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ADAMS SPECTRAL SEQUENCE AND HIGHER TORSION IN MSp_*

Boris I. Botvinnik and Stanley O. Kochman

Abstract

In this paper we study higher torsion in the symplectic cobordism ring. We use Toda brackets and manifolds with singularities to construct elements of higher torsion and use the Adams spectral sequence to determine an upper bound for the order of these elements.

1. Introduction

The symplectic cobordism ring MSp_* is the homotopy of the Thom spectrum MSp and classifies up to cobordism the ring of smooth manifolds with a symplectic structure on their stable normal bundles. Although MSp_* only has two-torsion, its ring structure is very complicated and is only completely understood through the 100 stem [**7**], [**13**], [**15**]. In [**2**], we proved that there are nontrivial elements in MSp_* of all orders 2^k. In this paper, we construct new elements of higher torsion by means of Toda brackets, and we study their properties using the Adams spectral sequence (ASS).

The following result provides the geometrical input we use to construct higher torsion elements. Its proof in Section 5 uses low dimensional calculations in the Atiyah-Hirzebruch spectral sequence for π_*MSp. Let $\phi_0 = \eta \in MSp_1$, and let $\phi_k \in MSp_{8k-3}$ for $k \geq 1$ denote the Ray elements [**12**]. The elements of MSp_* are built from the Ray elements using Toda brackets. The most elementary ones are $\langle \phi_m, 2, \phi_n \rangle$ for $0 \leq m < n$. Gorbounov [**1**, p. 139], [**4**] showed that these triple brackets contain zero when $m = 0$. On the other hand, it was shown in [**6**, Thm. 8.1 3(c)] that these triple brackets do not contain zero when $(m, n) \geq (3, 5)$ in the lexicographical order. The following theorem resolves the situation when $m = 1$ leaving open only the case $m = 2$.

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Theorem 1. *In MSp_*, the Toda brackets $\langle \phi_1, 2, \phi_n \rangle$ contain zero for all $n \geq 0$.*

Let $J = (j_1, \dots, j_s)$ with $0 < j_1 < j_2 < \cdots < j_s$. By induction on $s \geq 1$, we define elements $a[J] \in MSp_*$. The following theorem describes our elements of higher torsion $a[J]$. Although we show how the $a[J]$ decompose in terms of Toda brackets, the $a[J]$ will be defined by specific representative symplectic manifolds.

Theorem 2. *There exist elements $a[J] \in MSp_*$ with the following properties:*

(a) $a[j_1] = a[j_1, j_2] = a[j_1, j_2, j_3] = 0$;
(b) $a[j_1, j_2, j_3, j_4] \in \phi_{j_1}\phi_{j_2}\langle \phi_{j_3}, 2, \phi_{j_4}\rangle + \phi_{j_3}\phi_{j_4}\langle \phi_{j_1}, 2, \phi_{j_2}\rangle$;
(c) $a[2j_1, \dots, 2j_s]$ *is indecomposable for* $s \geq 5$;
(d) $\phi_1 a[J] = 0$ *and* $a[j_1, \dots, j_s] \in \langle \phi_{j_s}, 2, \phi_1, a[j_1, \dots, j_{s-1}]\rangle$ *for* $s \geq 5$;
(e) *for* $s \geq 7$ *and* $1 \leq i_1 < \cdots < i_s$ *the element* $a\left[2^{i_1}, \dots, 2^{i_s}\right] \in MSp_{4*+1}$ *has order at least* $2^{h(s)}$ *where* $h(s) = [(s+1)/2] - 2$.

Our main tool for proving Theorem 2 in Section 6 is the ASS which we apply to the spectrum MSp and the spectra $MSp_*^{\widehat{\Sigma}_n}$. The latter spectra classify bordism classes of symplectic manifolds with singularities $\widehat{\Sigma}_n = (P_2, \dots, P_n)$ where $[P_i] = \phi_{2^{i-2}}$ for $2 \leq i \leq n$. The spectrum $MSp_*^{\widehat{\Sigma}_3}$ is especially useful to us. Let

$$MSp_*^{\widehat{\Sigma}_3} \xrightarrow{\widehat{\beta}_3} MSp_*^{\widehat{\Sigma}_2} \xrightarrow{\widehat{\beta}_2} MSp_*$$

be the Bockstein operators. Using the ASS we first construct higher torsion elements $t[J]$ in the ring $MSp_*^{\widehat{\Sigma}_3}$ using Toda brackets. Then we define the elements $a[J] \in MSp_*$ as $\widehat{\beta_2}\left(\widehat{\beta_3}\,(t[J])\right)$. We prove Theorem 2 by identifying the projections of elements $t[J]$ and $a[J]$ in the ASS. In particular, we show that the elements $2^k a[J]$ for $0 \leq k \leq s-4$ determine towers of infinite cycles in $\mathbb{E}_2^{4*+1,2k+4}$ of the ASS for MSp_*. These towers are very interesting: their heights give *upper bounds* for the orders of our elements. However, we show that their top halves bound by higher differentials, so only their bottom halves survive. This explains why our elements of higher torsion only have half of their potential order.

To analyze the *lower bounds* of the orders of the $a[J]$ we use the results of [**2**] which were proved using the Adams-Novikov spectral sequence. Let $MSp_*^{\Sigma_n}$ denote the bordism theory with singularities $\Sigma_n = (P_1, \dots, P_n)$ where $[P_1] = \eta$. If $J = (2^{i_1}, \dots, 2^{i_s})$, let $\mathfrak{i} = (i_1, \dots, i_s)$. In [**2**] we constructed the higher order elements $\tau_3(\mathfrak{i}) \in MSp_*^{\Sigma_3}$ of order at least $2^{[(s+1)/2]}$ which defined elements $\alpha(\mathfrak{i}) \in MSp_*$ of order at least $2^{[(s+1)/2]-3}$. We show that the elements $2a[2^{i_1}, \dots, 2^{i_s}]$ may be identified with the $\alpha(\mathfrak{i})$.

Our analysis in the ASS and the ANSS is far from low-dimensional. For example, the first element of order eight in MSp_* given by Theorem 2 has degree 16,377. However, if the following conjecture were true then the first of these elements of order eight would be in degree 729.

