Ref.2364-CERN

ACTION MINIMA AMONG SOLUTIONS TO A CLASS OF EUCLIDEAN SCALAR FIELD EQUATIONS

S. Coleman *), V. Glaser and A. Martin
CERN -- Geneva

A B S T R A C T

We show that for a wide class of Euclidean scalar field equations, there exist non-trivial solutions, and the non-trivial solution of lowest action is spherically symmetric. This fills a gap in a recent analysis of vacuum decay by one of us.

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*) Current address : Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA. Work supported in part by the US National Science Foundation under grant Nr. PHY75-20427.

Ref.TH.2364-CERN
2 September 1977

1. INTRODUCTION

In the course of a study of vacuum instability [1], one of us encountered the differential equation in four-dimensional Euclidean space,

$$\Delta\Phi = U'(\Phi) \quad . \tag{1.1}$$

Here Δ is the usual Euclidean Laplace operator, U is a quartic polynomial in the single real field Φ, and the prime denotes differentiation with respect to Φ. This equation admitted a trivial solution, Φ a constant. In Ref. 1, a spherically symmetric non-trivial solution was constructed, and it was conjectured that this solution had the lowest action of any non-trivial solution. The purpose of this note is to supply the proof of this conjecture.

More precisely, we prove that, for a wide class of functions U, the non-trivial solution to Eq. (1.1) of smallest action is necessarily spherically symmetric. Our proof is valid for any number of Euclidean dimensions greater than two, although the class of admissible U's does depend upon the dimension[1].

The remainder of this section is a statement of our main result with some comments on its meaning. Sections 2 and 3 consist of the proof.

1.1 Statement of the theorem

Definition. We will say a real function of a single real variable $U(\Phi)$ is admissible in N dimensions if:

1) U is continuously differentiable for all Φ;

2) $U(0) = U'(0) = 0$;

3) U is somewhere negative;

4) There exist positive numbers a, b, α, and β such that

$$\alpha < \beta < 2N/(N-2) \quad , \tag{1.2}$$

and

$$U - a|\Phi|^{\alpha} + b|\Phi|^{\beta} \geq 0 \tag{1.3}$$

for all Φ.

The main theorem. In N-dimensional Euclidean space, N > 2, for any admissible U, Eq. (1.1) possesses at least one monotone spherically symmetric solution vanishing at infinity[2], other than the trivial solution $\Phi = 0$. Furthermore, this solution has Euclidean action,

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1. In particular, our theorem applies to the non-polynomial U considered by Frampton [2].

2. "Spherically symmetric" means that Φ is a function only of Euclidean distance from *some* point. "Vanishing at infinity" means that for any positive ε the set of all points for which $|\Phi| \geq \varepsilon$ has finite Lebesgue measure.

$$S = \int d^N x \left[\tfrac{1}{2} (\nabla \phi)^2 + U(\phi) \right] , \tag{1.4}$$

less than or equal to that of any other solution vanishing at infinity. If the other solution is not both spherically symmetric and monotone, the action is strictly less than that of the other solution.

1.2 Comments

i) It is difficult to imagine weakening the first three of the four conditions for admissibility. Condition (1) ensures that Eq. (1.1) makes sense, condition (2) is necessary if we are to have any hope of finding finite-action solutions that vanish at infinity, and condition (3) is necessary if there are to be any non-trivial solutions at all [3].

ii) Condition (4) is another story. It certainly cannot be dropped altogether[3], but it is possible that it could be weakened by better analysts than us. At any rate, we have stated it in the weakest form that we can, perhaps at the cost of making the statement somewhat clumsy. There are certainly conditions that are simpler to phrase and that imply our condition, though they are not implied by it. For example, it is easy to see that condition (4) is implied by the following: a) U is twice continuously differentiable, b) $U''(0) > 0$, and c) U is positive outside of some finite interval.

