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Acyclic Calabi–Yau categories

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With an appendix by Michel Van den Bergh

Abstract

We prove a structure theorem for triangulated Calabi–Yau categories: an algebraic 2-Calabi–Yau triangulated category over an algebraically closed field is a cluster category if and only if it contains a cluster-tilting subcategory whose quiver has no oriented cycles. We prove a similar characterization for higher cluster categories. As an application to commutative algebra, we show that the stable category of maximal Cohen–Macaulay modules over a certain isolated singularity of dimension 3 is a cluster category. This implies the classification of the rigid Cohen–Macaulay modules first obtained by Iyama and Yoshino. As an application to the combinatorics of quiver mutation, we prove the non-acyclicity of the quivers of endomorphism algebras of cluster-tilting objects in the stable categories of representation-infinite preprojective algebras. No direct combinatorial proof is known as yet. In the appendix, Michel Van den Bergh gives an alternative proof of the main theorem by appealing to the universal property of the triangulated orbit category.

1. Introduction

Cluster algebras were introduced and studied by Fomin and Zelevinsky and Berenstein, Fomin and Zelevinsky in a series of articles [FZ02, FZ03a, BFZ05, FZ07]. It was the discovery of Marsh, Reineke and Zelevinsky [MRZ03] that they are closely connected to quiver representations. This link is similar to the one between quantum groups and quiver representations discovered by Ringel [Rin90] and investigated by Kashiwara, Lusztig, Nakajima and many others. The link between cluster algebras and quiver representations becomes especially beautiful if, instead of categories of quiver representations, one considers certain triangulated categories deduced from them: the so-called *cluster categories*. These were introduced in [BMRRT06] and, for Dynkin quivers of type A_n, in [CCS06]. They were shown to be triangulated in [Kel05]. If k is a field and Q a quiver without oriented cycles, the associated cluster category $\mathcal{C}_Q$ is the 'largest' 2-Calabi–Yau category under the derived category of representations of Q over k. It was shown [BMRRT06, BMR06, BMR08, BMRT05, CC06, CK05, CK06] that this category fully determines the combinatorics of the cluster algebra associated with Q and carries considerably more information. This was used to prove significant new results on cluster algebras. For example, if Q is a connected acylic quiver, then:

(i) the cluster variables of the associated cluster algebra are in bijection with the almost positive Schur roots of the associated Kac–Moody Lie algebra (this results from [CK06] by the method of [IY06, § 7]);

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(ii) the mutation class of the quiver Q is finite if and only if the underlying graph of Q has two vertices, is a Dynkin diagram or is an extended Dynkin diagram of type A, D or E (cf. [BR06]);

(iii) the coefficients appearing in the numerators of the cluster variables are positive integers [CR06], as conjectured by Fomin and Zelevinsky [FZ02] for arbitrary cluster algebras.

We refer to [FZ03b] and [Zel04] for more background on cluster algebras and to [ABST06], [ABS06], [BMR08], [BR05], [BRS07], [BM07], [DWZ07], [GLS07c], [GLS06], [Iya05a], [IR06], [Rin07], [Tho07], [Wra07] and [Zhu08] for recent developments in the study of their links with representations of quivers and finite-dimensional algebras.

The question arises as to whether, for a given quiver Q without oriented cycles, the cluster category is the 'unique model' of the associated cluster algebra. In other words, if we view the cluster algebra as a combinatorial invariant associated with the cluster category, is the category determined by this invariant?

In this paper, we show that, surprisingly, this question has a positive answer. Namely, we prove that if k is an algebraically closed field and $\mathcal{C}$ an algebraic 2-Calabi–Yau category containing a cluster-tilting object T whose endomorphism algebra has a quiver Q without oriented cycles, then $\mathcal{C}$ is triangle-equivalent to the cluster category $\mathcal{C}_Q$. Notice that this result is 'of Morita type', but much stronger than typical Morita theorems, since we only need to know the *quiver* of the endomorphism algebra, not the algebra itself.

We give several applications. First, we show on an example that cluster categories naturally appear as stable categories of Cohen–Macaulay modules over certain singularities. This yields an alternative proof of Iyama and Yoshino's [IY06] classification of rigid Cohen–Macaulay modules over a certain isolated singularity. More examples may be obtained from [IR06] and [IY06] (cf. [BIKR07]).

Secondly, we show that the quivers associated in [GLS07a] with representation-infinite finite-dimensional preprojective algebras are not mutation-equivalent to quivers without oriented cycles. This last result was obtained independently by Geiss (private communication). It has been used in [DWZ07, Example 8.7], to show that the class of rigid quivers with potential is strictly greater than the class of quivers with potential mutation-equivalent to acyclic ones. An application to the realization of cluster categories as stable categories of Frobenius categories with finite-dimensional morphism spaces is given in [BIRS07] and [GLS07b]. In [HJ06a] and [HJ06b], the authors use our results to determine which stable categories of representation-finite self-injective algebras of type A and D are higher cluster categories. More generally, in [Ami06], Amiot obtains a classification of 'most' triangulated categories with finitely many indecomposables by methods similar to ours.

The main difficulty in the proof is the construction of a triangle functor between the cluster category and the given Calabi–Yau category. Our construction is based on the description [Kel05] of the cluster category as a stable derived category of a certain differential graded category. This approach leads to interesting connections between Calabi–Yau categories of dimensions 2 and 3, which have been further investigated in [Tab07]. It turns out that each algebraic Calabi–Yau category of dimension 2 containing a cluster-tilting subcategory is equivalent to a stable derived category of a differential graded category whose perfect derived category is Calabi–Yau of dimension 3.

A more direct approach, based on the universal property of the cluster category [Kel05], has been discovered by Michel Van den Bergh, who has kindly agreed to include his proof as an appendix to this paper.

It turns out that the main theorem and its proofs can be generalized almost without effort to Calabi–Yau categories of any dimension $d \geqslant 2$. However, one has to take into account that, in the d-cluster category, the self-extensions of the canonical cluster-tilting object vanish in degrees $-d+2, \ldots, -1$. This condition therefore has to be added to the hypotheses of the generalized main theorem.

2. The main theorem and two applications

2.1 Statement

Let k be a perfect field. Let $\mathcal{E}$ be a k-linear Frobenius category. Recall [Hap88] that this means that $\mathcal{E}$ is an exact category in the sense of Quillen, that it has enough projectives and enough injectives and that an object of $\mathcal{E}$ is projective if and only if it is injective. We assume that $\mathcal{E}$ has split idempotents (i.e. every idempotent morphism admits a kernel in $\mathcal{E}$). The stable category $\mathcal{C} = \underline{\mathcal{E}}$ is the quotient of $\mathcal{E}$ by the ideal of morphisms factoring through a projective object. It is a triangulated category [Hap88]. We suppose that the stable category $\mathcal{C}$ has finite-dimensional Hom-spaces and is Calabi–Yau of CY-dimension 2, i.e. we have bifunctorial isomorphisms

$$D\mathcal{C}(X, Y) \xrightarrow{\sim} \mathcal{C}(Y, S^2X), \quad X, Y \in \mathcal{C},$$

where D is the duality functor $\mathrm{Hom}_k(?, k)$ and S the suspension of $\mathcal{C}$.

