceed the phase velocity in the ambient medium (typically, air), electromagnetic cloaking is impossible in the regime $\lambda_{\text{vac}} \gg \Lambda$. This limitation does not apply to acoustic cloaking, which can utilize ‘hard’ materials with the group (and phase) velocity of sonic waves greatly exceeding its value in the ambient medium.

The goal of this work is to suggest a methodology and a practical recipe for ultra-broadband acoustic cloaking devices that can be made of industrially available materials. Recently, the feasibility of acoustic cloaking has been the subject of many theoretical [15]–[22] and experimental [7, 8] studies. Theoretically, acoustic cloaking with fluid-like metamaterials should be possible [7, 20, 21], [23]–[25] due to the direct analogy between linearized fluid dynamics (acoustics) and electrodynamics [18, 19, 26], for which the cloaking effect was demonstrated experimentally [5]. The analogy becomes especially straightforward for two-dimensional (cylindrical) acoustic cloaks, which are the subject of this study. Cloaking devices that operate on the principle of *isotropic* compression of the apparent object size require an *anisotropic* effective mass density matrix. Its degree of anisotropy (characterized by the ratio of principal values $\rho_1^{\text{eff}}/\rho_2^{\text{eff}}$) relates to performance characteristics of the cloak, as explained in section 2.

Several implementations of anisotropic mass density have been proposed for the long-wavelength regime that could potentially lead to ultra-broadband acoustic cloak designs. For instance, several groups [20, 25] have suggested that a perfect cloak can be implemented as a series of homogeneous concentric layers with prescribed density $\rho(r)$ and bulk modulus $K(r)$ (see figure 2). However, the literature still lacks any practical recipes for making a material with the required values of (ρ, K) from naturally available components. In reality, achieving

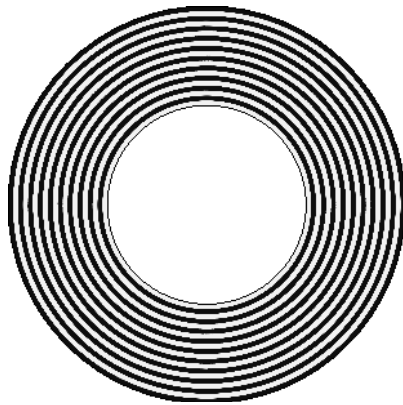


Figure 2. Geometry of a cylindrical cloak based on concentric layers of a bi-layer metamaterial.

independent control of the static density and static bulk modulus of a material is a very difficult task; generally, for natural materials the bulk modulus K is a monotonically increasing function of the density ρ . This correlation is often expressed in the form of an Ashby chart [27]. Independent variation of ρ and K may be unnecessary if one resorts to building an *eikonal* cloak, which controls only the dispersion relation $\omega(k)$ (which in the long-wavelength limit depends solely on the speed of sound $c_p = \sqrt{K/\rho}$, as explained above), but not the acoustic impedance $Z = \sqrt{K\rho}$. Thus, only the two ratios $\rho_{r,\theta}/K$ must be controlled in an eikonal cloak. We note that the principal values of the speed of sound $c_{r,\theta}$ are not determined solely by c_p of the components of the mixture; generally, two of the three quantities ρ , K , c_p must be known for each component. Therefore, correlations between ρ and K in the component materials could prohibit designing a composite with the desired pair $c_{r,\theta}$ [23]. Below, we show how to overcome the conflict between these correlations and the desired cloaking transformation. In short, our solution is to choose the cloaking transformation that fits the material model, instead of seeking a material model that can implement a particular cloaking transformation.

The flexibility of cloaking transformations has been previously exploited in the context of electromagnetic cloaks. However, in previous works the authors typically selected the transformation based on optimization of a particular goal, such as minimum impedance mismatch at the surface [28], or the constancy of one or more constitutive parameters [29]. Calculation of a transformation from the boundaries of attainable material properties (known as Ashby charts in mechanical engineering [27]) is a new paradigm introduced by this work. Although independent control over constitutive parameters of electromagnetic metamaterials is relatively easy [5], this new concept may prove useful for the design of electromagnetic cloaks, as well as acoustic.

This paper is organized as follows. In section 2, a general procedure for designing an omnidirectional acoustic cloak is introduced. Section 3 illustrates this procedure using a basic, concentric bi-layer metamaterial. Section 4 discusses the advantages and limitations of tri-layer composites. The transition from fluid-like to truly elastic media and the role of shear waves are studied in section 5. A summary of the results is listed in section 6.

2. Construction of the cloaking transformation from achievable ranges of material properties

Here we present a general method for *feasibility assessment* and subsequent *construction* of a cylindrical acoustic cloaking transformation, given a particular material model and ranges of practically available material parameters. This procedure can be readily modified to design a spherically symmetric cloak. In what follows, we neglect the curvature of the unit cells, and refer to the two principal values of the effective density matrix as ρ_θ and ρ_r .

It has been shown [21, 26] that an acoustic metamaterial with cylindrically symmetric distributions $\rho_r(r)$, $\rho_\theta(r)$ and $K(r)$ can implement a transformation $r' = q(r)$ of the physical radius r onto the radial coordinate r' in an imaginary space where the material acts as a uniform medium with $\rho'_r(r') = \rho'_\theta(r') = \rho_0 = \text{const}$ and $K'(r') = K_0 = \text{const}$. Here, ρ_0 and K_0 are the density and bulk modulus of ambient medium filling the exterior of the cloak. Specifically, an annulus $a \leq r \leq b$ corresponding to the cloak in the physical space is mapped onto the annulus $a' < r' < b$, if the material properties in $a \leq r \leq b$ are given by the following formulae (see [21] for a d -dimensional generalization):

$$\begin{aligned}\rho_\theta(r)/\rho_0 &= \left(\frac{q}{r}\right) \left(\frac{\partial q}{\partial r}\right)^{-1}, \\ \rho_r(r)/\rho_0 &= (\rho_\theta(r)/\rho_0)^{-1} = \left(\frac{q}{r}\right)^{-1} \left(\frac{\partial q}{\partial r}\right), \\ K(r)/K_0 &= \left(\frac{q}{r}\right)^{-1} \left(\frac{\partial q}{\partial r}\right)^{-1}.\end{aligned}\tag{1}$$

For the principal values of the speed of sound in the cloak, one has

$$\begin{aligned}c_0/c_\theta &= \left(\frac{\rho_\theta(r)/\rho_0}{K/K_0}\right)^{1/2} = \frac{q}{r}, \\ c_0/c_r &= \left(\frac{\rho_r(r)/\rho_0}{K/K_0}\right)^{1/2} = \frac{\partial q}{\partial r}.\end{aligned}\tag{2}$$

When a cylindrically symmetric material distribution implements the speed of sound (2), it is known as an *eikonal cloak*. If it also satisfies a more stringent set of requirements (1), the cloak is referred to as *impedance-matched*. Note that if an eikonal cloak satisfies a constraint

$$\rho_r(r)\rho_\theta(r) = \rho_0^2\tag{3}$$

for all r , it is impedance-matched. In the geometric optics (GO) limit, an eikonal cloak produces the same ray trajectories as the ideal (impedance-matched) cloak with the same coordinate transformation [5, 30]; however, outside of that limit the eikonal cloak always has a non-vanishing scattering cross-section. The degree to which the eikonal approximation increases the scattering cross-section depends not only on the wavelength, but also on the choice of the cloaking transformation [28]. For discussions of the eikonal cloak and its scattering pattern, see [5, 28, 31, 32]. In sections 3 and 4, we assume that the size of the cloak is substantially larger than the wavelength, such that the GO (or eikonal) approximation is valid. In the remainder of this section, we show how either an ideal or an eikonal cloak can be constructed from attainable material properties.