iii) Because U can assume both positive and negative values, one might worry that S might be ill-defined, and our main theorem thus meaningless. This is not a problem for continuous functions [and therefore for solutions of Eq. (1.1)] that vanish at infinity. This is because condition (4) implies that U is non-negative in some neighbourhood of $\Phi = 0$; hence U is a continuous function of x which is negative only on a set of finite measure, and the integral of U over all space is therefore unambiguous. (Of course, it might be unambiguously plus infinity, but this is no problem, since in this case we unambiguously do not have a minimum of S).

iv) For certain U's, there exist a set of numbers c_i, such that $U(\Phi - c_i)$ is admissible in our sense for each i. In this case, we can apply our main theorem to solutions such that $\Phi - c_i$ vanishes at infinity. We stress that although our main theorem tells us that in each of these classes the solution of minimum action is monotone and spherically symmetric, it gives us no way of comparing the actions of solutions in different classes. In particular, it does not exclude the possibility that an asymmetric solution in one class might have lower action than the symmetric solution of minimum action in another class.

3. Consider $U = -\Phi^2$.

v) The omission of two dimensions is caused by technical details of our proof, not by any deep insight on our part. Our main theorem (with appropriate admissibility conditions) might well be true in two dimensions; we know of no counter-examples.

vi) We should emphasize that the problem handled here is not a conventional minimization problem. We are *not* searching for a function which minimizes S; we are searching for a function which minimizes S *restricted to its stationary points*. If S were bounded below, these two problems would be equivalent, but it is not[4], and therefore they are not.

vii) It is for this reason that our first step is to reduce our problem to a genuine minimization problem. This is done in Section 2 below. We then handle the reduced problem with the standard methods of functional analysis in Section 3.

2. THE REDUCED PROBLEM

Our first step is to divide S into two parts,

$$S = T + V \tag{2.1}$$

where

$$T = \frac{1}{2}\int d^N x\, (\nabla\Phi)^2 , \tag{2.2}$$

and

$$V = \int d^N x\, U(\Phi) . \tag{2.3}$$

If we define a scale transformation by

$$\Phi_\sigma(x) = \Phi(x/\sigma) , \tag{2.4}$$

where σ is a positive number, then these objects have simple scaling properties:

$$V[\Phi_\sigma] = \sigma^N V[\Phi] , \tag{2.5}$$

and

$$T[\Phi_\sigma] = \sigma^{N-2} T[\Phi] . \tag{2.6}$$

As a first application, we observe that any solution of Eq. (1.1) makes S stationary. Thus, in particular, S must be stationary under scale transformations; whence,

$$(N-2)T + NV = 0 \tag{2.7}$$

or, equivalently,

$$S = 2T/N , \tag{2.8}$$

for any solution of Eq. (1.1).

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4. It is easy to see that, since U is somewhere negative, the infimum of S is minus infinity.

Definition: "The reduced problem" is the problem of finding a function vanishing at infinity which minimizes T for some fixed negative V [5].

From Eqs. (2.5) and (2.6), it is obvious that if we can find a solution to the reduced problem for some negative V we can find a solution for any negative V; the solutions are just scale transforms of each other. Indeed, all the solutions have the same value of the scale-invariant ratio,

$$R = -T^{(N/N-2)}/V, \tag{2.9}$$

and the reduced problem can equivalently be stated as the problem of finding a function with negative V that minimizes R.

Theorem A: If a solution of the reduced problem exists, then, for appropriately chosen V, it is a solution of Eq. (1.1) that has action less than or equal to that of any non-trivial solution of Eq. (1.1).

Proof: First we shall show that a solution of the reduced problem can always be scale-transformed into a solution of Eq. (1.1). A solution of the reduced problem Φ is a function which stationarizes

$$S' = T + \lambda^2 V, \tag{2.10}$$

where λ^2 is a Lagrange multiplier. By the same arguments as those which led to Eq. (2.7),

$$(N-2)T + \lambda^2 N V = 0. \tag{2.11}$$

Because T is positive and V is negative, λ^2 must be positive and our notation is not deceptive. Then the scale transformed function Φ_λ is a solution of Eq. (1.1).