Let $\mathcal{T} \subset \mathcal{C}$ be a cluster-tilting subcategory. Recall from [KR07] that this means that $\mathcal{T}$ is a k-linear subcategory which is functorially finite in $\mathcal{C}$ and such that an object X of $\mathcal{C}$ belongs to $\mathcal{T}$ if and only if we have $\mathrm{Ext}^1(T, X) = 0$ for all objects T of $\mathcal{T}$. As shown in [KR07], the category $\mathrm{mod}\,\mathcal{T}$ of finitely presented $\mathcal{T}$-modules is then abelian. If it is hereditary (i.e. for all $\mathcal{T}$-modules L and M, all extension groups $\mathrm{Ext}^i_{\mathrm{mod}\,\mathcal{T}}(L, M)$ vanish for $i \geqslant 2$), the cluster category $\mathcal{C}_{\mathcal{T}}$, as defined in [BMRRT06], is the orbit category of the bounded derived category $\mathcal{D}^b(\mathrm{mod}\,\mathcal{T})$ under the action of the autoequivalence $S^2 \circ \Sigma^{-1}$, where S is the suspension and Σ the Serre functor of $\mathcal{D}^b\,(\mathrm{mod}\,\mathcal{T})$.

Theorem. *If $\mathrm{mod}\,\mathcal{T}$ is hereditary, then $\mathcal{C}$ is triangle-equivalent to the cluster category $\mathcal{C}_{\mathcal{T}}$.*

We will prove the theorem in § 3 below. Now assume that k is algebraically closed. Let $\mathcal{R}$ be the radical of $\mathcal{T}$, i.e. the ideal such that, for two indecomposables X, Y, the space $\mathcal{R}(X, Y)$ is formed by the non-isomorphisms from X to Y. Let Q be the quiver of $\mathcal{T}$. Its vertices are the isomorphism classes of indecomposables of $\mathcal{T}$ and the number of arrows from the class of an indecomposable X to an indecomposable Y is the dimension of the vector space $\mathcal{R}(X, Y)/\mathcal{R}^2(X, Y)$.

Corollary. *If k is algebraically closed and, for each vertex x of Q, only finitely many paths start in x and only finitely many paths end in x, then $\mathcal{C}$ is triangle-equivalent to the cluster category $\mathcal{C}_Q$.*

Note that under the assumptions of the corollary, the projective (right) kQ-module $kQ(?, x)$ and the injective kQ-module $DkQ(x, ?)$ are of finite total dimension and that the category $\mathrm{mod}\,kQ$ of finitely presented kQ-modules coincides with the category of modules of finite total dimension. We will prove the corollary in § 3 below.

2.2 Application: Cohen–Macaulay modules

Suppose that k is algebraically closed of characteristic 0. Let the cyclic group $G = \mathbb{Z}/3\mathbb{Z}$ act on the power series ring $S = k[[X, Y, Z]]$ such that a generator of G multiplies each indeterminate by the same primitive third root of unity. Then the fixed point ring $R = S^G$ is a Gorenstein ring, cf. e.g. [Wat74], and an isolated singularity of dimension 3 (cf. e.g. [IY06, Corollary 8.2]). The category $\mathsf{CM}(R)$ of maximal Cohen–Macaulay modules is an exact Frobenius category. By Auslander's results [Aus78], cf. [Yos90, Lemma 3.10], its stable category $\mathcal{C} = \underline{\mathsf{CM}}(R)$ is 2-Calabi–Yau. By work of Iyama [Iya05b], the module $T = S$ is a cluster-tilting object in $\mathcal{C}$. The endomorphism ring of T over R is the skew group ring $S * G$. Under the action of G, the module T decomposes into three indecomposable direct factors $T = T_1 \oplus T_2 \oplus S^G$ and we see that its endomorphism ring

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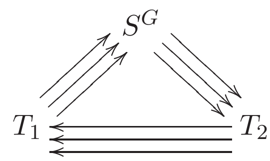
$S * G$ is isomorphic to the completed path algebra of the quiver



subject to all the 'commutativity relations' obtained by labelling the three arrows between any consecutive vertices by X, Y and Z. The stable endomorphism ring of T is thus isomorphic to the path algebra of the following generalized Kronecker quiver.

$$Q:\ 1 \equiv\!\!> 2$$

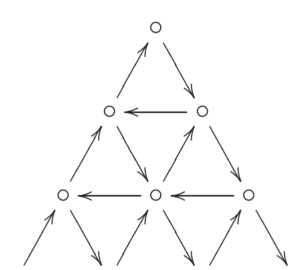
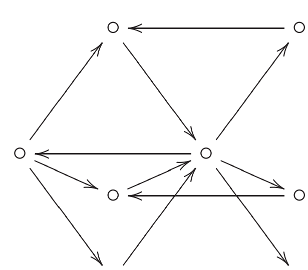
The theorem now shows that the stable category of Cohen–Macaulay modules $\underline{\mathsf{CM}}(R)$ is triangle-equivalent to the cluster category $\mathcal{C}_Q$.

As a further application, we give an alternative proof of a theorem from [IY06], stating that the indecomposable non-projective rigid modules in $\mathsf{CM}(R)$ are exactly the modules of the form $\Omega^i(T_1)$ and $\Omega^i(T_2)$ for $i \in \mathbb{Z}$. For this, note that the indecomposable rigid objects in $\mathcal{C}_Q$ are exactly the images of the indecomposable rigid kQ-modules and the SP, for P an indecomposable projective kQ-module [BMRRT06]. So they correspond to the vertices of the component of the AR-quiver of $\mathcal{C}_Q$ containing the indecomposable projective kQ-modules. The corresponding component of the AR-quiver of $\underline{\mathsf{CM}}(R)$ is the one containing T_1 and T_2. Hence the indecomposable rigid objects in $\underline{\mathsf{CM}}(R)$ are all τ-shifts of these. Finally, we use that $\tau = \Omega^{-1}$ in this case [Aus78].

Note that this application does not need the full force of the main theorem. For we only use that the AR-quivers of $\mathcal{C}_Q$ and $\underline{\mathsf{CM}}(R)$ are isomorphic, with the component of the projective kQ-modules for $\mathcal{C}_Q$ corresponding to the component of T_1 and T_2 for $\underline{\mathsf{CM}}(R)$, and it is easy to see that this follows from [KR07, Proposition 2.1(c) and Lemma 3.5], cf. also [BMR07].

2.3 Application: non-acyclicity

Let k be an algebraically closed field and Λ the preprojective algebra of a simply laced Dynkin diagram Δ. Then Λ is a finite-dimensional self-injective algebra and the stable category $\mathcal{C}$ of finite-dimensional Λ-modules is 2-Calabi–Yau, cf. [Cra00], and admits a canonical cluster-tilting subcategory $\mathcal{T}'$ with finitely many indecomposables, cf. [GLS07a]. Let Q' be its quiver. For example, by [GLS07a], the quivers Q' corresponding to $\Delta = A_5$ and $\Delta = D_4$ are respectively as shown below.

Part (b) of the following proposition was obtained independently by Geiss (private communication).