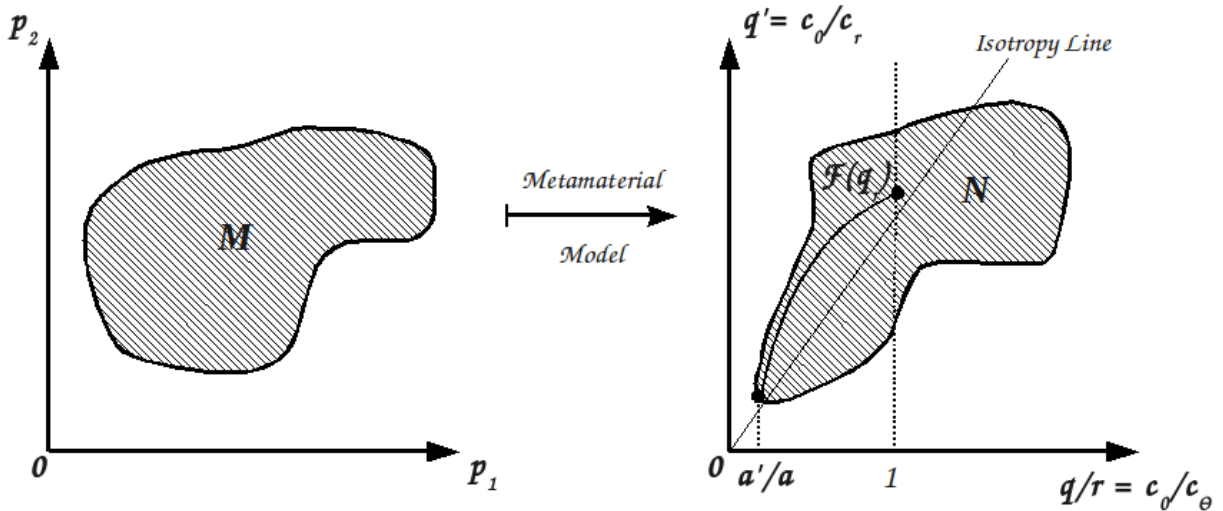


Figure 3. Schematic depiction of the cloaking transformation design based on a hypothetical metamaterial model and practically attainable ranges of parameters. Left: the region in the metamaterial parameter space dictated by fabrication constraints (cartoon). Right: Ashby chart [27] for anisotropic speed of sound, utilized to design an acoustic cloaking transformation. For a sample chart describing a realistic metamaterial, see figure 8.

The cloak design begins with identifying n_1 *variable parameters* p_i ($i = 1, \dots, n_1$) of the metamaterial unit cell, which can be varied continuously. For example, the unit cell can consist of two or more uniform materials, and the variable parameters are volume fractions and geometric aspect ratios describing the size and shape of the inclusions. Additionally, one can have n_2 design parameters that are constant throughout the transformation, but can be chosen judiciously to optimize performance or the cost of fabrication. Those parameters do not participate in the transformation-finding procedure described below. In the n_1 -dimensional space of variable parameters, fabrication constraints restrict one to a certain manifold M , whose dimension is $m = \dim M$, $1 \leq m \leq n_1$ (figure 3, left).

As the second step, one has to choose a metamaterial homogenization model that estimates $\rho_{\theta,r}$ and bulk modulus K for any $\{p_i\} \in M$. This homogenization model maps M onto another manifold D in the three-dimensional space of $(\rho_{\theta}, \rho_r, K)$. Its dimension is $d = \min(3, m)$ for a non-degenerate mapping; the situation $d < \min(3, m)$ signals that some of the tunable parameters p_i are useless, and it will not be considered here.

The third step is somewhat different, depending on whether an eikonal or an impedance-matched cloak is desired. If only an eikonal cloak is sought, one maps the manifold D onto some manifold N in the two-dimensional space of variables $(c_0/c_{\theta}, c_0/c_r)$ using equations (2); the dimension of N is $n = \min(2, d) = \min(2, m)$. For an impedance-matched cloak, one first finds the subset $D' \subset D$ by intersecting D with the surface $\rho_r \rho_{\theta} = \rho_0^2$; apparently, $\dim D' \equiv d' = d - 1$. Then, D' is mapped onto N' in the plane of variables $(c_0/c_{\theta}, c_0/c_r)$ using the same equations (2), and $\dim N' = n' = \min(2, d') = \min(2, m - 1)$.

The fourth step of this procedure is to relate the abscissa c_0/c_{θ} and ordinate c_0/c_r on the speed-of-sound chart (figure 3, right) with the two quantities $q_r \equiv q/r$ and $q' \equiv dq/dr$,

respectively. If $\dim N \geq 1$ (or $\dim N' \geq 1$), it may be possible to choose a continuous single-valued function $q' = F(q_r)$ within the manifold N (respectively, N') that

- (I) starts from $q_r = q_r^{\min}$ (where $0 \leq q_r^{\min} \ll 1$),
- (II) ends at $q_r = 1$ and
- (III) lies entirely above the isotropy line $q' = q_r$, i.e. in the region with

$$c_r < c_\theta. \quad (4)$$

Apart from conditions (I–III), the function $F(q_r)$ is completely *arbitrary*. It is not required to be monotonic; inequality (4) alone guarantees that $q(r)$ is a monotonically increasing, continuous function of r . If one can choose a curve satisfying all three conditions (I–III), cloaking with such a metamaterial is *feasible*.

The desired cloaking transformation $q(r) = r q_r(r)$, where $a \leq r \leq b$, is finally recovered using a simple quadrature:

$$\ln \frac{r}{a} = \int_{q_r^{\min}}^{q_r(r)} \frac{dq_r}{F(q_r) - q_r}. \quad (5)$$

The linear cloaking transformation $q(r) = b(r - a)/(b - a)$ widely used in the literature [6] corresponds to the choice $q' = F(q_r) = \text{const} = b/(b - a)$ and $q_r^{\min} = 0$. The most important feature of our procedure is that it enables cloaking with metamaterials that cannot implement a linear transformation. As we shall see in the following, the part of the manifold N within the stripe of interest, $0 \leq q_r \leq 1$, is typically an elongated shape extended along the line $q' = q_r$; a *horizontal* line with $q' = \text{const}$ within N cannot extend from a very small q_r to $q_r = 1$, but a *tilted* line with $\partial q'/\partial q_r > 0$ can.

The final, and trivial, step of the design procedure is to find the values of variable material parameters that correspond to each point on the curve $q' = F(q_r)$. Note that if the mapping $M \rightarrow N'$ is not an isomorphism, i.e. $\dim N' < \dim M$, there may be additional freedom in choosing the points in the $\{p_i\}$ space.

A number of observations can be made based on this general analysis. *Firstly*, the minimum number of degrees of freedom needed for a cylindrical eikonal cloak is $m = 1$, and an impedance-matched cloak needs $m \geq 2$. The fact that eikonal cloaking can be achieved with just one degree of freedom will be used in the next section.

