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ACTUARIAL VALUES CALCULATED USING THE INCOMPLETE GAMMA FUNCTION

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1. INTRODUCTION

Complete expectation-of-life $\overline{e}_x$ for a x-aged life is the expected value of his time-until-death

$$\overline{e}_x = \int_0^{\omega - x} \frac{S(x+t)}{S(x)} dt ,$$

with $x \in]0, \omega]$, being ω the limiting age, for whichever possible survival function $S(x)$ measuring the probability a newborn survives until to the age of x. If we discard the limiting age, then

$$\overline{e}_x = \int_0^{\infty} \frac{S(x+t)}{S(x)} dt .$$

Let us consider the Makeham survival function

$$S(x) = e^{-Ax - m(e^x - 1)};$$

it will be

$$\overline{e}_x = \int_0^{\omega - x} e^{-At - mc^x (c^t - 1)} dt .$$

This integral has not received any explicit evaluation until now: as a matter of fact the complete expectation-of-life problem has been usually bypassed by calculating the curtate expectation-of-life instead of the complete one.

As well, if we consider the actuarial present value of a continuous whole life annuity

$$\overline{a}_x = \int_0^{\omega - x} v^t \frac{S(x+t)}{S(x)} dt , \; 0 < v \leq 1,$$

plugging the Makeham function there, one again gets an integral not explicitly evaluated up to now: it became a common practice indeed, to avoid such obstacles, determining $\overline{a}_x$ by using the so called "Euler approximation"

$$\overline{a}_x \cong \frac{1}{2} + a_x - \frac{1}{12}(\delta + \mu_x),$$

being a_x, δ and μ_x some actuarial figures out of interest here. In this paper the closed form integrations for both $\overline{e}_x$ and $\overline{a}_x$ are provided by means of the incomplete Gamma function.

2. COMPLETE EXPECTATION-OF-LIFE

Let $s=\exp(-A)$, $g=\exp(-m)$. Adopting the four-parameters Makeham's survival law

$$S(x) = S_M(x;k,g,c,s) = kg^{c^x}s^x, \; x \in \;]0\,,\,\omega],$$

with $k > 0$, $0 < g < 1$, $c > 1$, $0 < s < 1$, then $\overline{e}_x$ becomes:

$$\overline{e}_x = \int_0^{\omega-x} g^{c^x(c^t-1)}s^t\,dt \tag{1}$$

which is deemed not integrable in the literature on this subject (Bowers *et al.*, 1986).

We recall the incomplete Gamma function definition:

$$\Gamma(z,\beta) = \int_\beta^\infty e^{-t}t^{z-1}dt,$$

and the relationship (Gradshteyn and Ryzhik, 1965, page 308, formula 3.331-2) which holds for Re $\beta > 0$

$$\int_0^\infty \exp[-\beta e^t - \mu t]dt = \beta^\mu \Gamma(-\mu,\beta), \tag{2}$$

obtained by putting there $\beta e^t = t$. For further details, the reader should look at special treatises on the Gamma function (Campbell, 1966, Temme, 1996 and Chaundry and Syed, 2002). We then arrive at our first result.

Theorem 1. For any $x \in [0\,,\,\omega]$ we have

$$\overline{e}_x = \frac{(-c^x \ln g)^{-\frac{\ln s}{\ln c}}}{g^{c^x}\ln c}\left\{\Gamma\left(\frac{\ln s}{\ln c}, -c^x \ln g\right) - \Gamma\left(\frac{\ln s}{\ln c}, -c^\omega \ln g\right)\right\}. \tag{3}$$

Proof. First we have

$$g^{c^x(c^t-1)}s^t = \frac{g^{c^x c^t}s^t}{g^{c^x}} = \frac{\exp[c^x c^t \ln g + t\ln s]}{g^{c^x}}.$$

Write $c^t = \exp[t \ln c]$ so that

$$\overline{e}_x = \frac{1}{g^{c^x}}\int_0^{\omega-x}\exp[c^x e^{t\ln c}\ln g + t\ln s]dt =$$

$$= \frac{1}{g^{c^x}\ln c}\int_0^{(\omega-x)\ln c}\exp\left[c^x e^{\tau}\ln g + \tau\frac{\ln s}{\ln c}\right]d\tau$$

notice that we have used the substitution $t = t \ln c$. Observe that

$$\overline{e}_x = \frac{1}{g^{c^x}\ln c}\left\{\int_0^{\infty}A(x,c,g,s;\tau)d\tau - \int_{(\omega-x)\ln c}^{\infty}A(x,c,g,s;\tau)d\tau\right\}, \tag{4}$$

where

$$A(x,c,g,s;\tau) = \exp\left[c^x e^{\tau}\ln g + \tau\frac{\ln s}{\ln c}\right].$$

Now we evaluate the integrals in (4) separately. Being $0 < g < 1$, we have $c^x \ln g < 0$; in the same way $0 < s < 1$ means $\ln s < 0$.