Proposition. *Suppose that Λ is representation-infinite.*

(a) *The stable category $\mathcal{C} = \underline{\mathrm{mod}}\,\Lambda$ is not equivalent to the cluster category $\mathcal{C}_Q$ of a finite quiver Q without oriented cycles.*

(b) *The quiver Q' of the canonical cluster-tilting subcategory of [GLS07a] is not mutation-equivalent to a quiver Q without oriented cycles.*

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In particular, it follows that the two above quivers are not mutation-equivalent to quivers without oriented cycles. In the proof of the proposition, we use the main theorem. Let us stress that, as at the end of § 2.2, we do not need its full force but only use the isomorphism of AR-quivers. This is the variant of the proof also given by Geiss (private communication).

Proof. (a) Recall first that the AR-translation τ is isomorphic to the suspension in any 2-Calabi–Yau category, so that it is preserved under triangle equivalences. We know from [AR94] that the AR-translation τ of $\mathcal{C}$ is periodic of period dividing 6. In particular, we have $\tau^6(X) \xrightarrow{\sim} X$ for each indecomposable X of $\mathcal{C}$. But in $\mathcal{C}_Q$, for each indecomposable X that is the image of a preprojective kQ-module, the iterated translates $\tau^{-n}(X)$, $n \geqslant 0$, are all pairwise non-isomorphic since Q is representation-infinite.

(b) Suppose that Q' is mutation-equivalent to a quiver Q. By one of the main results of [GLS06], it follows that $\mathcal{C}$ contains a cluster-tilting subcategory $\mathcal{T}$ whose quiver is Q. If Q does not have oriented cycles, it follows from the main theorem that $\mathcal{C}$ is triangle-equivalent to $\mathcal{C}_Q$ in contradiction to part (a). □

3. Proofs

3.1 Proof of the corollary

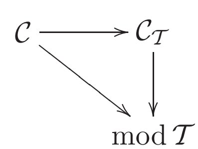
First recall from [KR07] that the category $\operatorname{mod}\mathcal{T}$ of finitely presented $\mathcal{T}$-modules is abelian and Gorenstein of dimension at most 1. It follows from our hypothesis that each object of $\operatorname{mod}\mathcal{T}$ has a finite composition series all of whose subquotients are simple modules

$$S_M = \mathcal{T}(?, M)/\mathcal{R}(?, M)$$

associated with indecomposables M of $\mathcal{T}$ and that each of these simple modules is of finite projective dimension. Thus each object of $\operatorname{mod}\mathcal{T}$ is of finite projective dimension so that $\operatorname{mod}\mathcal{T}$ has to be hereditary. Since k is algebraically closed, it follows that $\operatorname{mod}\mathcal{T}$ is equivalent to $\operatorname{mod} kQ$ and the claim of the corollary follows from the theorem.

3.2 Plan of the proof of the theorem

Our aim is to construct a triangle equivalence $\mathcal{C} \to \mathcal{C}_\mathcal{T}$ such that the triangle



becomes commutative, where the diagonal functor takes X to $\mathcal{C}(?, X)|\mathcal{T}$. To construct the triangle equivalence $\mathcal{C} \to \mathcal{C}_\mathcal{T}$, we use the construction of $\mathcal{C}_\mathcal{T}$ given in [Kel05], namely, the category $\mathcal{C}_\mathcal{T}$ is the stable derived category $\mathsf{stab}(\mathcal{T} \oplus D\mathcal{T}[-3])$ of the differential graded (dg) category whose objects are the objects of $\mathcal{T}$ and whose morphism complexes are given by the graded modules

$$\mathcal{T}(x, y) \oplus (D\mathcal{T}(y, x))[-3]$$

endowed with the vanishing differential (the construction of the stable derived category is recalled in § 3.3 below). Thus, we have to construct an equivalence

$$\mathcal{C} \to \mathsf{stab}(\mathcal{T} \oplus (D\mathcal{T})[-3]).$$

We proceed in three steps:

(1) We construct a dg category $\mathcal{A}$ and a triangle functor

$$\mathcal{C} \to \mathsf{stab}(\mathcal{A}).$$

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We show moreover that the subcategory of indecomposables of the homology $H^*\mathcal{A}$ is isomorphic to $\mathcal{T} \oplus (D\mathcal{T})[-3]$.

(2) Using the fact that k is perfect we show that the dg category $\mathcal{A}$ is formal, i.e. linked to its homology by a chain of quasi-isomorphisms. This yields the required triangle functor

$$\mathcal{C} \to \mathsf{stab}(\mathcal{A}) \xrightarrow{\sim} \mathsf{stab}(H^*\mathcal{A}) \xrightarrow{\sim} \mathsf{stab}(\mathcal{T} \oplus (D\mathcal{T})[-3]) = \mathcal{C}_{\mathcal{T}}.$$

(3) In a final step, we show that the composed functor $\mathcal{C} \to \mathcal{C}_{\mathcal{T}}$ is fully faithful and that its image generates $\mathcal{C}_{\mathcal{T}}$.

3.3 The proof

Let $\mathcal{M} \subset \mathcal{E}$ be the preimage of $\mathcal{T}$ under the projection functor. In particular, $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of the projective-injective objects in $\mathcal{M}$. Note that $\mathcal{T}$ equals the quotient $\underline{\mathcal{M}}$ of $\mathcal{M}$ by the ideal of morphisms factoring through a projective-injective. For each object M of $\mathcal{T}$, choose an $\mathcal{E}$-acyclic complex A_M of the form

$$0 \to M_1 \to M_0 \to P \to M \to 0,$$

where P is $\mathcal{E}$-projective and M_0 and M_1 are in $\mathcal{M}$, cf. [KR07]. Note that, if ΩM denotes the kernel of $P \to M$, the induced morphism $M_0 \to \Omega M$ is automatically a right $\mathcal{M}$-approximation of ΩM. Let $\mathcal{A}$ be the dg subcategory of the dg category $\mathcal{C}(\mathcal{E})_{\mathrm{dg}}$ of complexes over $\mathcal{E}$ whose objects are these acyclic complexes. Thus, for two objects A_L and A_M of $\mathcal{A}$, we have

$$H^n\mathcal{A}(A_L, A_M) = \mathrm{Hom}_{\mathcal{HE}}(A_L, A_M[n]),$$

where $\mathcal{HE}$ denotes the homotopy category of complexes over $\mathcal{E}$. To compute this space, let G_1 be the functor $\mathcal{E} \to \mathrm{Mod}\,\mathcal{M}$ taking an object X to $\mathcal{E}(?, X)|\mathcal{M}$. The image of A_M under G_1 is a projective resolution of the $\mathcal{M}$-module $\underline{M} = \underline{\mathcal{E}}(?, M)$. Thus we have

$$\mathrm{Hom}_{\mathcal{HE}}(A_L, A_M[n]) \xrightarrow{\sim} (\mathcal{D}\mathcal{M})(\underline{L}, \underline{M}[n]) = \mathrm{Ext}^n_{\mathcal{M}}(\underline{L}, \underline{M}),$$

where $\mathcal{D}\mathcal{M}$ denotes the (unbounded) derived category of $\mathrm{Mod}\,\mathcal{M}$. Notice that by the Yoneda lemma, for each object N of $\mathcal{M}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(G_1 N, \underline{M}) \xrightarrow{\sim} \underline{M}(N) = \underline{\mathcal{E}}(N, M) = \mathcal{C}(N, M).$$