We shall now show that the solution we have constructed has S less than or equal to that of any solution of Eq. (1.1). Let $\bar{\Phi}$ be some non-trivial solution of Eq. (1.1). Since $\bar{\Phi}$ is non-trivial, $T[\bar{\Phi}]$ is not zero, and by Eq. (2.7), $V[\bar{\Phi}]$ is negative. Now, let Φ be the solution to the reduced problem with

$$V[\Phi] = V[\bar{\Phi}]. \tag{2.12}$$

By the definition of the reduced problem

$$T[\Phi] \leq T[\bar{\Phi}]. \tag{2.13}$$

5. We emphasize that in defining the reduced problem we do not necessarily restrict ourselves to continuous functions. The minimizing function is indeed continuous, but this is something we prove, not something we assume. However, one might worry that, in the absence of a continuity assumption V, and therefore the reduced problem, might be ill-defined. In fact, this is not a problem: if Φ vanishes at infinity, and T is finite, then a standard Sobolev inequality implies that Φ is in $L^{2N/(N-2)}$. Condition (4) then implies that the integral of the negative part of U is finite.

Comparing Eqs. (2.7) and (2.11), we see that

$$\lambda \leqslant 1 . \tag{2.14}$$

As before, Φ_λ satisfies Eq. (1.1). But

$$T[\Phi_\lambda] = \lambda^{(N-2/2)} T[\Phi] \leqslant T[\bar{\Phi}] \tag{2.15}$$

Thus, by Eq. (2.8),

$$S[\Phi_\lambda] \leqslant S[\bar{\Phi}] \tag{2.16}$$

This completes the proof of Theorem A.

It is clear from this proof that Theorem A is valid for a system of many coupled scalar fields. Unfortunately, the same is not true for the rest of our argument, which is very much restricted to the case of a single field.

3. ANALYSIS OF THE REDUCED PROBLEM

Theorem B: There exists at least one solution to the reduced problem. All solutions to the reduced problem are spherically symmetric and monotone.

Theorems A and B imply our main theorem. The argument for Theorem B is somewhat lengthy, and thus we have organized it as a long sequence of statements with short proofs.

We begin by reminding the reader of our fourth condition on the function U: that there exist positive numbers a, b, α, and β with

$$2N/(N-2) > \beta > \alpha \tag{3.1}$$

such that

$$U - a|\Phi|^\alpha + b|\Phi|^\beta \geqslant 0 . \tag{3.2}$$

Statement 1: For any function Φ such that $V[\Phi]$ is negative,

$$\int d^N x\, |\Phi|^\beta \geqslant \frac{a}{b} \int d^N x\, |\Phi|^\alpha . \tag{3.3}$$

Proof: Integrating Eq. (3.2), we find

$$b \int d^N x\, |\Phi|^\beta \geqslant a \int d^N x\, |\Phi|^\alpha - V[\Phi] . \tag{3.4}$$

But V is negative.

Statement 2: For any function Φ such that $V[\Phi]$ is negative, and for any $\gamma > \beta$,

$$\int d^N x\, |\Phi|^\gamma \geqslant \left(\frac{a}{b}\right)^{(\gamma-\alpha)/(\beta-\alpha)} \int d^N x\, |\Phi|^\alpha . \tag{3.5}$$

Proof: By Hölder's inequality,

$$\int d^N x\, |\Phi|^\beta \leqslant \left[\int d^N x\, |\Phi|^\alpha\right]^{(\gamma-\beta)/(\gamma-\alpha)} \left[\int d^N x\, |\Phi|^\gamma\right]^{(\beta-\alpha)/(\gamma-\alpha)} \tag{3.6}$$

Combined with Statement 1, this yields the desired result.

Having established these preliminary inequalities, we can now begin our attack on the reduced problem. Since T is a positive functional, it is bounded below on the set of all functions with fixed negative V, and thus has a greatest lower bound, inf T. Thus we can construct a minimizing sequence, an infinite sequence of functions Φ_n, such that $V[\Phi_n]$ is a fixed negative number and such that

$$\lim_{n\to\infty} T[\Phi_n] = \inf T\,. \tag{3.7}$$

Our task is to show that we can choose the elements of the minimizing sequence such that they converge to an actual minimum of T.