Then, if $b = -c^x \ln g$, $m = -\ln s / \ln c$, we can use (2) to infer that

$$\int_0^{\infty}A(x,c,g,s;\tau)d\tau = \beta^{\mu}\Gamma(-\mu,\beta) \tag{5}$$

For the second integral we set $s = t - (\omega\text{-}x) \ln c$, so that

$$\int_{(\omega-x)\ln c}^{\infty}A(x,c,g,s;\tau)d\tau = e^{-\mu(\omega-x)\ln c}\int_0^{\infty}\exp[-\beta c^{\omega-x}e^{\sigma} - \mu\sigma]d\sigma.$$

Using (2) again and simplifying the exponential factor, we obtain:

$$\int_{\omega-x}^{\infty}A(x,c,g,s;\tau)d\tau = c^{-\mu(\omega-x)}(\beta c^{\omega-x})^{\mu}\Gamma(-\mu,\beta c^{\omega-x}). \tag{6}$$

Therefore, by (5), (6) and (4), collecting the common factor $\beta^{\mu} = (-c^x \ln g)^{-\frac{\ln s}{\ln g}}$ we have proved (3).

If we do not assume a limiting age, it is easy to state the following:

Theorem 2. The following formula holds:

$$\overline{e}_x = \frac{(-c^x \ln g)^{-\frac{\ln s}{\ln c}}}{g^{c^x} \ln c} \Gamma\left(\frac{\ln s}{\ln c}, -c^x \ln g\right). \tag{7}$$

3. THE CONTINUOUS WHOLE LIFE ANNUITY

It is well known that

$$\overline{a}_x = \int_0^{\omega - x} v^t \frac{S(x+t)}{S(x)} dt, \tag{8}$$

for whichever survival function $S(x)$ one applies.

In the previous integral for the annuity, it is $v = (1+i)^{-1} > 0$ being i the rate of interest. For the Makeham case we again get

$$\frac{S_M(x+t)}{S_M(x)} = g^{c^x(c^t - 1)} s^t = \frac{g^{c^{x+t}}}{g^{c^x}} s^t,$$

and therefore

$$\overline{a}_x = \frac{1}{g^{c^x}} \int_0^{\omega - x} g^{c^x(c^t - 1)} (sv)^t \, dt. \tag{9}$$

Being the former integral (9) of the same kind as (1), we are able to establish immediately a further explicit formula giving $\overline{a}_x$ in closed form.

Theorem 3. Let $b = -c^x \ln g$, $s = -\ln(sv) / \ln c$, then for any $x \in [0, \omega]$:

$$\overline{a}_x = \frac{(-c^x \ln g)^{-\frac{\ln(sv)}{\ln c}}}{g^{c^x} \ln c} \left\{ \Gamma\left(\frac{\ln(sv)}{\ln c}, -c^x \ln g\right) - \Gamma\left(\frac{\ln(sv)}{\ln c}, -c^\omega \ln g\right) \right\}. \tag{10}$$

4. THE RELEVANT MORTALITY TABLE

Formulæ (3) and (10) allow a mortality table to be written which for the first time holds the exact values of $\overline{e}_x$ and $\overline{a}_x$. Let us provide a sample case using the 1991 Italian male census and the King and Hardy method, after establishing four age ranges: 20-34, 35-49, 50-64, 65-79, we found:

$c = 0.896507729$;
$g = 0.000816777$;
$s = 1.000799587626$;
$k = 0.000193377$.

In such a way, a table has been created by using Mathematica, whose library holds the incomplete Gamma function, to evaluate the formulæ (3) and (10) for x variable from 0 to ω=109, the last age group included in the Italian tables. We used the rate r=0.02. The mortality table and a graphic are included in the Appendix.

Comparing the approximate actuarial table with the exact one, we observe that the values coming from the discrete approximation are always slightly greater than the exact ones.