Using this we see that the vector space $\mathrm{Ext}^n_{\mathcal{M}}(\underline{L}, \underline{M})$ is the homology in degree n of the complex

$$0 \to \mathcal{C}(L, M) \to 0 \to \mathcal{C}(L_0, M) \to \mathcal{C}(L_1, M) \to 0.$$

Clearly it is isomorphic to $\mathcal{C}(L, M)$ for $n = 0$. Using the triangle

$$S^{-2}L \to L_1 \to L_0 \to S^{-1}L$$

and the fact that $\mathcal{C}(S^{-1}L_0, M) = 0$ and $\mathcal{C}(S^{-1}L, M) = 0$, we see that the homology is isomorphic to $\mathcal{C}(S^{-2}L, M) = D\mathcal{C}(M, L)$ for $n = 3$ and vanishes for all other $n \neq 0$. More precisely, we see that the map $M \mapsto A_M$ extends to an equivalence whose target is the (additive) graded category $H^*\mathcal{A}$ and whose source is the graded category $\mathcal{T} \oplus (D\mathcal{T})[-3]$ whose objects are those of $\mathcal{T}$ and whose morphisms are given by

$$\mathcal{T}(L, M) \oplus (D\mathcal{T}(M, L))[-3].$$

In particular, we have a faithful functor $\mathcal{T} \to H^*\mathcal{A}$ that yields an equivalence from $\mathcal{T}$ to $H^0\mathcal{A}$. We denote by $\mathcal{D}^b(\mathcal{A})$ the full subcategory of the derived category $\mathcal{D}\mathcal{A}$ whose objects are the dg modules X such that the restriction of the sum of the H^nX, $n \in \mathbb{Z}$, to $\mathcal{T}$ lies in the category $\mathrm{mod}\,\mathcal{T}$ of finitely presented $\mathcal{T}$-modules (by [KR07, Proposition 2.1(a)] this category is abelian). In particular, each representable $\mathcal{A}$-module lies in $\mathcal{D}^b(\mathcal{A})$ (by [KR07, Proposition 2.1(b)]) and thus the perfect derived category $\mathsf{per}(\mathcal{A})$ is contained in $\mathcal{D}^b(\mathcal{A})$. We denote by $\mathsf{stab}(\mathcal{A})$ the triangle quotient $\mathcal{D}^b(\mathcal{A})/\mathsf{per}(\mathcal{A})$. Recall from [KV87] and [Ric89] that we have a triangle equivalence

$$\underline{\mathcal{E}} \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P}).$$

Let $G : \mathcal{H}^b(\mathcal{E}) \to \mathcal{D}\mathcal{A}$ be the functor that takes a bounded complex X over $\mathcal{E}$ to the functor

$$A_M \mapsto \mathsf{Hom}^\bullet{}_{\mathcal{E}}(A_M, X),$$

where $\mathsf{Hom}^\bullet{}_{\mathcal{E}}$ is the complex whose nth component is formed by the morphisms of graded objects, homogeneous of degree n, and the differential is the supercommutator with the differentials of A_M and X. We will show that G takes $\mathcal{D}^b(\mathcal{P})$ to zero, and that it maps $\mathcal{H}^b(\mathcal{E})$ to $\mathcal{D}^b(\mathcal{A})$ and the subcategory of acyclic complexes to $\mathsf{per}(\mathcal{A})$. Thus it will induce a triangle functor

$$\mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P}) \to \mathcal{D}^b(\mathcal{A})/\mathsf{per}(\mathcal{A})$$

and we will obtain the required functor as the composition

$$\underline{\mathcal{E}} \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P}) \to \mathcal{D}^b(\mathcal{A})/\mathsf{per}(\mathcal{A}) = \mathsf{stab}(\mathcal{A}).$$

First recall that, if A is an acyclic complex and I a left bounded complex of injectives, then each morphism from A to I is nullhomotopic. In particular, the complex $\mathsf{Hom}^\bullet{}_{\mathcal{E}}(A_M, P)$ is nullhomotopic for each P in $\mathcal{H}^b(\mathcal{P})$. Thus G takes $\mathcal{H}^b(\mathcal{P})$ to zero. Now, we would like to show that G takes values in $\mathcal{D}^b(\mathcal{A})$ and that the image of each bounded acyclic complex is in $\mathsf{per}(\mathcal{A})$. For this, we need to compute

$$(GX)(A_L) = \mathsf{Hom}^\bullet{}_{\mathcal{E}}(A_L, X)$$

for L in $\mathcal{M}$ and X in $\mathcal{H}^b(\mathcal{E})$. To show that the restriction of the sum of the homologies of GX lies in $\operatorname{mod}\mathcal{T}$, it suffices to show that this holds if X is concentrated in one degree. Moreover, if we have a conflation

$$0 \to M_1 \to M_0 \to X \to 0$$

of $\mathcal{E}$ with M_i in $\mathcal{M}$, it induces a short exact sequence of complexes

$$0 \to GM_1 \to GM_0 \to GX \to 0.$$

So we may suppose that X is an object of $\mathcal{M}$ considered as a complex concentrated in degree 0. Then one computes that the space

$$\operatorname{Hom}_{\mathcal{H}\mathcal{E}}(A_L, X[n])$$

is isomorphic to the homology in degree n of the complex

$$0 \to \mathcal{C}(L, X) \to 0 \to \mathcal{C}(L_0, X) \to \mathcal{C}(L_1, X) \to 0,$$

where $\mathcal{C}(L, X)$ is in degree 0. For $n = 0$, we find that the homology is $\mathcal{C}(L, X)$. Using the triangle

$$S^{-2}L \to L_1 \to L_0 \to S^{-1}L$$

and the vanishing of $\mathcal{C}(S^{-1}L, X)$ and $\mathcal{C}(S^{-1}L_0, X)$, we see that the homology in degree n is $\mathcal{C}(L, S^2X)$ for $n = 3$ and vanishes for all other $n \neq 0$. This shows that the restriction of the sum of the homologies of GX to $\mathcal{T}$ lies in $\operatorname{mod}\mathcal{T}$ since the restriction of $\underline{\mathcal{E}}(?, Y)$ to $\mathcal{T}$ lies in $\operatorname{mod}\mathcal{T}$ for each Y in $\underline{\mathcal{E}}$.