Secondly, since $q_r^{\min} = 0$ corresponds to infinite speed of sound c_θ , in practice the transformation $q_r(r)$ would have to start at some small value $q_r^{\min} = a'/a$, which will determine the degree by which the cloak compresses the apparent size of the cloaked object. We therefore term this ratio a'/a the *cloaking deficiency*. It is easy to see that the smallest possible cloaking deficiency is determined by the maximum azimuthal speed of sound, c_θ^{\max} , in the metamaterial. In the long-wavelength regime, the principal values of the homogenized (effective) density matrix are bounded by the Voigt–Reiss inequalities, well known from the standard homogenization theory of elliptic operators [33]. Specifically, for an elliptic differential operator $\partial A \partial$, where $A(\vec{x})$ is a symmetric, positive-definite matrix that is a periodic function of \vec{x} , the homogenized matrix A_{eff} is bounded as follows:

$$\langle A^{-1} \rangle^{-1} \leq A_{\text{eff}} \leq \langle A \rangle, \quad (6)$$

where $\langle A^{-1} \rangle$ implies averaging over one unit cell of $A(\vec{x})$. Applied to one-dimensional, layered, fluid-like media, these bounds become

$$\rho_{\text{VR}}^{\min} \leq \rho_{\theta,r} \leq \rho_{\text{VR}}^{\max}, \quad (7)$$

where

$$\begin{aligned}\rho_{\text{VR}}^{\min} &= \left(\sum_i f_i \rho_i^{-1} \right)^{-1}, \\ \rho_{\text{VR}}^{\max} &= \sum_i f_i \rho_i,\end{aligned}\quad (8)$$

$\rho_{\theta,r}$ are the principal values of the homogenized density matrix and f_i are the filling fractions of the mixture components with constant isotropic density ρ_i . Combining this with a long-wavelength homogenization formula for the bulk modulus of a fluid-like composite,

$$K_{\text{eff}}^{-1} = \sum_i f_i K_i^{-1}, \quad (9)$$

one can show that the speed of sound cannot exceed the maximum speed of sound $c_{\text{max}} = \max_i c_i = \max_i \sqrt{K_i/\rho_i}$ in the metamaterial components:

$$\begin{aligned}c^2 &= K_{\text{eff}}/\rho \leq K_{\text{eff}}/\rho_{\text{VR}}^{\min} = \frac{\sum_i f_i \rho_i^{-1}}{\sum_i f_i K_i^{-1}} \\ &= \frac{\sum_i f_i K_i^{-1} c_i^2}{\sum_i f_i K_i^{-1}} \leq \frac{\sum_i f_i K_i^{-1} c_{\text{max}}^2}{\sum_i f_i K_i^{-1}} = c_{\text{max}}^2.\end{aligned}\quad (10)$$

Among naturally available materials, including those with Poisson's ratio $\nu < 0.5$, diamond has the highest speed of sound $c_p = 1.2 \times 10^4 \text{ m s}^{-1}$ due to its extremely high bulk modulus ($K = 4.42 \times 10^{11} \text{ Pa}$) and relatively low density ($\rho = 3.5 \text{ g cm}^{-3}$). Assuming that the ambient medium is air with $c_0 = 340 \text{ m s}^{-1}$, this puts the lower bound on cloaking deficiency in the long-wavelength regime: $\min(a'/a) = \min(c_0/c_p) \approx 0.028$. Considering that naturally available materials with $\nu \approx 0.5$ have a substantially lower speed of sound than diamond, the realistic lower bound is even higher.

Thirdly, we note that the curve $q' = F(q_r)$ cannot be arbitrarily close to the line $q' = q_r$; the latter corresponds to isotropic materials. When $F(q_r)$ approaches the isotropy line $q' = q_r$, the aspect ratio of the cloak $b/a = \exp \int_{a'/a}^1 dq_r / (q' - q_r)$ increases rapidly. In particular, if $0 < q' - q_r \leq \Delta$ with some constant $\Delta > 0$, then

$$b/a \geq \exp(1/\Delta). \quad (11)$$

This is another manifestation of the well-known fact that isotropic cloaking requires anisotropic material properties [3, 4]. The inverse ratio a/b , which we term the *cloaked payload*, determines the volume fraction of the useful volume of the cloak (payload), and in practice one wants $a/b \leq 1$ to be as large as possible. As we shall see in what follows, realistic material models typically impose a correlation between the minimum cloaking deficiency a'/a and the maximum cloaked payload a/b . An optimized trade-off between these figures of merit should be found for one's own optimization goal.

3. Implementation using the bi-layer metamaterial model

Using the general methodology of the previous section, we design an acoustic cloak based on a particular acoustic metamaterial model.

Layered metamaterials traditionally attract much theoretical and experimental interest, due to the relative ease of fabrication and the existence of closed-form analytic homogenization models. It has been shown that the anisotropic mass density matrix can be implemented in a composite consisting of alternating layers of two isotropic, homogeneous media [20, 24, 25, 34]. An additional reason for layered materials to be important for cloaking devices is that they implement both the upper and the lower quasi-static Voigt–Reiss bounds for effective density (see equation (8)):

$$\begin{aligned}\rho_{\parallel}^{\text{eff}} &= \rho_{\text{VR}}^{\text{min}} = (f_1 \rho_1^{-1} + f_2 \rho_2^{-1})^{-1}, \\ \rho_{\perp}^{\text{eff}} &= \rho_{\text{VR}}^{\text{max}} = f_1 \rho_1 + f_2 \rho_2,\end{aligned}\quad (12)$$

where $\rho_{\parallel, \perp}^{\text{eff}}$ are the principal values of the effective density matrix corresponding to the directions parallel and perpendicular to the layers, $\rho_{1,2}$ are the densities and $f_1, f_2 = 1 - f_1$ are the filling fractions of the two homogeneous, isotropic materials. This means that a bi-layer metamaterial exhibits the maximum level of density anisotropy obtainable by mixing two isotropic materials with given densities at a given volume fraction. Other shapes of inclusions, such as finite-width blocks [35] or ellipsoids, cannot provide a greater degree of anisotropy than the flat bi-layer, at least not in the quasi-static limit needed for ultra-broadband cloaking. Therefore, in the remainder of this paper we will focus on layered metamaterials.

From (12) it follows that $\rho_{\parallel}^{\text{eff}} \leq \rho_{\perp}^{\text{eff}}$, i.e. $c_{\parallel} \geq c_{\perp}$. Combined with the cloaking requirement $c_r < c_{\theta}$ (4), this dictates that the material layers must be concentric, as shown in figure 2. In the remainder of this paper, it is assumed that $c_{\theta} = c_{\parallel}$. Numerical finite-element simulations of fluid-like acoustic cloaks with anisotropic, continuously varying density have been reported previously [20]. We have verified, by solving the acoustic Helmholtz equation, that fluid-like cloaks with a finite number of isotropic-density layers are a good approximation to the continuous, graded-index version, as long as the number of layers is sufficiently large. This confirms the validity of the approach proposed in [20, 24, 25].

As previously observed, it should be possible, in principle, to implement an eikonal cloaking transformation using a metamaterial with only one degree of freedom. Here, we concentrate on a layered metamaterial consisting of two, fluid-like, isotropic and homogeneous media, whose density $\rho_{1,2}$ and bulk modulus $K_{1,2}$ do not vary in the cloak. The continuous degree of freedom (p_1 from figure 3) in this case is the filling fraction of one of the materials (f_1 or $f_2 = 1 - f_1$), and the manifold M is the one-dimensional interval $[0; 1]$. The choice of the two materials is dictated mostly by the need for (a) very large azimuthal speed of sound at the inner radius of the cloak, relative to the speed of sound in ambient medium, and (b) sufficiently strong anisotropy of effective density. Therefore, one must have $c_1 \equiv \sqrt{K_1/\rho_1} \gg c_0$ and $\rho_1/\rho_2 \gg 1$, where we labeled the denser, harder layers with index 1.