Conjecture. *The elements $a[J] \in MSp_*$ of Theorem 2 are indecomposable of order $2^{[(s+1)/2]-2}$ for all sequences J of distinct positive even integers of length at least 7.*

All groups, rings and spectra are two-local throughout this paper. By [**14**], [**16**], the theories $MSp_*^{\Sigma_n}(\cdot)$ and $MSp_*^{\widehat{\Sigma}_n}(\cdot)$ have admissible commutative and associative product structures. In particular, the associativity, commutativity and Toda bracket constructions as well as all the results of [**2**, Section 3] are valid for all of these theories.

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2. May Spectral Sequence for MSp^{Σ_n}

Let MSp^{Σ_n}, $n \geq 1$, be the spectrum defined in the Introduction with singularities $\Sigma_n = (P_1, \dots, P_n)$, and let MSp^{Σ_0} denote MSp. In this section we compute the $\mathbb{E}_2$-term of the Adams spectral sequence (ASS):

$$\mathbb{E}_2^{s,t} = \mathrm{Cotor}^s_{\mathcal{A}} \left(H_* MSp^{\Sigma_n}, \mathbb{Z}/2\right)_t \Longrightarrow MSp_*^{\Sigma_n}. \tag{1}$$

Our approach is analogous to that used in [**5**] in the case $n = 0$. In particular, we use a change of rings theorem to reduce the problem of calculating $\mathbb{E}_2$ to computing

$$\mathrm{Cotor}_{B(n)} (\mathbb{Z}/2, \mathbb{Z}/2) . \tag{2}$$

Here $B(n)$ is a truncated polynomial algebra which we define as a quotient of the dual of the Steenrod algebra below. Then we use the May spectral sequence to compute the algebra (2). We compute $\mathcal{E}_2$ of these May spectral sequences using the resolution constructed by May in [**11**]. Then we construct filtered polynomial DGA algebras $\mathfrak{P}_n$ as quotients of the cobar construction which induce these May spectral sequences. We prove this from the case $n = 0$ of [**5**] by using induction on n and a generalized Five Lemma. Then for $n \geq 1$ we define representative cycles of the algebra generators of $\mathcal{E}_2$ to show that these May spectral sequences collapse and that all the algebra extensions from $\mathcal{E}_\infty$ to (2) are trivial. Thus, when $n \geq 1$ the situation is much simpler than the case $n = 0$ where there are nonzero d_2-differentials and nontrivial extensions. Consequently, for $n \geq 1$ we can describe $\mathbb{E}_2$ of the ASS (1) in terms of five families of algebra generators and four families of relations while for $n = 0$ nine families of algebra generators and forty families of relations were required.

We begin by recalling the structure of the homology of MSp^{Σ_n} as a comodule over the dual of the Steenrod algebra $\mathcal{A}_* = \mathbb{Z}/2\,[\xi_1, \dots, \xi_k, \dots]$. Let S be the $\mathcal{A}_*$-primitive polynomial algebra:

$$S = \mathbb{Z}/2\,[V_2, V_4, V_5, \dots, V_m, \dots]$$

where $m = 2, 4, 5, \dots$, $m \neq 2^l - 1$, and $\deg V_m = 4m$. V. Vershinin [**14**], [**16**] proves that there is an isomorphism of $\mathcal{A}_*$-comodules:

$$H_* MSp^{\Sigma_n} \cong \mathbb{Z}/2\left[\xi_1^2, \dots, \xi_n^2, \xi_{n+1}^4, \dots, \xi_k^4, \dots\right] \otimes S \tag{3}$$

for $n \geq 0$. Define the $\mathbb{Z}/2$-Hopf algebra

$$B\,(n) = \mathcal{A}_* / \left(\xi_h^2,\, \xi_k^4 \mid 1 \leq h \leq n \text{ and } n < k\right)$$

with coproduct ψ induced from the coproduct of $\mathcal{A}_*$. Note that in [**5**] the Hopf algebra $B\,(0)$ is denoted as B. By (3), the problem of computing $\mathbb{E}_2$ of the ASS (1) is greatly simplified by Liulevicius's interpretation [**10**, Corollary I.5] of the Cartan-Eilenberg change of rings theorem [**3**, Proposition VI.4.1.3] which gives an isomorphism of $\mathbb{Z}/2$-algebras:

$$\mathbb{E}_2 = \mathrm{Cotor}_{\mathcal{A}_*}\left(H_* MSp^{\Sigma_n}, \mathbb{Z}/2\right) \cong \mathrm{Cotor}_{B(n)}\,(\mathbb{Z}/2, \mathbb{Z}/2) \otimes S. \tag{4}$$

To compute the cohomology of the $B\,(n)$ we use the May spectral sequence [**11**]:

$$\mathcal{E}_2 = \mathrm{Cotor}_{E^0 B(n)}\,(\mathbb{Z}/2, \mathbb{Z}/2) \Longrightarrow \mathrm{Cotor}_{B(n)}\,(\mathbb{Z}/2, \mathbb{Z}/2)\,.$$

Recall that this spectral sequence is defined by giving $B\,(n)$ the coproduct filtration

$$F^0 B\,(n) \subset F^{-1} B\,(n) \subset \cdots \subset F^{-p} B\,(n) \subset \cdots$$

where by induction on $p \geq 1$

$$F^0 B\,(n) = \mathbb{Z}/2,\; F^{-p} B\,(n) = \left\{ b \in B\,(n) \mid \overline{\psi}\,(b) \in F^{-p+1} B\,(n) \otimes IB\,(n) \right\}.$$

Here $\overline{\psi}$ denotes the reduced coproduct: $\overline{\psi}(b) = \psi(b) - b \otimes 1 - 1 \otimes b$, and $IB\,(n)$ denotes the augmentation ideal of $B\,(n)$. The following lemma describes the structure of the Hopf algebra $E^0 B\,(n)$. It is an immediate consequence of the coalgebra structure of $\mathcal{A}_*$ and the definition of the $B\,(n)$.