Now we have to show that G takes acyclic bounded complexes to perfect dg $\mathcal{A}$-modules. For this, we first observe that we have a factorization of G as the composition

$$\mathcal{H}^b(\mathcal{E}) \xrightarrow{G_1} \mathcal{D}\mathcal{M} \xrightarrow{G_2} \mathcal{D}\mathcal{A},$$

where G_1 sends X to $\mathsf{Hom}^\bullet(?, X)|\mathcal{M}$ and G_2 sends Y to the dg module

$$A_L \mapsto \mathsf{Hom}^\bullet(G_1 A_L, Y).$$

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Clearly the functor G_1 sends $\mathcal{E}$-acyclic bounded complexes to bounded complexes whose homology modules are in $\operatorname{mod}\underline{\mathcal{M}}$. Since $\operatorname{mod}\underline{\mathcal{M}}$ lies in $\mathsf{per}\,\mathcal{M}$, it follows that G_1 sends bounded acyclic complexes to objects of $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Under the functor G_2, the module $\underline{\mathcal{M}}(?,L)$ is sent to A_L and G_2 restricted to the triangulated subcategory generated by the $\underline{\mathcal{M}}(?,L)$ is fully faithful. We claim that this subcategory equals $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Indeed, each object in $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ is an iterated extension of its homology objects placed in their respective degrees. So it suffices to show that each object concentrated in degree 0 is the cone over a morphism between objects $\underline{\mathcal{M}}(?,L)$, $L \in \mathcal{M}$. But this is clear since $\operatorname{mod}\underline{\mathcal{M}}$ is equivalent to $\operatorname{mod}\mathcal{T}$, which is hereditary. It follows that G_2 induces an equivalence from $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ to $\mathsf{per}\,\mathcal{A}$ and thus $G = G_2G_1$ sends bounded acyclic complexes to $\mathsf{per}\,\mathcal{A}$. Thus, we obtain the required triangle functor $F : \mathcal{C} \to \mathsf{stab}(\mathcal{A})$.

In § 3.4 below, we will show that $\mathcal{A}$ is formal. Thus we get an isomorphism

$$\mathcal{T} \oplus (D\mathcal{T})[-3] \xrightarrow{\sim} \mathcal{A}$$

in the homotopy category of small dg categories. This yields an equivalence

$$\mathcal{C}_{\mathcal{T}} = \mathsf{stab}(\mathcal{T} \oplus (D\mathcal{T})[-3]) \xrightarrow{\sim} \mathsf{stab}(\mathcal{A}).$$

By construction, it takes each object T of $\mathcal{T}$ to the module $T^{\wedge} = \mathcal{T}(?,T)$ in $\mathcal{C}_{\mathcal{T}}$. Since $\mathcal{T}$ generates $\mathcal{C}$ and the $T^{\wedge}$, $T \in \mathcal{T}$, generate $\mathcal{C}_{\mathcal{T}}$, it is enough to show that F is fully faithful. We thank Michel Van den Bergh for simplifying our original argument: For each object X of $\mathcal{C}$, we have a triangle

$$T_1 \to T_0 \to X \to ST_1$$

with T_0, T_1 in $\mathcal{T}$. Thus, to conclude that F induces a bijection

$$\mathcal{C}(T,X) \to \mathcal{C}_{\mathcal{T}}(FT,FX)$$

for each $T \in \mathcal{T}$, it suffices to show that F induces bijections

$$\mathcal{C}(T,T'[i]) \xrightarrow{\sim} \mathcal{C}_{\mathcal{T}}(FT,FT'[i])$$

for T, T' in $\mathcal{T}$ and $0 \leqslant i \leqslant 1$. This is clear for $i = 1$ and not hard to see for $i = 0$. We conclude that, for each Y of $\mathcal{C}$, F induces bijections

$$\mathcal{C}(T',Y[i]) \xrightarrow{\sim} \mathcal{C}_{\mathcal{T}}(FT',FY[i])$$

for all T' in $\mathcal{T}$ and all Y in $\mathcal{C}$ and $i \in \mathbb{Z}$. By the above triangle, it follows that F induces bijections

$$\mathcal{C}(X,Y) \to \mathcal{C}_{\mathcal{T}}(FX,FY)$$

for all X, Y in $\mathcal{C}$.

3.4 Formality

For categories $\mathcal{T}$ given by 'small enough' quivers Q, one can use the argument of Seidel and Thomas [ST01, Lemma 4.21] to show that the category $\mathcal{T} \oplus (D\mathcal{T})[-3]$ is intrinsically formal and thus $\mathcal{A}$ is formal. We thank Van den Bergh for pointing out that, for general categories $\mathcal{T}$ with hereditary module categories, the Seidel and Thomas argument cannot be adapted. Instead, we show directly that $\mathcal{A}$ is formal (we do not know if $\mathcal{T} \oplus (D\mathcal{T})[-3]$ is intrinsically formal). Of course, it suffices to show that the full subcategory $\mathcal{A}'$ whose objects are the A_M with indecomposable M is formal.

Since k is perfect, the category of bimodules over a semi-simple k-category is still semi-simple. From this, one deduces that the category $\mathcal{T}$ is equivalent to the tensor category of a bimodule over the semi-simplification of $\mathcal{T}$ (cf. [Ben98, Proposition 4.2.5]). Using this we can construct a lift of the functor $\mathsf{ind}\,\mathcal{T} \to H^0(\mathcal{A}')$ to a functor $\mathsf{ind}\,\mathcal{T} \to Z^0(\mathcal{A}')$, where $\mathsf{ind}\,\mathcal{T}$ denotes the full subcategory of $\mathcal{T}$ formed by a set of representatives of the isomorphism classes of the indecomposables. We define a $\mathcal{T}$-bimodule by

$$X(L,M) = \mathsf{Hom}^{\bullet}{}_{\mathcal{E}}(A_L, M), \quad L, M \in \mathcal{M},$$

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where we consider M as a subcomplex of A_M. Note that X is a right ideal in the category $\mathcal{A}'$, that it is a kQ-subbimodule of $(L,M) \mapsto \mathcal{A}'(A_L, A_M)$ and that we have $fg = 0$ for all homogeneous elements f, g of X of degree greater than 0. The computation made above in the proof that G takes $\mathcal{H}^b(\mathcal{E})$ to $\mathcal{D}^b(\mathcal{A})$ shows that X has homology only in degree 3 and that we have a bimodule isomorphism

$$D\mathcal{T}(M,L) \xrightarrow{\sim} H^3X(L,M), \quad L, M \in \mathcal{M}.$$

Thus we have an isomorphism

$$D\mathcal{T}[-3] \xrightarrow{\sim} X$$

in the derived category of $\mathcal{T}$-bimodules. We choose a projective bimodule resolution P of $D\mathcal{T}[-3]$ whose non-zero components are concentrated in degrees 1, 2 and 3 (note that this is possible since the bimodule category is of global dimension 2). We obtain a morphism of complexes of bimodules

$$P \to X$$

inducing an isomorphism in homology. We compose it with the inclusion $X \to \mathcal{A}'$. All products of elements in the image of P vanish since they all lie in components of degree greater than 0 of X. Thus we obtain a morphism of dg categories

$$\mathcal{T} \oplus P \to \mathcal{A}'$$

inducing an isomorphism in homology. This clearly shows that $\mathcal{A}'$ is formal.