In order to illustrate the upper and, perhaps, unattainable bound on the ultra-broadband cloaking deficiency, consider a hypothetical, nearly incompressible (i.e. having Poisson's ratio $\nu \approx 0.5$) material with the speed of sound comparable to that in diamond. A typical speed-of-sound diagram of a bi-layer medium with large density contrast ($\rho_1/\rho_2 \gg 1$) is shown in figure 4. The diagram is a curve parameterized by the filling fraction f_2 of the low-density fluid. This curve represents the manifold N from figure 3 (right); this manifold is one-dimensional because $\dim M = 1$. Our choice of the cloaking transformation is restricted to that curve. Therefore, we only have the freedom to choose the start point q_r^{min} of the $q' = F(q_r)$ curve, i.e. the choice of f_2^{min} ; recall that $q_r^{\text{max}} \equiv 1$ due to the requirement $c_{\theta} = c_0$ at the surface of the cloak, which is necessary for the continuity of the transformation function $q(r)$ at $r = b$.

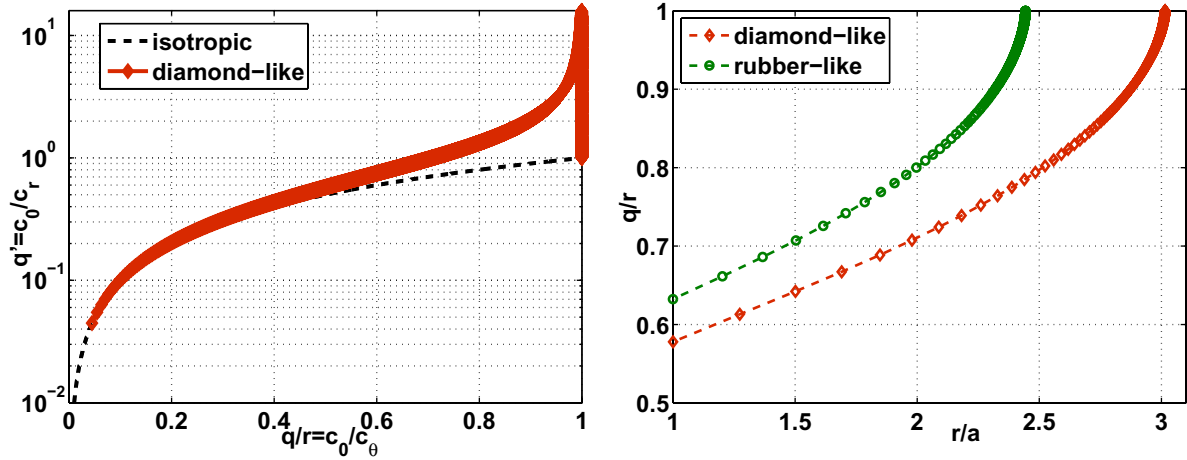


Figure 4. Left: speed of the p-wave diagram of bi-layer metamaterial composed of two fluid-like media. The curve is obtained by varying the filling fraction $f_1 \in [0, 1]$ of medium 1. High-density layers have $\rho_1 \text{ g cm}^{-3}$, comparable to the density of liquids, rubber-like materials and diamond. Low-density layers are filled with a gaseous medium, the same as the ambient medium. Density ratios are $\rho_1/\rho_0 = 10^3$ and $\rho_2/\rho_0 = 1$; bulk modulus ratios are $K_1/K_0 = 10^6$ (diamond-hard material in air) and $K_2/K_0 = 1$. The curve starts at min $q/r = c_0/c_1 \approx 0.03$. For $K_1/K_0 = 10^4$ (rubber-hard material in air), the curve looks almost the same, except that it starts at min $q/r \approx 0.3$. Right: the cloaking transformation obtained from the curve on the left using a numerical quadrature of equation (5). Plotted is the normalized transformation function, $r'/r \equiv q(r)/r$, versus normalized physical radius, r/a , where a is the inner radius of the cloak. The diamond-marked curve refers to the same parameters as for the curve on the left; the circle-marked curve is the same except that $K_1/K_0 = 10^4$. The transformation begins at the filling fraction of the gaseous fluid $f_2^{\min} = 5 \times 10^{-4}$ in order to avoid the singularity at $c_\theta = c_r$. The ordinate of the intersection of each curve with the vertical axis is the cloaking deficiency, $a'/a = q(a)/a$, and the abscissa of the intersection with the horizontal line $q/r = 1$ is the cloak aspect ratio, b/a .

At $f_2 = 0$, the effective medium is isotropic and its azimuthal speed of sound is maximal: $c_\theta = c_r = c_1$. At $f_2 = 1$, the effective medium is again isotropic. Therefore, the curve in figure 4 returns to the point $(q_r = 1, q' = 1)$ on the isotropy line, after going through a sharp maximum. For simplicity, figure 4 assumes that the low-density layers are filled with the ambient medium ($\rho_2/\rho_1 = 1$). In general, $\rho_2 = \rho_0$ and $K_2 = K_0$ (or $c_2 = c_0$) is not required for an eikonal cloak; however, the requirement $c_\theta(r = b) = c_0$ translates to a constraint

$$c_2 \leq c_0. \quad (13)$$

Indeed, if both $c_{1,2}$ exceeded c_0 , it would be impossible to implement $c_\theta = c_0$ in this metamaterial.

The value f_2^{\max} must be chosen such that $c_0/c_\theta = 1$. If $c_2 < c_0$, one always finds $f_2^{\max} < 1$; however, the case $c_2 = c_0$ plotted in figure 4 needs special consideration. In this situation, $q' = q_r$ at $f_2 = 1$; at the same time, $f_2^{\max} = 1$ corresponds to isotropic material and the

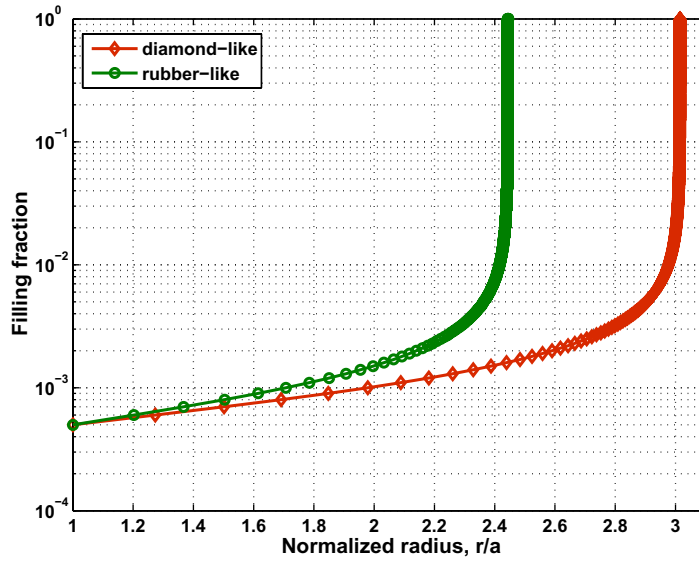


Figure 5. Filling fraction of the gaseous fluid required to implement the transformation in figure 4; circle- and diamond-like curves refer to the two cases described in the caption of figure 4.

transformation (5) diverges. Fortunately, the curve q' versus q_r is almost vertical at that point: its slope is $\frac{\partial q'}{\partial q_r}|_{f_2=1} \approx -(\rho_1/\rho_2)^2$, which is a very large number. The integral $\int \frac{dq_r}{q'-q_r} = \int \frac{\partial q_r}{\partial q'} \frac{dq'}{q'-q_r}$ can be made convergent by choosing $q_r^{\max} = 1 - (\rho_2/\rho_1)^2 \approx 1 - 10^{-6}$. The resulting transformation function is shown in figure 4, and the required filling fraction of the solid as a function of the physical radius is shown in figure 5.