Lemma 2.1. *There is an isomorphism of Hopf algebras:*

$$E^0B(n) \cong E\left(\xi_j^{(1)} \mid 1 \le j\right) \otimes E\left(\xi_k^{(2)} \mid n < k\right)$$

where the elements $\xi_j^{(1)}$, $1 \le j \le n+1$, $\xi_k^{(2)}$, $n < k$, *are primitive and*

$$\overline{\psi}\left(\xi_j^{(1)}\right) = \xi_{j-1}^{(2)} \otimes \xi_1^{(1)} \text{ for } j \ge n+2.$$

As in [**5**, Section 1], we compute the $\mathcal{E}_2$-term of these May spectral sequences by using the methods of May [**11**, Section 5] to construct a DGA $D(n)$ whose homology is isomorphic to

$$\operatorname{Cotor}_{E^0B(n)}(\mathbb{Z}/2, \mathbb{Z}/2).$$

In the notation of [**5**] and [**11**], we define the DGA

$$D(n) = \mathbb{Z}/2\left[s\xi_j^{(1)},\, s\xi_k^{(2)} \mid j \ge 1,\, k > n\right]$$

with differential:

$$d\left(s\xi_j^{(1)}\right) = \begin{cases} 0 & \text{for } j \le n+1 \\ s\xi_1^{(1)} s\xi_{j-1}^{(2)} & \text{for } j \ge n+2 \end{cases},$$

$$d\left(s\xi_k^{(2)}\right) = 0 \text{ for } k > n.$$

The following lemma is a straightforward generalization of [**5**, Lemmas 1.4, 1.5 and Theorem 1.6].

Lemma 2.2. *There is an isomorphism of algebras:*

$$H_*D(n) \cong \operatorname{Cotor}_{E^0B(n)}(\mathbb{Z}/2, \mathbb{Z}/2).$$

We will use the elements defined below to compute the homology of the $D(n)$.

Definition 2.3. In the algebra $\operatorname{Cotor}_{E^0B(n)}(Z_2, Z_2) \cong H_*D(n)$ define the following elements:

$$h = \left[s\xi_1^{(1)}\right], \qquad r_k = \left[s\xi_{k+1}^{(2)}\right] \quad \text{for } k \ge n,$$

$$q_j = \left[s\xi_{j+2}^{(1)}\right] \text{ for } 0 \le j < n, \qquad \left[q_k^2\right] = \left[\left(s\xi_{k+2}^{(1)}\right)^2\right] \text{ for } k \ge n,$$

$$p(m_1, \ldots, m_s) = \left[\sum_{i=1}^s s\xi_{m_i+1}^{(2)} s\xi_{m_1+2}^{(1)} \ldots \widehat{s\xi}_{m_i+2}^{(1)} \ldots s\xi_{m_s+2}^{(1)}\right]$$

for $0 \leq m_1 < \cdots < m_s$.

Note 2.1. We will also need the following degenerate cases of these elements:

$$\begin{aligned} & r_m = 0 \text{ for } m < n, && \left[q_m^2\right] = q_m^2 \text{ for } m < n, \\ & p\,(m) = r_m, && p\,(m,m) = 0, \\ & p\,(m_1, \dots, m_s, m, m) = p\,(m_1, \dots, m_s)\left[q_m^2\right] \text{ for } s \geq 1. \end{aligned} \tag{5}$$

The homology of the $D\,(n)$ can be computed as in [**10**, Proposition I.11].

Lemma 2.4. *For $n \geq 1$, the elements*

$$h,\ r_k,\ q_j,\ \left[q_k^2\right],\ p\,(m_1, \dots, m_s)$$

for $k \geq n$, $0 \leq j < n$, $0 \leq m_1 < \cdots < m_s$ are generators of the algebra

$$\mathrm{Cotor}_{E^0 B(n)}\,(\mathbb{Z}/2, \mathbb{Z}/2)\,.$$

A complete set of relations among these generators is given by the degeneracy relations of Note 2.1 and by:

(1) $p\,(m, m_1, \dots, m_s) = p\,(m_1, \dots, m_s)\, q_m$ *for $m < n$ and $s \geq 1$;*
(2) $hp\,(m_1, \dots, m_s) = 0$*;*
(3) $\sum_{i=1}^{s} r_{m_i} p\,(m_1, \dots, \widehat{m_i}, \dots, m_s) = 0$*;*
(4) $p\,(m_1, \dots, m_s)\, p\,(g_1, \dots, g_t)$
$= \sum_{i=1}^{t} r_{g_i} p\,(m_1, \dots, m_s, g_1, \dots, \widehat{g}_i, \dots, g_t)$.

We use the methods of [**5**, Section 3], to show that $\mathcal{E}_2 = \mathcal{E}_\infty$ and that all the extensions are trivial in the May spectral sequence of $B(n)$ for $n \geq 1$. That is, we construct polynomial DGAs $\mathfrak{P}_n$ whose homology is $\mathrm{Cotor}_{B(n)}\,(Z_2, Z_2)$. To avoid repeating an analogue of the proof given in [**5**, Section 3], we use the following lemma which shows how we automatically obtain the $\mathfrak{P}_n$, $n \geq 1$, with the required properties from the $\mathfrak{P}$ constructed in [**5**, Section 3].

Let $C\,(\mathbb{Z}/2, A, \mathbb{Z}/2)$ denote the cobar construction for A, a connected $\mathbb{Z}/2$-Hopf algebra. Suppose we have a DGA P and a $\mathbb{Z}/2$-linear map $\overline{\lambda} : A \to P$. The map $\overline{\lambda}$ induces an algebra homomorphism $\lambda : C\,(Z_2, A, Z_2) \to P$ which we assume is a map of DGAs. We also assume that the algebra homomorphism

$$\lambda_* : \mathrm{Cotor}_A\,(\mathbb{Z}/2, \mathbb{Z}/2) \to H_* P$$

induced by λ is an isomorphism. Suppose that we have a primitive element x in the center of A which is not a zero-divisor. Let

$$A_1 = A/(x), \quad y = \overline{\lambda}(x) \text{ and } P_1 = P/(y).$$

Lemma 2.5 (Generalized Five Lemma). *Assume that we have a $\mathbb{Z}/2$-Hopf algebra A, a DGA P, a $\mathbb{Z}/2$-linear map $\overline{\lambda} : A \to P$ and elements x, y as above which satisfy the following additional conditions:*

(i) $x^2 = 0$;
(ii) *y is central in P;*
(iii) $\overline{\lambda}\,(IA \cdot x) = 0$.

Then λ induces a map of DGAs $\lambda_1 : C\,(\mathbb{Z}/2, A_1, \mathbb{Z}/2) \to P_1$ such that

$$\lambda_{1*} : \mathrm{Cotor}_{A_1}\,(\mathbb{Z}/2, \mathbb{Z}/2) \to H_* P_1$$

is an algebra isomorphism.