4. A generalization to higher dimensions

4.1 Negative extension groups

Let k be a field and H a finite-dimensional hereditary k-algebra. We write ν for the Serre functor of the bounded derived category $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} H)$ and S for its suspension functor. Let $d \geqslant 2$ be an integer. Let $\mathcal{C} = \mathcal{C}_H^{(d)}$ be the *d-cluster category*, i.e. the orbit category of $\mathcal{D}$ under the action of the automorphism $\nu^{-1}S^d$, and $\pi : \mathcal{D} \to \mathcal{C}$ the canonical projection functor. We know from [Kel05] that $\mathcal{C}$ is canonically triangulated and d-Calabi–Yau and that π is a triangle functor. Moreover, the image $\pi(H)$ of H in $\mathcal{C}$ is a d-cluster-tilting object, cf. e.g. [KR07]. The fact that the module H is projective and concentrated in degree 0 yields vanishing properties for the negative self-extension groups of $\pi(H)$ if $d \geqslant 3$.

Lemma. *We have*

$$\operatorname{Hom}(\pi(H), S^{-i}\pi(H)) = 0$$

for $1 \leqslant i \leqslant d-2$.

Proof. Put $T = H$. For $p \in \mathbb{Z}$, let $\mathcal{D}_{\leqslant p}$ and $\mathcal{D}_{\geqslant p}$ be the $(-p)$th suspensions of the canonical left, respectively right, aisles of $\mathcal{D}$ (cf. [KV88]). We have to show that the groups

$$\operatorname{Hom}(T, \nu^{-p}S^{pd-i}T)$$

vanish for all $p \in \mathbb{Z}$ and all $1 \leqslant i \leqslant d-2$. Suppose that $p = -q$ for some $q \geqslant 0$. Then we have

$$\operatorname{Hom}(T, \nu^{-p}S^{pd-i}T) = \operatorname{Hom}(T, \nu^{q}S^{-qd-i}T)$$

and the last group vanishes since T lies in $\mathcal{D}_{\leqslant 0}$ and $\nu^q S^{-qd-i}T$ lies in $\mathcal{D}_{\geqslant q(d-1)+i}$ and we have $q(d-1)+i > 0$. Now suppose that $p \geqslant 1$. Then we have

$$\operatorname{Hom}(T, \nu^{-p}S^{pd-i}T) = \operatorname{Hom}(\nu^p T, S^{pd-i}T) = \operatorname{Hom}(\nu^{p-1}(\nu T), S^{pd-i}T)$$

and this group vanishes since we have $\nu^{p-1}(\nu T) \in \mathcal{D}_{\geqslant p-1}$ (because $\nu T = \nu H$ is in $\operatorname{mod} H$) and

$S^{pd-i}T \in \mathcal{D}_{\leqslant -(pd-i)}$ and

$$(pd-i)-(p-1) = p(d-1)-i+1 \geqslant d-1-i+1 \geqslant d-i \geqslant 2. \qquad \square$$

4.2 A characterization of higher cluster categories

Let $d \geqslant 2$ be an integer, k an algebraically closed field and $\mathcal{C}$ a Hom-finite algebraic d-Calabi–Yau category containing a d-cluster-tilting object T.

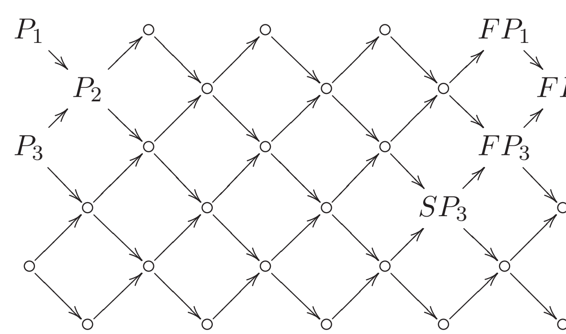
Theorem. *Suppose that* $\mathrm{Hom}(T, S^{-i}T) = 0$ *for* $1 \leqslant i \leqslant d-2$*. If* $H = \mathrm{End}(T)$ *is hereditary, then, with the notation of § 4.1, there is a triangle equivalence* $\mathcal{C} \xrightarrow{\sim} \mathcal{C}_H^{(d)}$ *taking* T *to* $\pi(H)$.

Notice that, by the lemma above, the assumption on the vanishing of the negative extension groups is necessary. These assumptions imply that the endomorphism algebra is Gorenstein of dimension at most $d-1$, as we show in the lemma in § 4.6 below. For $d \geqslant 3$, this does not, of course, imply that the endomorphism algebra is hereditary if its quiver does not have oriented cycles, but it implies that the global dimension is at most $d-1$.

We will prove the theorem below in § 4.4. In [IY06, Theorem 1.3], the reader will find an example from the study of rigid Cohen–Macaulay modules which shows that the vanishing of the negative extension groups does not follow from the other hypotheses. The following simple example, based on an idea of Van den Bergh, is similar in spirit.

4.3 Example

Let $\widetilde{H}$ be the path algebra of a quiver with underlying graph A_6 and alternating orientation. Put $\widetilde{\mathcal{D}} = \mathcal{D}^b(\mathrm{mod}\, \widetilde{H})$ and let $\mathcal{C}$ be the orbit category of $\widetilde{\mathcal{D}}$ under the automorphism $F = \tau^4$ (where $\tau = S^{-1}\nu$). Then $\mathcal{C}$ is 3-Calabi–Yau. Indeed, one checks that $F^2 = \tau^{-1}S^2$ in $\widetilde{\mathcal{D}}$, which clearly yields $\nu = S^3$ in $\mathcal{C}$. The following diagram shows a piece of the Auslander–Reiten quiver of $\widetilde{\mathcal{D}}$ which is a 'fundamental domain' for F. To obtain the Auslander–Reiten quiver of $\mathcal{C}$, we identify the left and right borders.



Using the mesh category of this quiver, it is not hard to check that the sum of the images of the indecomposable projectives P_1, P_2, P_3 in $\mathcal{C}$ is a 3-cluster-tilting object whose endomorphism ring H is the path algebra on the full subquiver with the corresponding three vertices. On the other hand, the image of P_3 in $\mathcal{C}$ has a one-dimensional space of (-1)-extensions. Note that $\mathcal{C} = \widetilde{\mathcal{D}}/F$ is nevertheless an orbit category and admits the 3-cluster category

$$\mathcal{C}_{\widetilde{H}}^{(3)} = \widetilde{\mathcal{D}}/F^2$$

as a '2-sheeted covering'.