The limiting case with $c_2 = c_0$ (medium 2 is the same as ambient medium) allows a perfect match in both the speed of sound and acoustic impedance at the surface of the cloak, simply because $\rho_r = \rho_\theta = \rho_0$ and $K = K_0$ at $f_2 \approx 1$. Unfortunately, this design is not the easiest to implement. The gradient of the transformation function, dq/dr , reaches a large value $\approx \frac{1}{2}\sqrt{\rho_1/\rho_2}$, of order 16 for $\rho_1/\rho_2 = 10^3$. This large gradient would have to be resolved by sufficiently thin unit cells of the metamaterial, which may be technologically expensive. To avoid exceedingly large values of q' , the speed of sound c_2 should be adjusted. The maximum of $q' = F(q_r)$ is located at $q_r = q_r^0 \approx (c_0/c_2)^2(1 - \rho_2/\rho_1)$. By choosing c_2 slightly less than c_0 , such that $(c_2/c_0)^2 < 1 - \rho_2/\rho_1$, the location of this maximum is pushed away into the region $q_r > 1$, which is not involved in the cloaking transformation. This is illustrated by figure 6, where c_2/c_0 is decreased by choosing $\rho_2/\rho_0 = 1.1$.

4. Tri-layer metamaterial

As shown above, a large anisotropy of effective density is required for high-quality cloaking devices. One obvious generalization of the concentric bi-layer metamaterial would be to use components that are anisotropic to begin with. However, highly anisotropic materials typically do not have a very small shear modulus, which would complicate the design. The effect of anisotropic components can be achieved by using another bi-layer metamaterial for one or both of the layers. Naturally, we come to the idea of multi-layer materials. Here, we evaluate the performance of a triple layer composite versus a bi-layer studied above.

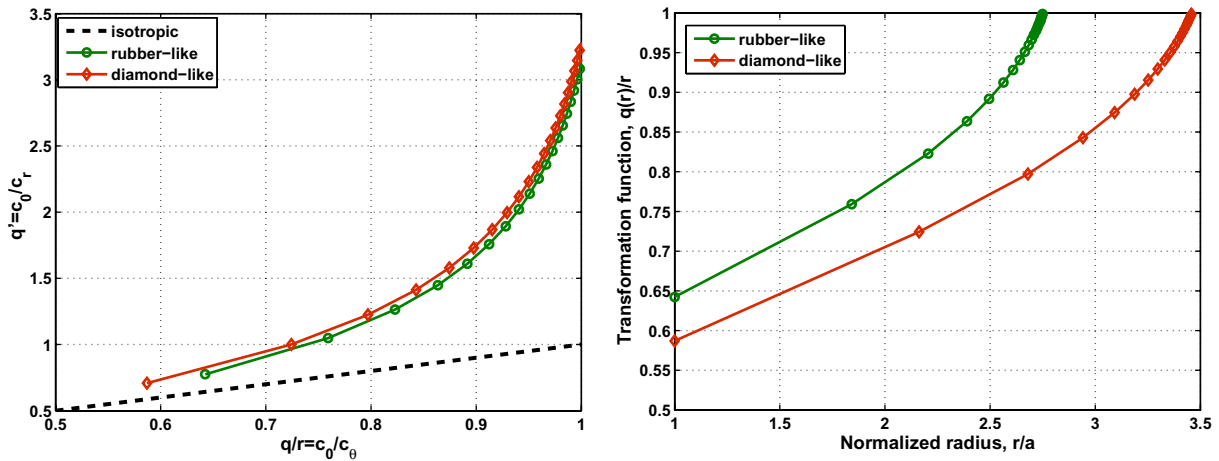


Figure 6. Left: speed of the p-wave diagram of bi-layer metamaterial composed of two fluid-like media. Circle-marked curve: a rubber-like or water-like material is assumed for the high-density layers, and gaseous fluid for the low-density layers; $\rho_1/\rho_0 = 10^3$, $\rho_2/\rho_0 = 1.1$, $K_1/K_0 = 10^4$, $K_2/K_0 = 1$. Diamond-marked curve: same as the circle-marked curve except that $K_1/K_0 = 10^6$ (diamond-hard solid). Right: transformation of the radial coordinate versus normalized physical radius inside the cloak; circle- and diamond-like curves refer to the same parameters. The transformation is assumed to begin at the filling fraction of the gaseous fluid $f_2^{\min} = 5 \times 10^{-4}$.

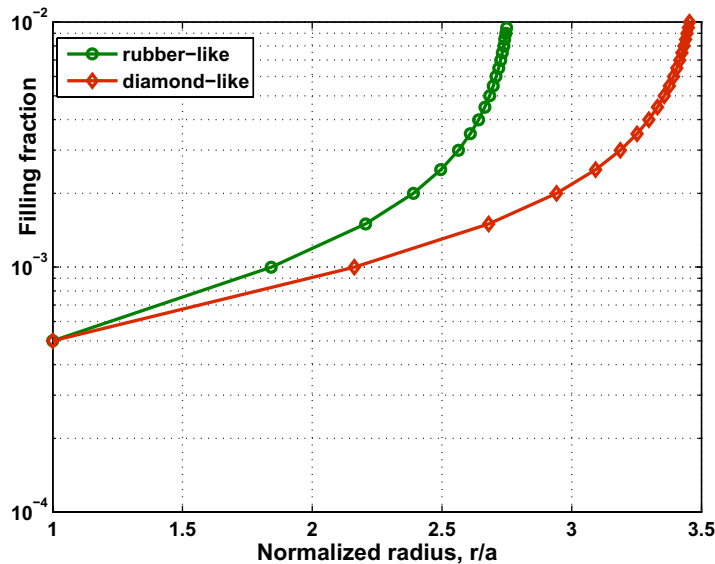


Figure 7. Filling fraction of the gaseous fluid required to implement the transformation in figure 6; circle- and diamond-like curves refer to the two cases described in figure 6.

One positive effect of an extra layer per unit cell is that it has two degrees of freedom: the filling fractions f_1 and f_2 ($f_3 = 1 - f_1 - f_2$). Consequently, the allowed domain for cloaking transformations has a finite area (see figure 8). This gives flexibility in choosing the