It will be convenient to choose the elements of our minimizing sequence to be differentiable functions of compact support. (It is easy to see that this is always possible.) Of course, this does not imply that their limit (if it exists) is such a function.

Statement 3: Either there exists a minimizing sequence such that $\Phi_n(x)$ is greater than or equal to zero for all n and all x, or there exists a minimizing sequence such that $\Phi_n(x)$ is less than or equal to zero for all n and all x.

Proof: Any function can be written as the sum of its positive and negative parts:

$$\Phi(x) = \Phi_+(x) + \Phi_-(x) \tag{3.8}$$

where

$$\Phi_+(x) = \max\{\Phi(x), 0\} \tag{3.9a}$$

and

$$\Phi_-(x) = \min\{\Phi(x), 0\}\,. \tag{3.9b}$$

Clearly,

$$T[\Phi] = T[\Phi_+] + T[\Phi_-]\,. \tag{3.10}$$

Also, because U(0) = 0,

$$V[\Phi] = V[\Phi_+] + V[\Phi_-]\,. \tag{3.11}$$

Now let us construct the scale-invariant ratio of Eq. (2.9),

$$R[\Phi] = \frac{-(T[\Phi_+] + T[\Phi_-])^{N/(N-2)}}{V[\Phi_+] + V[\Phi_-]} \tag{3.12}$$

From this, it is easy to see that either

$$R[\Phi] \geq R[\Phi_+] \quad \text{if} \quad V[\Phi_-] \geq 0, \tag{3.13a}$$

$$R[\Phi] \geq R[\Phi_-] \quad \text{if} \quad V[\Phi_+] \geq 0, \tag{3.13b}$$

or

$$R[\Phi] \geq \min\{R[\Phi_+], R[\Phi_-]\} \quad \text{if} \quad V[\Phi_\pm] < 0. \tag{3.13c}$$

These inequalities must be obeyed by each function in a minimizing sequence; thus, either the minimizing sequence has an infinite subsequence for which

$$R[\Phi_n] \geq R[\Phi_{n+}], \tag{3.14a}$$

or it has an infinite subsequence for which

$$R[\Phi_n] \geq R[\Phi_{n-}]. \tag{3.14b}$$

For the moment, let us assume it is the first alternative that prevails. The subsequence is then a minimizing sequence for which Eq. (3.14a) holds throughout. We now define a new sequence of functions Φ_n', where each Φ_n' is a scale transform of Φ_{n+}, with the scale transformation chosen such that

$$V[\Phi_n'] = V[\Phi_n] \tag{3.15}$$

But, by Eq. (3.14a),

$$T[\Phi_n'] \leq T[\Phi_n] \tag{3.16}$$

Thus we have constructed a minimizing sequence composed exclusively of non-negative functions. Identical reasoning applies if (3.14b) holds for an infinite subsequence.

To keep our arguments as simple as possible, we will assume from now on that we are dealing with a minimizing sequence composed of non-negative functions. The arguments for the alternative case can be constructed trivially by replacing Φ by $-\Phi$ everywhere.

<u>Statement 4</u>: There exists a minimizing sequence such that $\Phi_n(x)$ is spherically symmetric and monotone for all n.

Proof: We remind the reader of the definition of the spherical rearrangement Φ_R of a non-negative function Φ. Φ_R is a spherically symmetric function, monotone decreasing as one moves away from the origin, such that, for any positive number M,

$$\mu\{x \mid \Phi_R(x) \geq M\} = \mu\{x \mid \Phi(x) \geq M\}\,, \tag{3.17}$$

where μ denotes the Lebesgue measure. It is trivial that

$$V[\Phi_R] = V[\Phi]\,. \tag{3.18}$$

It is not trivial, but it is known [4], that

$$T[\Phi_R] \leq T[\Phi]\,. \tag{3.19}$$

Thus, the spherical rearrangement of our minimizing sequence is a minimizing sequence.