Proof: Let $M = \mathbb{Z}/2 \oplus \mathbb{Z}/2(X)$, with $\deg X = \deg x$, denote a comodule over the algebra A. The comodule structure on M, $\psi : M \to M \otimes A$, is induced by:

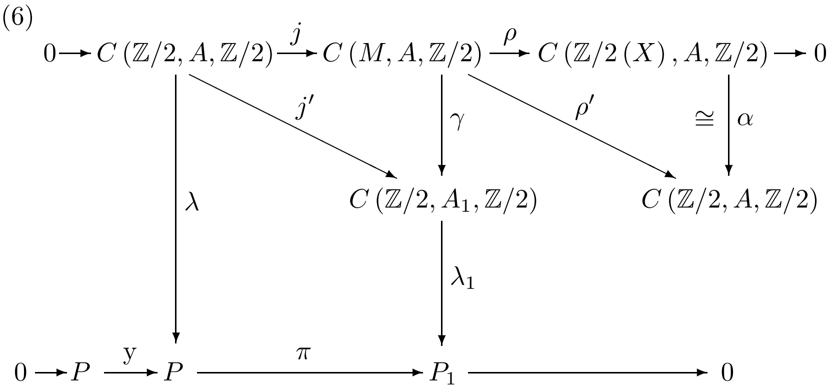
$$\psi(X) = X \otimes 1 + 1 \otimes x.$$

Then the following cobar constructions give a short exact sequence of DGAs:

$$0 \to C\,(\mathbb{Z}/2, A, \mathbb{Z}/2) \xrightarrow{j} C\,(M, A, \mathbb{Z}/2) \xrightarrow{\rho} C\,(\mathbb{Z}/2\,(X)\,, A, \mathbb{Z}/2) \to 0$$

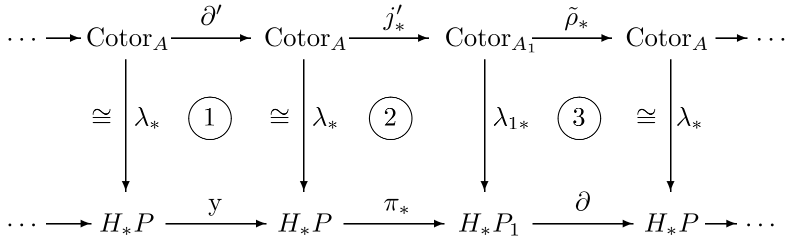
where $j\,(a) = a + 0X$ and $\rho\,(a + bX) = bX$ for a, $b \in \mathbb{Z}/2$. Consider the diagram (6) below. In this diagram, $j' = \gamma \circ j$, $\rho' = \alpha \circ \rho$, $\alpha\,(aX) = a$ and $\gamma\,(a + bX) = \pi'\,(a)$ where $\pi : P \to P_1$ and $\pi' : A \to A_1$ are the canonical projection maps. By condition (iii), λ induces a map of DGAs λ_1 making the trapezoid in (6) commute. By condition (i), the exterior algebra $E\,(x)$ is a sub-Hopf algebra of A. Therefore, γ_* in (6) is an isomorphism by the change of rings theorem [**10**, I.5]. We use the abbreviations $\mathrm{Cotor}_A = \mathrm{Cotor}_A\,(\mathbb{Z}/2, \mathbb{Z}/2)$ and $\mathrm{Cotor}_{A_1} = \mathrm{Cotor}_{A_1}\,(\mathbb{Z}/2, \mathbb{Z}/2)$.

(6)



The short exact sequences on the top and bottom rows of this diagram induce the following long exact sequences in homology.

(7)



In this diagram $\partial' = \partial \circ \alpha_*^{-1}$ and $\tilde{\rho}_* = \rho'_* \circ \gamma_*^{-1}$. We show that diagram (7) commutes. It then follows from the usual Five Lemma that λ_{1*} is an isomorphism. Square 1 commutes because

$$\begin{aligned}\lambda_* \partial' \{Z\} &= \lambda_* \left\{ j^{-1} d \rho^{-1}(XZ) \right\} = \lambda_* \left\{ j^{-1} d(XZ) \right\} \\ &= \lambda_* \left\{ j^{-1}(xZ) \right\} = \lambda_* \{xZ\} = \lambda_* \{x\} \lambda_* \{Z\} = y \lambda_* \{Z\}.\end{aligned}$$

Square 2 commutes because the trapezoid in (6) commutes. Note that

$$d(Z' + XZ'') = d(Z') + xZ'' + Xd(Z'')$$

in $C(M, A, \mathbb{Z}/2)$. Thus, if $Z' + XZ''$ is a cycle then Z'' is a cycle and $d(Z') = xZ''$. Therefore, Square 3 commutes because

$$\begin{aligned}&\lambda_* \tilde{\rho}_* \gamma_* \{Z' + XZ''\} = \lambda_* \alpha_* \{XZ''\} = \lambda_* \{Z''\} \text{ and} \\ &\partial \lambda_{1*} \gamma_* \{Z' + XZ''\} = \partial \lambda_{1*} \{Z'\} = \partial \{\pi \lambda(Z')\} = \{d\lambda(Z')/y\} = \{\lambda d(Z')/y\} \\ &\qquad = \{\lambda(xZ'')/y\} = \{\lambda(x)\lambda(Z'')/y\} = \lambda_* \{Z''\}. \ \blacksquare\end{aligned}$$

We will need the following generalization of the previous lemma which follows from it by induction on $n \geq 1$.

Lemma 2.6. *Let the $\mathbb{Z}/2$-Hopf algebra A, the algebra P and the $\mathbb{Z}/2$-linear map $\overline{\lambda} : A \to P$ be as above. Let $x_1, \ldots, x_n, \ldots$ be a sequence of elements in the center of A. Let $I_0 = 0$, $I_n = (x_1, \ldots, x_n)$ for $n \geq 1$ and $A_n = A/I_n$. Let $y_n = \overline{\lambda}(x_n)$, $J_n = (y_1, \ldots, y_n)$, and $P_n = P/J_n$. Assume that:*

(i) *the ideals I_n are prime and invariant;*
(ii) *$x_n^2 \in I_{n-1}$ for $n \geq 1$;*
(iii) *the y_n are central in P;*
(iv) *$\overline{\lambda}(IA \cdot x_n) = 0$ for $n \geq 1$.*

Then for $n \geq 1$, λ induces maps of DGAs $\lambda_n : C(Z_2, A_n, Z_2) \to P_n$ such that the

$$\lambda_{n*} : \mathrm{Cotor}_{A_n}(Z_2, Z_2) \to H_* P_n$$

are algebra isomorphisms.