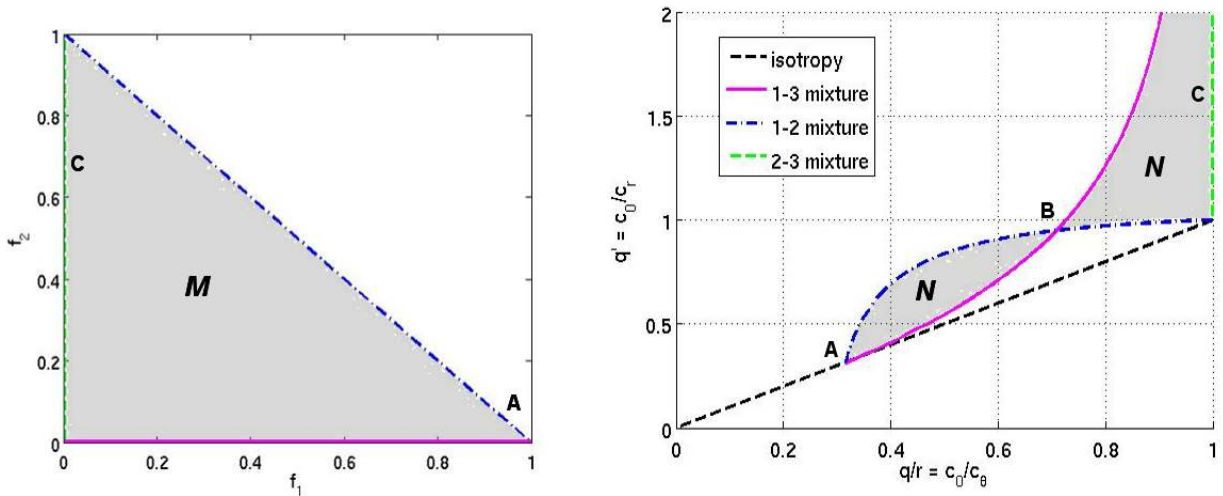


Figure 8. Left: manifold M in the two-dimensional material parameter space of a tri-layer metamaterial, corresponding to the schematic in figure 3 (left). The dimensions are the filling fractions f_1, f_2 of the two components; the constraint $f_3 = 1 - f_1 - f_2 \in [0; 1]$ defines the triangular shape of M . Right: speed-of-sound chart corresponding to figure 3 (right) for a particular tri-layer metamaterial with normalized densities $\rho_1/\rho_0 = 10^3$, $\rho_2/\rho_0 = 10^4$, $\rho_3/\rho_0 = 1$ and normalized bulk moduli $K_1/K_0 = K_2/K_0 = 10^4$, $K_3/K_0 = 1$. The speed of sound in components 2 and 3 is chosen as equal ($c_2 = c_3 = c_0$); thus, the 2–3 mixture line is vertical with $c_\theta \equiv c_0$ and $c_r \leq c_0$.

transformation for an eikonal cloak. For example, in figure 8, one can choose a transformation that starts from point A , follows the upper line corresponding to $f_3 = 0$, and then follows an arbitrary path from point B to point C while staying between the two borderlines, which correspond to $f_2 = 0$ and $f_3 = 0$, respectively.

However, we find that tri-layer design has little to offer beyond this flexibility: the allowed domain is bounded by three curves representing bi-layer metamaterials. Near point A , which is the crucial point of the design, the maximum speed of sound is still bounded by the speed of sound in the components of the mixture. Likewise, the maximum anisotropy is determined by anisotropy in binary mixtures, and it is bounded by $\max\{c_\theta(1-2)/c_r(1-2), c_\theta(1-3)/c_r(1-3)\}$, if material 1 has the largest speed of sound of all three. In other words, what is generally impossible with bi-layers is also impossible with tri-layers. By induction, this applies to multi-layer designs with more layers per cell.

5. The effect of shear modulus

In this section, we study the effect of finite shear modulus on the performance of acoustic cloaks.

Although acoustic cloaking with fluid-like materials is a theoretically valid concept, any realistic cloaking device would require some solid components, merely to maintain its structural integrity. Metamaterials with hard, solid inclusions that are acoustically fluids have been designed for two-dimensional (cylindrical) geometry [35, 36], with applications in acoustical focusing [36] and beam bending [35]. However, generalizing such metamaterials to three

dimensions while preserving the structural strength of the entire device may be challenging. This motivates our study of acoustic metamaterials made entirely of *elastic solids*, and thus having a non-vanishing effective shear modulus.

To establish the connection with the well-known theory of fluid-like acoustic metamaterials, we argue that an elastic medium with the stress tensor

$$\begin{aligned}\sigma_{ij} &= (\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})) u_{kl}, \\ u_{kl} &= \frac{1}{2}(\partial_k u_l + \partial_l u_k)\end{aligned}\quad (14)$$

(in Cartesian coordinates) and the density tensor ρ_{ij} is indistinguishable, in the limit $\mu \rightarrow 0$, from a fluid medium with the bulk modulus $K = \lambda$ and the same density. It is the second Lamé coefficient μ , also known as shear modulus G , that distinguishes an elastic solid from a fluid. Based on this physical argument, we expect that a fluid-like cloak designed in the previous sections still reduces the scattering of compression waves (p-sound) propagating in the ambient fluid, if the ratio of Lamé coefficients μ/λ is sufficiently small.

It is well known that, unlike Maxwell's equations, Navier equations (14) of vector elastodynamics are not form-invariant [17]; although they can be postulated in a covariant form, an arbitrary coordinate transformation induces changes in the equations that cannot be absorbed into appropriately redefined elastic properties. This means that arbitrary coordinate transformations cannot be implemented *precisely* as distributions of elastic properties, although in particular cases various approximations may be possible [22]. A 'viscoelastic' generalization of Navier equations, based on a new constitutive law that couples stress, strain, momentum and strain rate (velocity), has been proposed, and is shown to be form-invariant for an arbitrary coordinate transformation [17]. However, despite some theoretical effort [15], practical recipes for making elastic metamaterials with such exotic properties are yet to be developed.

One manifestation of the lack of form-invariance of the standard Navier equations is that they allow propagation of two different types of waves: (a) longitudinal *pressure waves* or p-sound, and (b) transverse *shear waves* or s-sound. Their phase velocities are typically quite different. Specifically, in thermodynamically stable and isotropic elastic media the speed of p-sound, $c_p = \sqrt{(\lambda + 2\mu)/\rho}$, strictly exceeds the speed of s-sound, $c_s = \sqrt{\mu/\rho}$. The ratio $c_p/c_s = \sqrt{2(1-\nu)/(1-2\nu)}$ cannot be less than $\sqrt{4/3}$, since Poisson's ratio ν is always greater than -1 . Therefore, matching the dispersion relation of the two branches, as one does in impedance-matched electromagnetic cloaks [6], is impossible in isotropic elastic media, at least in the quasi-static limit studied in this work. However, it may be possible to minimize the coupling between p- and s-sound, or perhaps even eliminate it completely. If that goal is achieved, solid acoustic devices placed in a fluid environment would need to control only the propagation of p-sound, which is the only kind of wave existing in ambient medium. Below, we assess the possibility of s-wave decoupling in elastic media with isotropic elastic moduli yet possibly anisotropic density.

Consider a linear elastic medium whose elastic properties are isotropic and therefore given by the stress tensor (14), in which the Lamé coefficients λ , μ may depend on coordinates. The divergence of this tensor may be written in the following form:

$$\begin{aligned}\partial_j \sigma_{ij} &= \partial_i(\lambda \partial_j u_j) + \partial_j(\mu \partial_i u_i) + \partial_j(\mu \partial_j u_i) \\ &= \partial_i((\lambda + \mu) \partial_j u_j) + \partial_j(\mu \partial_j u_i) + \{(\partial_j \mu)(\partial_i u_j) - (\partial_i \mu)(\partial_j u_j)\}.\end{aligned}\quad (15)$$

The last term in braces obviously vanishes if the shear modulus $\mu = \text{const}$, which we assume in what follows. Furthermore, the second term can be simplified using the identity $(\nabla \times \mu \nabla \times \vec{u})_i = \partial_i(\mu \partial_j u_j) - \partial_j \mu \partial_j u_i$:

$$\partial_j \sigma_{ij} = \partial_i((\lambda + 2\mu)(\nabla \cdot \vec{u})) - (\nabla \times \mu \nabla \times \vec{u})_i. \quad (16)$$