4.4 Proof

The proof of the theorem follows the lines of the one in § 3.3. Let $\mathcal{T}$ be the full subcategory of $\mathcal{C}$ whose objects are the direct sums of direct factors of T. Let M be an object of $\mathcal{T}$. We construct an

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$\mathcal{E}$-acyclic complex A_M

$$0 \to M_{d+1} \to M_d \to \cdots \to M_1 \to M_0 \to 0,$$

which yields a resolution of the $\mathcal{M}$-module

$$\underline{\mathcal{E}}(?, M) : \mathcal{M}^{\rm op} \to \operatorname{Mod} k$$

as in [KR07, Theorem 5.4(b)], Thus we can take $M_0 = M$ and the morphism $M_1 \to M_0$ is a deflation with projective M_1. Each morphism

$$M_i \to Z_{i-1} = \mathsf{ker}(M_{i-1} \to M_{i-2}), \quad i \geqslant 2,$$

yields a $\mathcal{T}$-approximation in $\mathcal{C}$. Our vanishing assumption then implies that $M_2, \ldots, M_{d-1}$ are projective. As in § 3.3, we let $\mathcal{A}$ be the dg subcategory of the dg category $\mathcal{C}(\mathcal{E})_{\rm dg}$ of complexes over $\mathcal{E}$ whose objects are these acyclic complexes. Thus, for two objects A_L and A_M of $\mathcal{A}$, we have

$$H^n\mathcal{A}(A_L, A_M) = \operatorname{Hom}_{\mathcal{H}\mathcal{E}}(A_L, A_M[n]).$$

One computes that this vector space is isomorphic to $\mathcal{C}(L, M)$ for $n = 0$, to $\mathcal{C}(L, \Sigma M) = D\mathcal{C}(M, L)$ for $n = d+1$ and vanishes for all other n. Here we use again our vanishing hypothesis. We see that the map $M \mapsto A_M$ extends to an equivalence whose target is the (additive) graded category $H^*\mathcal{A}$ and whose source is the graded category $\mathcal{T} \oplus (D\mathcal{T})[-(d+1)]$ whose objects are those of $\mathcal{T}$ and whose morphisms are given by

$$\mathcal{T}(L, M) \oplus (D\mathcal{T}(M, L))[-(d+1)].$$

Now the proof proceeds as in § 3.3 and we obtain a triangle functor $F : \mathcal{C} \to \mathcal{C}_{\mathcal{T}}$ taking the subcategory $\mathcal{T}$ to $\operatorname{add} \pi(H)$ and whose restriction to $\mathcal{T}$ is an equivalence. By Lemma 4.5 below, F is an equivalence.

4.5 Equivalences between d-Calabi–Yau categories

Let $d \geqslant 2$ be an integer, k a field and $\mathcal{C}$ and $\mathcal{C}'$ k-linear triangulated categories which are d-Calabi–Yau. Let $\mathcal{T} \subset \mathcal{C}$ and $\mathcal{T}' \subset \mathcal{C}'$ be d-cluster-tilting subcategories. Suppose that $F : \mathcal{C} \to \mathcal{C}'$ is a triangle functor taking $\mathcal{T}$ to $\mathcal{T}'$.

Lemma. *The triangle functor F is an equivalence if and only if the restriction of F to $\mathcal{T}$ is an equivalence.*

Proof. It follows from [KR07, Proposition 5.5(a)], cf. also [IY06, Theorem 3.1, part (1)], that $\mathcal{C}$ equals its subcategory

$$\mathcal{T} * S\mathcal{T} * \cdots * S^{d-1}\mathcal{T}$$

and similarly for $\mathcal{C}'$. Suppose that the restriction of F to $\mathcal{T}$ is an equivalence. Let $T \in \mathcal{T}$. By induction, we see that, for each $1 \leqslant i \leqslant d-1$, the map

$$\mathcal{C}(T, Y) \to \mathcal{C}'(FT, FY)$$

is bijective for each $Y \in \mathcal{T} * S\mathcal{T} * \cdots * S^i\mathcal{T}$. Thus the map

$$\mathcal{C}(S^jT, Y) \to \mathcal{C}'(S^jFT, FY)$$

is bijective for all $j \in \mathbb{Z}$ and Y in $\mathcal{C}$. Then it follows that the map

$$\mathcal{C}(X, Y) \to \mathcal{C}'(FX, FY)$$

is bijective for all X, Y in $\mathcal{C}$. Thus F is fully faithful. Since $\mathcal{T}'$ generates $\mathcal{C}'$, the functor F is an equivalence. Conversely, if F is an equivalence and takes $\mathcal{T}$ to $\mathcal{T}'$, then the image of $\mathcal{T}$ has to be $\mathcal{T}'$ since $F\mathcal{T}$ is maximal $(d-1)$-orthogonal in $\mathcal{C}'$. □

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4.6 The Gorenstein property for certain d-Calabi–Yau categories

Let $d \geqslant 2$ be an integer, k a field and $\mathcal{C}$ a k-linear triangulated category which is d-Calabi–Yau. Let $\mathcal{T} \subset \mathcal{C}$ be a d-cluster-tilting subcategory such that we have

$$\mathcal{C}(T, S^{-i}T') = 0$$

for all $1 \leqslant i \leqslant d-2$ and all T, T' of $\mathcal{T}$.

Lemma. *The category* $\operatorname{mod} \mathcal{T}$ *is Gorenstein of dimension less than or equal to* $d-1$.

Proof. As in [KR07], one sees that the functor $\operatorname{Hom}(T, ?)$ induces an equivalence from the category $S^d\mathcal{T}$ to the category of injectives of $\operatorname{mod} \mathcal{T}$. So we have to show that the $\mathcal{T}$-module $\operatorname{Hom}(?, S^dT)$ is of projective dimension at most $d-1$ for each T in $\mathcal{T}$. Put $Y = S^dT$. We proceed as in [KR07, § 5.5]. Let $T_0 \to Y$ be a right $\mathcal{T}$-approximation of Y. We define an object Z_0 by the triangle

$$Z_0 \to T_0 \to Y \to SZ_0.$$

Now we choose a right $\mathcal{T}$-approximation $T_1 \to Z_0$ and define Z_1 by the triangle

$$Z_1 \to T_1 \to Z_0 \to SZ_1.$$

We continue inductively constructing triangles

$$Z_i \to T_i \to Z_{i-1} \to SZ_i$$

for $1 < i \leqslant d-2$. By [KR07, Proposition 5.5], the object Z_{d-2} belongs to $\mathcal{T}$. We obtain a complex

$$0 \to Z_{d-2} \to T_{d-2} \to \cdots \to T_1 \to T_0 \to Y \to 0.$$

We claim that its image under the functor $F : \mathcal{C} \to \operatorname{mod} \mathcal{T}$ taking an object X of $\mathcal{C}$ to $\mathcal{C}(?, X)|\mathcal{T}$ is a projective resolution of FY. Indeed, by induction one checks that the object Z_i belongs to

$$S^{d-i-1}\mathcal{T} * S^{-i}\mathcal{T} * S^{-i+1}\mathcal{T} * \cdots * S^{-1}\mathcal{T} * \mathcal{T}.$$

Thus, for each $T \in \mathcal{T}$, we have $\mathcal{C}(T, S^{-1}Z_i) = 0$ by our vanishing assumptions. Moreover, the maps $FT_{i+1} \to FZ_i$ are surjective, by construction. Therefore, the triangle

$$S^{-1}Z_{i-1} \to Z_i \to T_i \to Z_{i-1}$$

induces a short exact sequence

$$0 \to FZ_i \to FT_i \to FZ_i \to 0$$

for each $0 \leqslant i \leqslant d-2$, where $Z_{-1} = Y$. This implies the assertion. ☐

Appendix. An alternative proof of the main theorem

Michel Van den Bergh

In this appendix we give a proof of Theorem 2.1 that is based on the universal property of orbit categories [Kel05]. We use the same notation as in the main text, but for the purposes of exposition we will assume that $\mathcal{T}$ consists of a single object T such that $B = \mathcal{C}(T, T) = kQ$ where Q is a (necessarily finite) quiver. The extension to more general $\mathcal{T}$ is routine.