We apply this lemma to complete our analysis of the May spectral sequence for

$$\mathrm{Cotor}_{B(n)}(\mathbb{Z}/2, \mathbb{Z}/2)$$

thereby computing $\mathbb{E}_2$ of the ASS (1). Recall from Lemma 4 that $\mathbb{E}_2 \cong \mathrm{Cotor}_{B(n)}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes S$, where S is the polynomial algebra with generators V_a, $a \neq 2^k - 1$. Let $|c|$ denote the degree of c.

Theorem 2.7. *Let $n \geq 1$. Then $\mathbb{E}_2$ of the ASS for $MSp_*^{\Sigma_n}$ is the algebra generated by*

$$V_a, \quad |V_a| = (0, 4a), \quad a \neq 2^l - 1;$$

$$h_0, \quad |h_0| = (0, 0);$$

$$R_k, \quad |R_k| = (1, 2^{k+2} - 3), \quad k \geq n;$$

$$Q_j, \quad |Q_j| = (1, 2^{j+2} - 2), \quad 0 \leq j < n;$$

$$\left[Q_k^2\right], \quad \left|\left[Q_k^2\right]\right| = (2, 2^{k+3} - 4), \quad k \geq n;$$

$$P(m_1, \ldots, m_s), \quad |P(m_1, \ldots, m_s)| = (s, 2^{m_1+2} + \cdots + 2^{m_s+2} - 2s - 1), \quad 0 \leq m_1 < \cdots < m_s.$$

A complete set of relations is given by:

(1) *$P(m, m_1, \ldots, m_s) = P(m_1, \ldots, m_s)\, Q_m$ for $m < n$ and $s \geq 1$;*
(2) *$h_0 P(m_1, \ldots, m_s) = 0$;*
(3) *$\sum_{i=1}^{s} R_{m_i} P(m_1, \ldots, \widehat{m}_i, \ldots, m_s) = 0$;*
(4) *$P(m_1, \ldots, m_s)\, P(g_1, \ldots, g_t)$ $= \sum_{i=1}^{t} R_{g_i} P(m_1, \ldots, m_s, g_1, \ldots, \widehat{g}_i, \ldots, g_t)$;*

and the degeneracy relations

$$R_m = 0 \text{ for } m < n, \qquad \left[Q_m^2\right] = Q_m^2 \text{ for } m < n,$$
$$P(m) = R_m, \qquad P(m,m) = 0,$$
$$P(m_1, \dots, m_s, m, m) = P(m_1, \dots, m_s)\left[Q_m^2\right] \text{ for } s \geq 1.$$

Proof: Recall the DGA $\mathfrak{P}$ constructed in [**5**, Section 3]. $\mathfrak{P}$ is the $\mathbb{Z}/2$-algebra with generators:

$$h_0,\ Q_k,\ R_k$$

for $k \geq 0$. h_0 and the R_k are cycles while $d(Q_k) = h_0 R_k$. The only relation in $\mathfrak{P}$ is

$$[h_0, Q_k] = R_0 R_k$$

for $k \geq 0$. Define a $\mathbb{Z}/2$-linear map $\overline{\lambda} : B \to \mathfrak{P}$ by $\overline{\lambda}(\xi_1) = h_0$, $\overline{\lambda}(\xi_{k+2}) = Q_k$, $\overline{\lambda}(\xi_{k+1}^2) = R_k$ for $k \geq 0$ and $\overline{\lambda}(\xi_{m_1}^{e_1} \dots \xi_{m_s}^{e_s}) = 0$ in all other cases. Then $\overline{\lambda}$ induces a map of DGAs $\lambda : C(\mathbb{Z}/2, B, \mathbb{Z}/2) \to \mathfrak{P}$. By [**5**, Theorem 3.4], λ_* is an algebra isomorphism. We apply Lemma 2.6 with

$$A = B, \quad P = \mathfrak{P} \text{ and } x_n = \xi_n^2 \text{ for } n \geq 1.$$

Then $y_n = R_{n-1}$, $I_n = \left(\xi_1^2, \dots, \xi_n^2\right)$, $A_n = B(n)$,

$$P_n = \mathfrak{P}_n = \mathfrak{P}_0 / (R_0, \dots, R_{n-1}) \text{ and } J_n = (R_0, \dots, R_{n-1}).$$

Observe that since $R_0 = 0$ in $\mathfrak{P}_n$ for $n \geq 1$, the algebra $\mathfrak{P}_n$ is the *commutative* polynomial algebra

$$\mathfrak{P}_n = \mathbb{Z}/2\left[h_0, Q_k, R_m \mid k \geq 0 \text{ and } m \geq n\right].$$

We check the hypotheses of Lemma 2.6.

(i) Since ξ_n^2, $n \geq 1$, is a regular sequence of primitive elements, the ideals $I_n = \left(\xi_1^2, \dots, \xi_n^2\right)$ are prime and invariant.
(ii) $E\left(\xi_1^2, \dots, \xi_n^2, \dots\right)$ is a sub-Hopf algebra of B.
(iii) Clearly the R_n are central in $\mathfrak{P}$.
(iv) By the definition of $\overline{\lambda}$, we see that $\overline{\lambda}\left(\alpha \xi_n^2\right) = 0$ for $\alpha \in IB$.

By Lemma 2.6, λ induces maps of DGAs $\lambda_n : C(Z_2, B(n), Z_2) \to \mathfrak{P}_n$ for $n \geq 1$ such that the $\lambda_{n*} : \mathrm{Cotor}_{B(n)}(\mathbb{Z}/2, \mathbb{Z}/2) \to H_*(\mathfrak{P}_n)$ are isomorphisms.