The linear elastic wave equation for monochromatic waves,

$$\partial_j \sigma_{ij} + \omega^2 \rho_{ij} u_j = 0, \quad (17)$$

can be written as

$$\rho_{ij}^{-1} \partial_k \sigma_{jk} + \omega^2 u_i = 0, \quad (18)$$

assuming that the density matrix ρ_{ij} is locally positively defined and thus invertible at each point. By taking first the divergence and then the curl of equation (18), we obtain a pair of coupled equations:

$$\begin{aligned} \nabla \rho^{-1} \nabla p + \frac{\omega^2}{\lambda + 2\mu} p &= -\nabla \rho^{-1} \nabla \times \vec{s}, \\ -\nabla \times \rho^{-1} \nabla \times \vec{s} + \frac{\omega^2}{\mu} \vec{s} &= \nabla \times \rho^{-1} \nabla p, \end{aligned} \quad (19)$$

where we have introduced the bulk pressure, $p = -(\lambda + 2\mu)(\nabla \cdot \vec{u})$, and the shear pressure, $\vec{s} = \mu \nabla \times \vec{u}$. In the case of two-dimensional wave propagation (plane strain), shear pressure has only the out-of-plane component $\vec{s} = s \vec{e}_z$, and the equations simplify to a more symmetric form:

$$\begin{aligned} \nabla \rho^{-1} \nabla p + \frac{\omega^2}{\lambda + 2\mu} p &= -\nabla \rho^{-1} (\nabla s \times \vec{e}_z), \\ \nabla \tilde{\rho}^{-1} \nabla s + \frac{\omega^2}{\mu} s &= \vec{e}_z \cdot \nabla \times \rho^{-1} \nabla p, \end{aligned} \quad (20)$$

where $\tilde{\rho}_{ij}^{-1} = -e_{im} e_{jn} \rho_{mn}^{-1}$, and e_{ij} is the two-dimensional Levi-Civita symbol. For isotropic density, $\tilde{\rho}^{-1} = \rho^{-1}$.

To reveal the structure of the coupling term, assume that the density matrix is diagonal. This can always be achieved at any given point by appropriate rotation of Cartesian coordinates. Then, the rhs of equations (20) become

$$\begin{aligned} -\nabla \rho^{-1} (\nabla s \times \vec{e}_z) &= -[\partial_1(\rho_{11}^{-1} \partial_2 s) - \partial_2(\rho_{22}^{-1} \partial_1 s)], \\ \vec{e}_z \cdot \nabla \times \rho^{-1} \nabla p &= \partial_1(\rho_{22}^{-1} \partial_2 p) - \partial_2(\rho_{11}^{-1} \partial_1 p). \end{aligned} \quad (21)$$

For these coupling terms to vanish with arbitrary s and p fields, the sufficient and necessary condition is that density must be isotropic and homogeneous. In the latter case, s -sound is completely decoupled from p -sound *regardless of the magnitude of shear modulus*. This calculation highlights an important feature of a medium with isotropic moduli and anisotropic density: s - and p -sound are coupled only through (a) anisotropy and (b) inhomogeneity of ρ and (c) inhomogeneity of μ .

With applications to acoustic cloaking, this finding is unfortunate: anisotropy of the speed of sound is required for cloaking. One could explore the possibility of using anisotropic elastic

modulus with isotropic density; this route is not covered by the formalism above and is outside the scope of this work. Alternatively, one could use relatively small anisotropy so that the s/p-sound coupling is sufficiently small. However, as we have seen in section 2, for weakly anisotropic cloaks there is a steep trade-off between cloaking accuracy and cloaked payload. The cloaking transformation corresponding to constant anisotropic density and variable bulk modulus is known [29]. The relationship between payload, deficiency and anisotropy ratio in a constant-density cloak is a power law: $a'/a = (a/b)^{(c_\theta/c_r)^{-1}}$, where $c_\theta/c_r = \sqrt{\rho_r/\rho_\theta} = \text{const} > 1$ and $a/b < 1$. A smaller density anisotropy ratio would minimize s/p coupling but would also simultaneously raise the cloaking deficiency a'/a .

Alternatively, we may expect that for sufficiently small values of μ , the magnitude of shear waves inside the cloak would be so small that they would not have a significant effect on the scattering cross-section of p-sound. This should be true regardless of the degree of anisotropy and inhomogeneity of the density matrix. This hypothesis has been verified in our numerical simulations as described below. Elastomers with the ratio $\mu/\lambda \equiv \beta \equiv 1 - 2\nu$ as small as 10^{-5} are not uncommon [37]; for many rubbers, including natural latex, this ratio is of order 10^{-4} – 10^{-3} [37]. It can be shown using the quasi-static Hashin–Shtrikman bounds that μ_{eff} of a well-ordered, elastically isotropic binary composite (with $\mu_1 \geq \mu_2 \geq 0$ and $\lambda_1 \geq \lambda_2 > 0$) does not exceed the largest local shear modulus of the components [38]: $\mu_{\text{eff}} \leq \mu_1$. This estimate suggests that an elastic metamaterial for acoustic cloaking can be as simple as a bi-layer metamaterial from section 3 consisting of a nearly ‘incompressible’ rubber with $\mu = \mu_1 \sim 10^{-3}\lambda_1$ and some fluid (liquid or gas) with $\mu = \mu_2 = 0$.

To test whether high-quality cloaking can be achieved with the β ratio in attainable range, we used a finite element solver, COMSOL Multiphysics with Acoustics and Structural Mechanics modules, which provides a predefined coupling between the two equations, known as *acoustic–structure interaction*. The simulation solves the vector equation for elastic strain (17) inside the cloak, the scalar acoustic Helmholtz equation for bulk pressure in the ambient medium, and imposes appropriate continuity conditions on the boundary between the fluid and the solid domains.

From figure 9 (left) it is evident that an elastic cloak behaves almost like its pure-fluid counterpart, when β is sufficiently small: $\beta \leq 0.02$ (i.e. $\nu \geq 0.49$). However, when $\beta = 0.1$, the cloaking effect is almost completely destroyed, as seen from figure 9 (right). To quantify the cloaking effect, we performed a parametric sweep by varying the parameter β . The figure of merit that quantifies the performance of an omnidirectional cloak is the TSCS [39]. The simulated TSCS plotted in figure 10 (left) as a function of β reveals an interesting structure in the scattering spectrum of this cloak. TSCS is negligible and practically flat for all values of β up to approximately 10^{-3} . In this regime, the cloaking effect is insensitive to the wavelength and frequency. Above that threshold, TSCS begins to grow slowly; however, it remains rather small up to roughly $\beta = 0.01$. Figure 10 (right) shows the distribution of shear stress at $\beta = 0.02$. The magnitude of s-waves does not exceed 0.15, which is an order of magnitude smaller than the magnitude of p-waves (1.2, according to figure 9, left). This validates our expectation that at sufficiently small μ/λ ratio, only a small fraction of incident energy is converted into s-waves inside the elastic cloak.