For spherically symmetric functions, it is a wise policy to rewrite things in terms of spherical coordinates. We define y by

$$r = \exp[y]\,, \tag{3.20}$$

where r is the usual distance from the origin. We also define $f_n(y)$ by

$$f_n = \Phi_n \exp\left[\tfrac{1}{2}(N-2)y\right]. \tag{3.21}$$

It is easy to see that

$$T[\Phi_n] = C_N \int dy \left[\frac{1}{2}\left(\frac{df_n}{dy}\right)^2 + \frac{(N-2)^2}{8} f_n^2\right], \tag{3.22}$$

where C_N is a positive constant, the result of integrating over angles.

Statement 5: There exists a minimizing sequence of non-negative spherically symmetric monotone functions such that all of the following are uniformly bounded from above:

$$\int dy\,(df_n/dy)^2\,, \tag{A}$$

$$\int dy\, f_n^2\,, \tag{B}$$

$$\frac{|f_n(y_1) - f_n(y_2)|}{|y_1 - y_2|^{1/2}}\,, \tag{C}$$

$$|f_n(y)|\,, \tag{D}$$

$$\int d^N x\, |\Phi_n|^{2N/(N-2)}\,, \tag{E}$$

$$\int d^N x\, |\Phi_n|^{\alpha}\,. \tag{F}$$

(This motley collection of bounds is arranged in this order because each is a consequence of the preceding ones, as we shall see.)

Proof: It is trivial that one can choose a minimizing sequence such that T is bounded; bounds (A) and (B) then follow from Eq. (3.22). Bound (C) then follows from bound (A) and the Schwarz inequality:

$$\begin{aligned} |f_n(y_1) - f_n(y_2)| &= \left| \int_{y_1}^{y_2} dy\,(df_n/dy) \right| \\ &\leqslant \left| \int_{y_1}^{y_2} dy\,(df_n/dy)^2 \right|^{1/2} \left| \int_{y_1}^{y_2} dy \right|^{1/2} \\ &\leqslant \left| \int_{y_1}^{y_2} dy\,(df_n/dy)^2 \right|^{1/2} |y_1 - y_2|^{1/2}. \end{aligned} \tag{3.23}$$

Because $f_n(y)$ vanishes at infinity,

$$|f_n(y_1)|^2 = \left| 2\int_{y_1}^{\infty} dy\; f_n\,(df_n/dy) \right| \leqslant \int dy \left[f_n^2 + (df_n/dy)^2 \right], \tag{3.24}$$

from which bound (D) follows. Bound (E) is now immediate[6]:

$$\int d^N x\, |\phi_n|^{2N/(N-2)} = \int dy\, |f_n|^{2N/(N-2)} \leqslant \sup |f_n|^{\frac{4}{N-2}} \int dy\, |f_n|^2 . \tag{3.25}$$

Bound (F) is a consequence of bound (E) and Statement 2.

Statement 6: There exists a minimizing sequence of spherically symmetric functions and a bounded continuous function f such that

$$\lim_{n \to \infty} f_n(y) = f(y), \tag{3.26}$$

pointwise for all y and uniformly on any finite interval.

Proof: By bounds (C) and (D), the minimizing sequence is a family of uniformly bounded equicontinuous functions; Ascoli's theorem then asserts the existence of a subsequence with the stated property.

If we define Φ by

$$\Phi = f \exp\left[-\tfrac{1}{2}(N-2)\, y \right] \tag{3.27}$$

then Statement 6 tells us that Φ_n converges to Φ pointwise almost everywhere (that is to say, except possibly at the origin). Because f is bounded, Φ vanishes at infinity. Note that we have not yet shown that Φ is not zero.

6. We are aware that bounds (C), (D), and (E) are standard Sobolev imbedding theorems [5], but, since they are easy to prove, we chose to include explicit proofs here.