We construct representative cycles in $\mathfrak{P}_n$ of the algebra generators of $\mathcal{E}_2$ of the May spectral sequence for $\mathrm{Cotor}_{B(n)}(Z_2, Z_2)$:

h is represented by h_0
r_k is represented by R_k for $k \geq n$;
$\left[q_k^2\right]$ is represented by Q_k^2 for $k \geq n$;
q_j is represented by Q_j for $0 \leq j < n$;
$p\left(m_1, \ldots, m_s\right)$ is represented by

$$P\left(m_1, \ldots, m_s\right) = \sum_{i=1}^{s} R_{m_i} Q_{m_1} \ldots \widehat{Q}_{m_i} \ldots Q_{m_s}.$$

It follows that $\mathcal{E}_2 = \mathcal{E}_\infty$ in the May spectral sequences for the $\mathrm{Cotor}_{B(n)}\left(Z_2, Z_2\right)$. Using these representative cycles of the algebra generators of $\mathcal{E}_\infty$, it is straightforward to check that all four families of relations in $\mathcal{E}_\infty$ are also valid in $\mathrm{Cotor}_{B(n)}\left(\mathbb{Z}/2, \mathbb{Z}/2\right)$. Thus, the structure of $\mathbb{E}_2$ of the ASS follows from (4) and Lemma 2.4. ■

Observe that the commutativity of the $\mathfrak{P}_n$ is the reason why the elements Q_k^2 and $P\left(m_1, \ldots, m_s\right)$ are cycles in $\mathfrak{P}_n$ for $n \geq 1$ while in $\mathfrak{P}$ they are not cycles and support nonzero d_2-differentials in the May spectral sequence for $\mathrm{Cotor}_B\left(\mathbb{Z}/2, \mathbb{Z}/2\right)$ when $s \geq 3$.

3. Adams Spectral Sequence for MSp^{Σ_n}

In the preceding section we obtained a concise algebraic description of $\mathbb{E}_2$ of the ASS (1) for MSp^{Σ_n}, $n \geq 1$. However, this algebraic description is not suitable for computing the differentials in the ASS or for understanding $MSp_*^{\Sigma_n}$ which is determined by the topology of the spectrum MSp^{Σ_n}. Thus, we begin this section with an alternate description of $\mathbb{E}_2$ in terms of the projections Φ_n of the Ray elements ϕ_n. Although this description may seems algebraicly awkward, it enables us to compute all of the d_2-differentials and some of the d_3-differentials. These d_3-differentials are used to prove a technical fact which we needed in [**2**, Section 6]. In addition, we will use this description of $\mathbb{E}_2$ in Section 6 to identify and analyze the elements of higher torsion we construct there.

Recall from [**5**, Theorem 5.3] that the Ray elements ϕ_k, $k \geq 1$, project to elements

(8) $$\Phi_k = \sum_{j \geq 0} R_j V_{I(k,j)} \in \mathbb{E}_2^{1, 8k-3}$$

of the ASS for MSp. In the following definition, as in [**5**, Section 4], we rewrite all the elements of $\mathbb{E}_2$ of the ASS for MSp^{Σ_n} in terms of the Ray elements.

Definition 3.1. Let $n \geq 1$. Define the following elements in $\mathbb{E}_2$ of the ASS for MSp^{Σ_n}:

(1) $\Psi_{-1} = \Psi_0 = Q_0$ and $\Psi_h = \sum_{j\geq 0} Q_j V_{I(h,j)} \in E_2^{1,8h-2}$ for $1 \leq h < 2^{n-1}$;

(2) $\left[\Psi_k^2\right] = \sum_{j=0}^{2^{n-1}-1} Q_j^2 V_{I(k,j)}^2 + \sum_{j\geq 2^{n-1}} \left[Q_j^2\right] V_{I(k,j)}^2$ for $k \geq 2^{n-1}$;

(3) $\rho(m_1, \dots, m_s)$
$= \sum_{j(1)\geq 0} \cdots \sum_{j(s)\geq 0} P(j(1), \dots, j(s)) V_{I(m_1,j(1))} \dots V_{I(m_s,j(s))}$ in $\mathbb{E}_2^{s,m}$ where $m = 4\delta_0^{m_1} + 8m_1 + \cdots + 8m_s - 2s - 1$ and $0 \leq m_1 < \cdots < m_s$.

In terms of these elements, the following description of $\mathbb{E}_2$ follows from Theorem 2.7.

Corollary 3.2. *$\mathbb{E}_2$ of the ASS for MSp^{Σ_n} is the algebra generated by:*

$$V_a,\ a \neq 2^q - 1,\ h_0,\ \Phi_k,\ k \geq 2^{n-1},\ \left[\Psi_k^2\right],\ k \geq 2^{n-1},\ \Psi_j,\ 0 \leq j < 2^{n-1}$$
$$\text{and } \rho(m_1, \dots, m_s),\ 0 \leq m_1 < \cdots < m_s.$$

A complete set of relations is given by:

(1) $\rho(m, m_1, \dots, m_s) = \rho(m_1, \dots, m_s)\Psi_m$ *for* $m < 2^{n-1}$ *and* $s \geq 1$*;*

(2) $h_0\rho(m_1, \dots, m_s) = 0$*;*

(3) $\sum_{i=1}^{s} \Phi_{m_i}\rho(m_1, \dots, \widehat{m_i}, \dots, m_s) = 0$*;*

(4) $\rho(m_1, \dots, m_s)\rho(g_1, \dots, g_t)$
$= \sum_{i=1}^{t} \Phi_{g_i}\rho(m_1, \dots, m_s, g_1, \dots, \widehat{g_i}, \dots g_t)$*;*

(5) *(a)* $\Psi_h = \sum_{j\geq 0} \Psi_{2^j-1} V_{I(h,j)}$ *for* $1 \leq h < 2^{n-1}$*;*

(b) $\left[\Psi_k^2\right] = \sum_{j=0}^{2^{n-1}-1} \Psi_{2^j-1}^2 V_{I(k,j)}^2 + \sum_{j\geq 2^{n-1}} \left[\Psi_{2^j-1}^2\right] V_{I(k,j)}^2$ *for* $k \geq 2^{n-1}$*;*

(c) $\rho(m_1, \dots, m_s) =$
$\sum_{j(1)\geq 0} \cdots \sum_{j(s)\geq 0} \rho\left(2^{j(1)-1}, \dots, 2^{j(s)-1}\right) V_{I(m_1,j(1))} \dots V_{I(m_s,j(s))}$*;*

and the degeneracy relations

$$\Phi_m = 0 \text{ for } m < 2^{n-1}, \qquad \left[\Psi_m^2\right] = \Psi_m^2 \text{ for } m < 2^{n-1},$$
$$\rho(m) = \Phi_m, \qquad \rho(m, m) = 0,$$
$$\rho(m_1, \dots, m_s, m, m) = \rho(m_1, \dots, m_s)\left[\Psi_m^2\right] \text{ for } s \geq 1.$$

Using the description of the elements of $\mathbb{E}_2$ given in Corollary 3.2, we compute the d_2-differentials.