Above $\beta = 0.02$, TSCS grows rapidly, and at $\beta \geq 0.1$, the cloaking effect is essentially absent. Additionally, the scattering spectrum shows multiple peaks and dips. We attribute those peaks and dips to macroscopic resonances of the shear waves in the cylindrical metamaterial body. The nature of these resonances deserves a separate study; here, we merely point out that

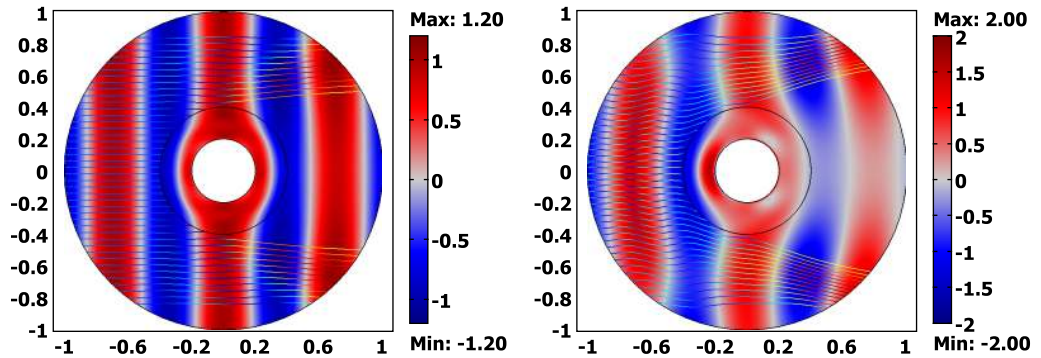


Figure 9. Plot of p-wave pressure from a finite-element simulation of an elastic cloak with anisotropic density $\rho_r = \rho_0 r / (r - a)$, $\rho_\theta = \rho_0 (r - a) / r$, first Lamé coefficient $\lambda = \lambda_0 ((b - a) / b)^2 r / (r - a)$ and shear modulus $\mu = \beta \lambda$, where $\beta = 0.02$ (left) and $\beta = 0.1$ (right). The cloak has $a = 0.2$ m, $b = 0.4$ m, and the wavelength in the fluid is $\lambda_{fl} = 0.686$ m, corresponding to the speed of sound 343 m s $^{-1}$ at the frequency 500 Hz. Also shown on the plots are the streamlines of acoustical energy flux in the fluid domain.

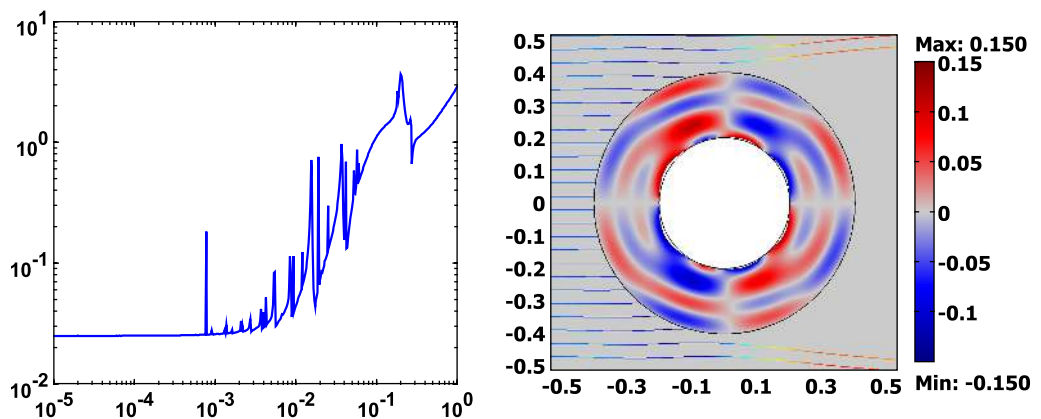


Figure 10. Left: TSCS (in arbitrary units) for the cloak described in figure 9, as a function of $\beta = \mu / \lambda$. Other properties of the cloak are defined in the caption to figure 9. Right: plot of s-wave pressure, defined in the solid as $s = \mu \vec{e}_z \cdot \vec{\nabla} \times \vec{u}$, at $\beta = 0.02$.

they must be similar to the geometric Mie resonances of cylindrical and spherical particles, whose spectra become even richer when the particle has either graded or layered material properties [40]. At small shear wavelength (i.e. at small β), these resonances are more similar to the whispering gallery resonances of rings; one such resonance, corresponding to the narrow peak in TSCS at $\beta = 7.76 \times 10^{-4}$, is shown in figure 11.

The deep minima in TSCS located between s-wave resonances, in particular the one at $\beta = 0.02$ (figure 10), can be useful for acoustic cloaking, as one can see from figure 9 (left). The sensitivity of these resonances to the wavelength-to-diameter ratio will determine the bandwidth of acoustic cloaks that utilize such resonances, and it should be the subject of additional studies.

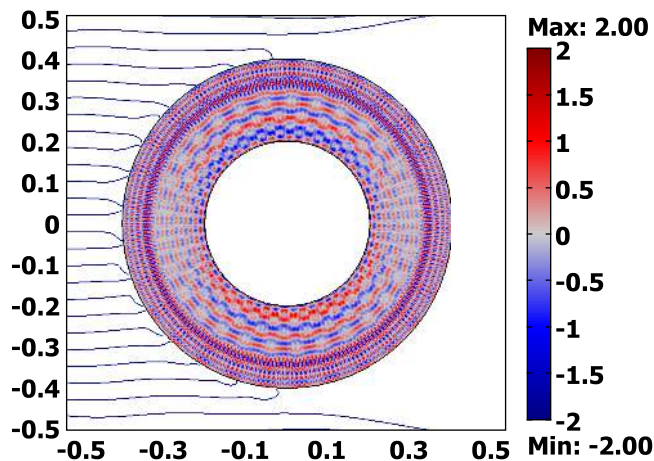


Figure 11. Plot of s-wave pressure at the resonant value of $\beta = 7.76 \times 10^{-4}$ for the cloak described in figure 9. The field structure typical of a high-order whispering gallery mode is revealed.

6. Conclusions

To summarize, we have studied the feasibility of a cylindrical ultra-broadband acoustic cloak based on the ideas of transformation acoustics. To maximize the generality of our analysis, we have introduced a broader class of acoustic cloaks. The generalization is achieved by (a) introducing general radial transformation, (b) lifting the requirement of acoustic impedance-matching in the entire volume of the device, (c) allowing incomplete compression of the cloaked volume and (d) allowing shear waves that are weakly coupled with longitudinal sound. Among these four generalizations, only (a) does not reduce the performance of the cloaking device, whereas (b)–(d) are introduced only as a trade-off between cloaking performance and construction difficulty. Two important figures of merit for generalized acoustic cloaks are defined: (i) cloaking deficiency and (ii) cloaked payload. It is shown that, for metamaterials operating in the quasi-static limit, the minimum cloaking deficiency is bounded by the fastest speed of sound in the individual components of an acoustic metamaterial. The maximum cloaked payload is limited by the minimum value of acoustic anisotropy ratio. These bounds motivate the search for materials with largest modulus-to-density ratio and materials with largest acoustic anisotropy. An imperfect, eikonal acoustic cloak with an extremely broad frequency band of operation is designed using the simple layered geometry of a unit cell. Estimates for the deficiency and payload of such acoustic cloaks are presented. It is found that it is beneficial to have the speed of sound in one of the components slightly lower (by only 0.05%) than the ambient speed of sound. We also show that including more than two layers in the unit cell increases design flexibility, although it does not push performance beyond the fundamental limitations of bi-layer metamaterials.

In addition, we have provided qualitative theoretical and quantitative numerical analysis of the role of shear waves in elastic cloaking devices. It is shown that for materials with Poisson's ratio $\nu > 0.49$, shear waves have little or no adverse effect on the performance of acoustic cloaks. We have found that isotropy and homogeneity of both density ρ and shear modulus μ is a necessary and sufficient condition for perfect decoupling of p- and s-waves in elastic

media with isotropic, *inhomogeneous* Lamé modulus λ . This finding opens the door to novel acoustic metamaterial devices besides the ultra-broadband acoustic cloak.

Acknowledgments

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