5. CONCLUSIONS

– Closed form integrations for both $\overline{e}_x$ and $\overline{a}_x$ have been calculated by means of the incomplete Gamma function $\Gamma(z,\beta)$, provided that the four-parameters Makeham law holds.

– Only three, g, c, s, of the four independent parameters characterizing the Makeham survival law are involved in the final formulæ for $\overline{e}_x$ and $\overline{a}_x$.

– Our purely analytical approach allows us to get rid of "expedients" (such as the numerical algorithm for evaluating the integral with loss of symbolic solution; the curtate expectation of life instead of the full one; the Euler approximation for the annuity) and to provide exact solutions involving the incomplete Gamma, available on computer programs such as Maple, Mathematica, and so on.

– The exact values of $\overline{e}_x$ and $\overline{a}_x$ are smaller than the approximate ones.

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APPENDIX

TABLE 1

The mortality table calculated with (3) and (10) to the age of 20

x	S(x)	$\overline{e}_x$	$\overline{a}_x$
0	99920.5	734.733	38.02
1	99839.	725.329	378.017
2	99757.4	715.918	375.789
3	99675.6	706.501	373.515
4	99593.7	697.078	371.195
5	99511.6	687.649	368.827
6	99429.3	678.214	366.411
7	99346.7	668.774	363.947
8	99263.8	659.328	361.432
9	99180.6	649.877	358.866
10	99097.	640.421	356.248
11	99012.9	630.961	353.577
12	98928.4	621.496	350.853
13	98843.3	612.026	348.074
14	98757.6	602.553	34.524
15	98671.2	593.076	342.349
16	98584.	583.597	33.94
17	98495.9	574.114	336.394
18	98406.8	564.629	333.328
19	98316.6	555.143	330.202
20	98225.2	545.655	327.015

Having the exact solutions we can plot $\overline{e}_x$ and $\overline{a}_x$.

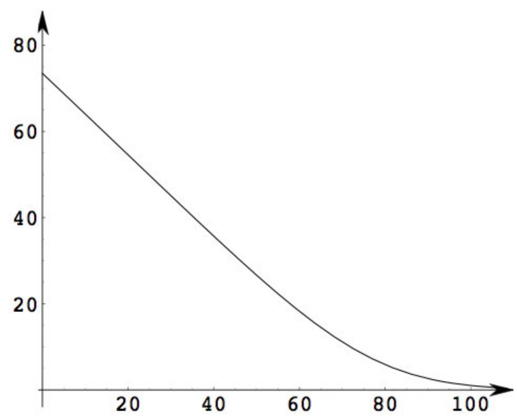

Figure 1 – $\overline{e}_x$.

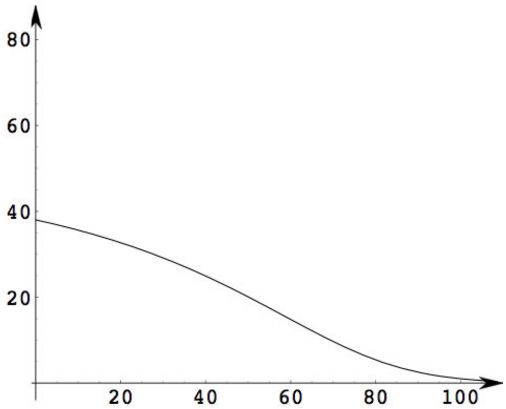

Figure 2 – $\overline{a}_x$.

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RIASSUNTO

Valori attuariali calcolati con la funzione Gamma incompleta

La vita media esatta di un individuo e i valori attuali delle rendite vitalizie continue sono definiti attraverso integrali in cui compare una funzione di sopravvivenza. Se si sceglie la funzione di sopravvivenza di De Moivre tali integrali sono facilmente calcolabili con gli strumenti dell'analisi matematica, ma se si sceglie la funzione di sopravvivenza di Makeham degli stessi integrali si possono calcolare soltanto approssimazioni numeriche. In questo lavoro si presenta il calcolo di detti integrali in forma chiusa attraverso la funzione Gamma incompleta.

SUMMARY

Actuarial values calculated using the incomplete Gamma function

The complete expectation-of-life for a person and the actuarial present value of continuous life annuities are defined by integrals. In all of them at least one of the factors is a

survival function value ratio. If de Moivre's law of mortality is chosen, such integrals can easily be evaluated; but if the Makeham survival function is adopted, they are used to be calculated numerically. For the above actuarial figures, closed form integrations are hereafter provided by means of the incomplete Gamma function.