Theorem 3.3. *Let $n \geq 1$. The d_2-differentials in the ASS for MSp^{Σ_n} are completely described below.*

(a) *If $k \neq 2^p$ then there is a choice of V_{2k} which is a d_2-cycle.*
(b) *For $k \geq 1$, there is a choice of V_{2^k} such that $d_2\left(V_{2^k}\right) = \Psi_0 \Phi_{2^{k-1}}$.*
(c) *Write $k = 2^{k_1} + \cdots + 2^{k_s} + 2^{k_{s+1}} + \cdots + 2^{k_t}$ where $0 \leq k_1 < \cdots < k_t$ and $k_s < n \leq k_{s+1}$. Then there is a choice of V_{2k-1} such that*

$$
\begin{aligned}
d_2\left(V_{2k-1}\right) = & \sum_{1 \leq i \leq s < j \leq t} \Psi_{2^{k_i}} \Phi_{2^{k_j}} V_{2^{k_1}+1} \ldots \widehat{V}_{2^{k_i}+1} \ldots \widehat{V}_{2^{k_j}+1} \ldots V_{2^{k_t}+1} \\
& + \sum_{s<i<j \leq t} \rho\left(2^{k_i}, 2^{k_j}\right) V_{2^{k_1}+1} \ldots \widehat{V}_{2^{k_i}+1} \ldots \widehat{V}_{2^{k_j}+1} \ldots V_{2^{k_t}+1}.
\end{aligned}
$$

(d) *h_0, the Ψ_i, the Φ_k, the $\left[\Psi_k^2\right]$ and the $\rho\left(m_1, \ldots, m_s\right)$ are infinite cycles for $i < 2^{n-1}$, $2^{n-1} \leq k$ and $0 \leq m_1 < \cdots < m_s$.*

Proof: Using the canonical map from the ASS for MSp to the ASS for MSp^{Σ_n}, the first three parts of this theorem follow from [**5**, Theorem 6.1]. It remains to prove (d). Clearly h_0 is an infinite cycle converging to 2. Since $\mathbb{E}_2^{*,2k-1} = 0$ for $2k-1 < 2^{n+2}-3$, the Ψ_i are infinite cycles. It remains to prove that the $\rho\left(m_1, \ldots, m_s\right)$ are infinite cycles.

Proposition 3.4. *For $n \geq 1$ and $2^{n-1} \leq m_1 < \cdots < m_t$, there exist elements $r_n\left(m_1, \ldots, m_t\right)$ in the ring $MSp_*^{\Sigma_n}$ such that:*

(i) *$r_n(m) = \phi_m$;*
(ii) *$r_n\left(m_1, \ldots, m_t\right) \in \left\langle \phi_{m_t}, 2, r_n\left(m_1, \ldots, m_{t-1}\right)\right\rangle$ for $t \geq 2$;*
(iii) $2 r_n\left(m_1, \ldots, m_t\right) = 0$.

Proof: We construct the elements $r_n\left(m_1, \ldots, m_t\right)$ by induction on $t \geq 1$. When $t = 1$ we use (i) to define $r_n\left(m_1\right)$. Assume that $t \geq 2$ and that this proposition is true for $t-1$. Select any element $r_n\left(m_1, \ldots, m_t\right)$ of the Toda bracket $\left\langle \phi_{m_t}, 2, r_n\left(m_1, \ldots, m_{t-1}\right)\right\rangle$. By [**2**, Lemma 3.4 and Note 3.1] we have:

$$
\begin{aligned}
2 r_n\left(m_1, \ldots, m_t\right) & \in 2\left\langle \phi_{m_t}, 2, r_n\left(m_1, \ldots, m_{t-1}\right)\right\rangle \\
& \subset \left\langle 2, \phi_{m_t}, 2\right\rangle r_n\left(m_1, \ldots, m_{t-1}\right).
\end{aligned}
$$

Note that the Σ_n-manifold $\Delta(2)$ is a representative of η, and $\eta = 0$ in the ring $MSp_*^{\Sigma_n}$ for $n \geq 1$. Thus, by [**2**, Lemma 3.3 and Note 3.1] and by our induction hypothesis we have:

$$
\left\langle 2, \phi_{m_t}, 2\right\rangle r_n\left(m_1, \ldots, m_{t-1}\right) = \left(\eta \phi_{m_t} + 2a\right) r_n\left(m_1, \ldots, m_{t-1}\right) = 0. \quad \blacksquare
$$

Proof of Theorem 3.3 continued: Clearly the element $r_n(m_1, \dots, m_s)$ projects to $\rho(m_1, \dots, m_s)$ in $\mathbb{E}_2$ of the ASS for MSp^{Σ_n}. ■

Next we compute the d_3-differentials on some of the polynomial generators of $\mathbb{E}_3^{0,4*}$ of the ASS for MSp^{Σ_n}. Recall from [**6**, Theorem 8.7(d)] the following d_3-differentials in the ASS for MSp:

$$d_3\left(V_{s,t}^2\right) = \widetilde{\Phi}_{2s}\Phi_t^2 + \Phi_s^2\widetilde{\Phi}_{2t} + \Phi_s\Phi_t\Sigma(0,s,t)$$

where

$$\begin{aligned} \widetilde{\Phi}_{2n} &= \Phi_{2n} + \Sigma_{k=1}^{n-1}\Sigma(0,k,2n-k) \text{ and} \\ \Sigma(a,b,c) &= \Phi_a V_{b,c} + \Phi_b V_{a,c} + \Phi_c V_{a,b}. \end{aligned} \tag{9}$$

Applying the canonical map from MSp to MSp^{Σ_n}, we obtain the following result.

Proposition 3.5. *In $\mathbb{E}_3$ of the ASS for MSp^{Σ_n}:*

(a) $d_3\left(V_{2^s,2^t}^2\right) = 0$ *if* $0 \le s < t$ *and* $s \le n-3$*;*

(b) $d_3\left(V_{2^{n-2},2^t}^2\right) = \Phi_{2^{n-1}}\Phi_{2^t}^2$ *if* $n-1 \le t$*;*

(c) $d_3\left(V_{2^s,2^t}^2\right) = \widetilde{\Phi}_{2^{s+1}}\Phi_{2^t}^2 + \Phi_{2^s}^2\widetilde{\Phi}_{2^{t+1}} + \Phi_{2^s}\Phi_{2^t}\Sigma(0,2^s,2^t)$ *if* $n-1 \le s < t$*;*

In order to identify the projection to the Adams Novikov spectral sequence of the elements of higher torsion which we constructed in [**2**, Section 6] we used the following technical fact.