A.1 The dualizing module

For use below we recall a version of the Gorensteinness result from [KR07]. Assume that $\mathcal{C}$ is a two-dimensional Ext-finite Krull–Schmidt Calabi–Yau category with a cluster-tilting object T. Let $B = \mathcal{C}(T, T)$.

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For a finitely generated projective right B-module we define $P \otimes_B T$ in the obvious way. For any $M \in \mathcal{C}$, there is a distinguished triangle (e.g. [KR07])

$$P'' \otimes_B T \xrightarrow{\phi} Q'' \otimes_B T \to M \to \ .$$

Now we apply this with $M = T[2]$. Consider a distinguished triangle

$$P'' \otimes_B T \to Q'' \otimes_B T \to T[2] \to \ .$$

Applying the long exact sequence for $\mathrm{Hom}_{\mathcal{C}}(T, -)$ we obtain a corresponding projective resolution as right module of the dualizing module of B:

$$0 \to P'' \to Q'' \to DB \to 0. \tag{A1}$$

If we choose any other right module resolution of DB,

$$0 \to P' \to Q' \xrightarrow{\alpha} DB \to 0, \tag{A2}$$

then it is equal to (A1) up to contractible summands. Hence we obtain a distinguished triangle

$$P' \otimes_B T \to Q' \otimes_B T \xrightarrow{\alpha'} T[2] \to \ . \tag{A3}$$

Changing, if necessary, α' by a unit in $B = \mathrm{End}(T[2])$ we may and we will assume that $\mathrm{Hom}(T, \alpha') = \alpha$ (under the canonical identifications $\mathrm{Hom}_{\mathcal{C}}(T, Q' \otimes_{BT}) = Q'$ and $\mathrm{Hom}_{\mathcal{C}}(T, T[2]) = DB$).

A.2 The proof

We now let $\mathcal{C}$ be as in the main text. By [Kel94, Theorem 4.3] we may assume that $\mathcal{C}$ is a strict (= closed under isomorphism) triangulated subcategory of a derived category $\mathcal{D}(\mathcal{A})$ for some dg category $\mathcal{A}$. We denote by ${}_B\mathcal{C}$ the full subcategory of $\mathcal{D}(B \otimes \mathcal{A})$ whose objects are differential graded $B \otimes \mathcal{A}$-modules which are in $\mathcal{C}$ when considered as $\mathcal{A}$-modules. Clearly ${}_B\mathcal{C}$ is triangulated.

LEMMA A.2.1. *Assume that $B = kQ$. Then the following hold:*

(a) *T may be lifted to an object in ${}_B\mathcal{C}$, also denoted by T; and*

(b) *there is an isomorphism in ${}_B\mathcal{C}$: $DB \overset{L}{\otimes}_B T \cong T[2]$.*

Proof. We may assume that T is a homotopy projective $\mathcal{A}$-module containing a summand for each of the vertices of Q. Then we may lift the action of the arrows in Q to an action of kQ on T. Hence part (a) holds.

To prove part (b), we choose a resolution of B-bimodules,

$$0 \to P' \to Q' \xrightarrow{\alpha} DB \to 0,$$

where P' and Q' are projective on the right. Such a resolution may be obtained by suitably truncating a projective bimodule resolution of DB. Derived tensoring this resolution on the right by T and comparing with (A3) we find an isomorphism in $\mathcal{C}$,

$$c : DB \overset{L}{\otimes}_B T \cong T[2], \tag{A4}$$

between objects in ${}_B\mathcal{C}$. Note that c satisfies $c \circ \alpha = \alpha'$.

We claim that c in (A4) is compatible with the left B-actions in $\mathcal{C}$ on both sides. Let $b \in B$. Then we have the following commutative diagram of right B-modules.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & Q' & \longrightarrow & DB & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b\cdot} & & \downarrow{\scriptstyle b\cdot} & & \downarrow{\scriptstyle b\cdot} & & \\ 0 & \longrightarrow & P' & \longrightarrow & Q' & \longrightarrow & DB & \longrightarrow & 0 \end{array} \tag{A5}$$

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Tensoring on the right by T we obtain the following morphism of triangles in $\mathcal{C}$, where $b' = c \circ (b \otimes \mathrm{id}_T) \circ c^{-1}$.

$$\begin{array}{ccccccc} P' \otimes_B T & \longrightarrow & Q' \otimes_B T & \xrightarrow{\alpha'} & T[2] & \longrightarrow & \\ {\scriptstyle b\cdot}\downarrow & & {\scriptstyle b\cdot}\downarrow & & {\scriptstyle b\cdot}\downarrow & & \\ P' \otimes_B T & \longrightarrow & Q' \otimes_B T & \xrightarrow{\alpha'} & T[2] & \longrightarrow & \end{array}$$

We need to prove that $b' = b$ under the identification $B = \mathrm{End}_{\mathcal{C}}(T[2])$. This follows easily by applying the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ and comparing to (A5) (using the fact that $\mathrm{Hom}(T, \alpha') = \alpha$).

The proof of part (b) can now be completed by invoking the following lemma. □

Lemma A.2.2. *Assume that B has Hochschild dimension 1. Let $M, N \in {}_B\mathcal{C}$. Then the map*

$$\mathrm{Hom}_{{}_B\mathcal{C}}(M, N) \to \mathrm{Hom}_{\mathcal{C}}(M, N)^B$$

is surjective (where $(-)^B$ denotes the B-centralizer).

Proof. Replacing M by a homotopy projective and N by a homotopy injective $B \otimes \mathcal{A}$-module, one easily obtains the identity

$$\mathsf{RHom}_{B\otimes\mathcal{A}}(M, N) = \mathsf{RHom}_{B^e}(B, \mathsf{RHom}_{\mathcal{A}}(M, N)),$$

which yields a spectral sequence

$$E_2^{pq} : \mathrm{Ext}^p_{B^e}(B, \mathrm{Ext}^q_{\mathcal{C}}(M, N)) \Rightarrow \mathrm{Ext}^n_{{}_B\mathcal{C}}(M, N).$$

Using the fact that B has projective dimension 1 as bimodule, this yields a short exact sequence

$$0 \to \mathrm{Ext}^1_{B^e}(B, \mathrm{Ext}^{-1}_{\mathcal{C}}(M, N)) \to \mathrm{Hom}_{{}_B\mathcal{C}}(M, N) \to \mathrm{Hom}_{B^e}(B, \mathrm{Hom}_{\mathcal{C}}(M, N)) \to 0,$$

which gives in particular what we wanted to show. □

We can now finish the proof of the main theorem. By Lemma A.2.1(a) we have a functor

$$F = - \overset{L}{\otimes}_B T : D^b(B) \to \mathcal{C},$$

which by Lemma A.2.1(b) satisfies

$$F \circ \Sigma[-2] = F \circ (- \otimes_B DB[-2]) \cong F.$$

By the universal property of orbit categories [Kel05], we obtain an exact functor

$$\mathcal{D}^b(B)/\Sigma[-2] \to \mathcal{C},$$

which sends B to T. This functor is then an equivalence by the lemma in § 4.5, which finishes the proof.

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