Statement 7: For the minimizing sequence of the preceding statement,

$$\lim_{n\to\infty} \int d^N x \, |\Phi_n|^\beta = \int d^N x \, |\Phi|^\beta . \tag{3.28}$$

Proof:

$$\int d^N x \, |\Phi_n|^\beta = C_N \int dy \, |f_n|^\beta \exp\left[\left(N - \tfrac{1}{2}\beta[N-2]\right) y\right] \equiv C_N \int dy \, h_n(y). \tag{3.29}$$

We break this integral into three parts:

$$\int dy \, h_n(y) = \int_{-\infty}^{y_1} dy \, h_n(y) + \int_{y_1}^{y_2} dy \, h_n(y) + \int_{y_2}^{\infty} dy \, h_n(y) \tag{3.30}$$

where y_1 and y_2 are numbers we shall fix shortly. Now,

$$\left| \int_{-\infty}^{y_1} dy \, h_n(y) \right| \leqslant \frac{\sup |f_n|^\beta \exp\left[\left(N - \frac{1}{2}\beta(N-2)\right) y_1\right]}{N - \frac{1}{2}\beta(N-2)} \tag{3.31}$$

Thus the first term in Eq. (3.30) may be made as small as we please, uniformly in n, by choosing y_1 sufficiently large and negative. Also,

$$\left| \int_{y_2}^{\infty} dy \, h_n(y) \right| \leqslant C_N^{-1} \sup |f_n|^{\beta-\alpha} e^{-\frac{1}{2}(N-2)(\beta-\alpha) y_2} \int d^N x \, |\Phi_n|^\alpha . \tag{3.32}$$

Thus the last term in Eq. (3.30) may also be made as small as we please, uniformly in n, by choosing y_2 sufficiently large and positive. Finally,

$$\lim_{n\to\infty} \int_{y_1}^{y_2} dy \, h_n(y) = \int_{y_1}^{y_2} dy \, |f|^\beta \exp\left[\left(N - \tfrac{1}{2}\beta[N-2]\right) y\right] \tag{3.33}$$

by Statement 6.

Statement 8: For the minimizing sequence of the preceding statement,

$$V[\Phi] \leqslant V[\Phi_n] . \tag{3.34}$$

(Remember: $V[\Phi_n]$ is independent of n.)

Proof: From Eq. (3.2),

$$U(\Phi) + b|\Phi|^\beta \geqslant 0$$

Thus, by Fatou's Lemma,

$$\begin{aligned} V[\Phi] + b\int d^N x \, |\Phi|^\beta &\leqslant \lim_{n\to\infty} \left[V[\Phi_n] + b\int d^N x \, |\Phi_n|^\beta \right] \\ &= V[\Phi_n] + b\int d^N x \, |\Phi|^\beta . \end{aligned} \tag{3.35}$$

We now know that Φ is not zero.

Statement 9: For the minimizing sequence of the preceding statement,

$$T[\Phi] \leq \lim_{n\to\infty} T[\Phi_n] \tag{3.36}$$

Proof: T can be thought of as defining a Hilbert space norm. In terms of this norm, our minimizing sequence is a sequence of bounded vectors; such a sequence always has a weakly convergent subsequence. Statement 9 is then just the well-known proposition that the norm in Hilbert space is weakly lower semi-continuous.

Statement 10: The function Φ defined above is a solution of the reduced problem.

Proof: By Statements 8 and 9,

$$R[\Phi] \leq \lim_{n\to\infty} R[\Phi_n]. \tag{3.37}$$

But the limit on the right is the infimum of the scale-invariant ratio R. Thus, "less than" is not a possibility, and Φ must attain the infimum, that is to say, be a minimum of R.

Thus we have almost proved Theorem B; we have constructed a monotone spherically-symmetric solution of the reduced problem. However, the remainder of Theorem B, the non-existence of non-spherically-symmetric (or non-monotone) solutions, is trivial, since it is known [4] that T for any function is equal to T for the spherical rearrangement of the function only if the original function is spherically symmetric and monotone. This completes the proof.

Acknowledgements

One of us (S.C.) thanks Ian Affleck for pointing out that his original attempt to prove a weaker version of this theorem was in error. He also thanks the CERN Theory Division for its hospitality and support during the time this work was done.

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