Corollary 3.6. *The cobordism class of the Σ_2-manifold $\Delta(W_2)$ equals ϕ_2.*

Proof: Recall that we defined $\Delta(W_2)$ in [**2**, Section 3] as

$$\Delta(W_2) = \mathfrak{m}_2\left(W_2^{(1)}, W_2^{(2)}\right) \times I \cup -\mathfrak{K}_2\left(W_2^{(2)}, W_2^{(1)}\right).$$

Since $MSp_{13}^{\Sigma_2} = \mathbb{Z}/2\phi_2$, the cobordism class of $\Delta(W_2)$ is either ϕ_2 or zero. By Proposition 3.5(b),

$$d_3\left(V_{1,2^n}^2\right) = \Phi_2\Phi_{2^n}^2$$

in the ASS for MSp^{Σ_2}. If we represent $V_{1,2^n} \in \mathbb{E}_2^{0,2^{n+3}+4}\left(MSp^{\Sigma_2}\right)$ by the Σ_2-manifold V_{2^n} of Lemma 5.3 then $V_{1,2^n}^2 \in \mathbb{E}_2^{0,2^{n+4}+8}\left(MSp^{\Sigma_2}\right)$ is represented by a Σ_2-manifold $V_{2^n}^{[2]}$ which can be defined as $\mathfrak{m}_2\left(V_{2^n}, V_{2^n}\right)$ union several manifolds of positive Adams filtration degree including $\mathfrak{K}_2\left(V_{2^n}, W_2\right) \times \phi_{2^n}$. Using the Hirsch formula, Lemma 3.1(a) and Note 3.1 of [**2**], $\delta V_{2^n}^{[2]}$ has as part of its boundary:

$$\mathfrak{K}_2\left(W_2, W_2\right) \times \phi_{2^n} \times \phi_{2^n}.$$

This unionand of $\delta V_{2^n}^{[2]}$ is the only one which could possibly project to $\Phi_2 \Phi_{2^n}^2$ in the ASS for MSp^{Σ_2}. Thus, $\Delta\left(W_2\right)$ must equal ϕ_2 and not zero. ■

4. Adams Spectral Sequence for $MSp^{\widehat{\Sigma}_n}$

Let $MSp^{\widehat{\Sigma}_n}$, $n \geq 2$, be the spectrum with singularities $\widehat{\Sigma}_n = \left(P_2, \ldots, P_n\right)$ defined in the Introduction, and let $MSp^{\widehat{\Sigma}_1}$ denote MSp. In this section we compute $\mathbb{E}_2$ of the ASS:

$$\mathbb{E}_2^{s,t} = \operatorname{Cotor}_{\mathcal{A}}^s\left(H_* MSp^{\widehat{\Sigma}_n}, \mathbb{Z}/2\right)_t \Longrightarrow MSp_*^{\widehat{\Sigma}_n}. \tag{10}$$

As in Section 2, we use a change of rings theorem to reduce the problem of calculating $\mathbb{E}_2$ to computing

$$\operatorname{Cotor}_{\widehat{B}(n)}\left(\mathbb{Z}/2, \mathbb{Z}/2\right) \tag{11}$$

where $\widehat{B}\left(n\right)$ is the truncated polynomial algebra which is defined as the following quotient Hopf algebra of the dual of the Steenrod algebra:

$$\widehat{B}\left(n\right) = \mathcal{A}_* / \left(\xi_1^4,\, \xi_h^2,\, \xi_k^4 \mid 2 \leq h \leq n \text{ and } n < k\right).$$

Note that in [**5**] the Hopf algebra $\widehat{B}\left(1\right)$ is denoted as B. We compute the algebra (11) by showing that it is the tensor product of a polynomial algebra and a direct summand of $\operatorname{Cotor}_B\left(\mathbb{Z}/2, \mathbb{Z}/2\right)$ which was computed in [**5**]. Thus, $\mathbb{E}_2$ of the ASS (10) for $n \geq 2$ has all the complexity of the $\mathbb{E}_2$-term of the ASS for MSp: it has nine families of algebra generators and forty families of relations. As in Section 3, we give an alternate description of $\mathbb{E}_2$ in terms of the projections Φ_n of the Ray elements ϕ_n into the ASS (10). In Section 6, we use this description to identify and analyze the elements of higher torsion which we construct there.

V. Vershinin [**14**], [**16**] showed that for $n \geq 2$ there is an isomorphism of $\mathcal{A}_*$-comodules:

$$H_* MSp^{\widehat{\Sigma}_n} \cong \mathbb{Z}/2\left[\xi_1^4, \xi_2^2, \ldots, \xi_n^2, \xi_{n+1}^4, \ldots, \xi_k^4, \ldots\right] \otimes S.$$

It follows from the change of rings theorem [**10**, Corollary I.5] that for $n \geq 1$ there is an isomorphism of $\mathbb{Z}/2$-algebras:

$$\mathbb{E}_2 = \mathrm{Cotor}_{\mathcal{A}_*}\left(H_* MSp^{\widehat{\Sigma}_n}, \mathbb{Z}/2\right) \cong \mathrm{Cotor}_{\widehat{B}(n)}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes S. \tag{12}$$

Define the sub-Hopf algebra $\widehat{C}(n)$ of $\widehat{B}(n)$ by

$$\widehat{C}(n) = \mathbb{Z}/2\left[\xi_1, \xi_k \mid n < k\right] / \left(\xi_1^4,\ \xi_k^4 \mid n < k\right).$$

Since the ξ_h, $2 \leq h \leq n$, are primitive in $\widehat{B}(n)$,

$$\widehat{B}(n) \cong \widehat{C}(n) \otimes E(\xi_2, \dots, \xi_n)$$

as Hopf algebras. Let Q_{h-1} denote the homology class of $[\xi_h]$ for $2 \leq h \leq n$. We thus have the following lemma.

Lemma 4.1. *For $n \geq 2$, there is an isomorphism of $\mathbb{Z}/2$-algebras:*

$$\mathrm{Cotor}_{\widehat{B}(n)}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes \mathbb{Z}/2\left[Q_1, \dots, Q_{n-1}\right]. \tag{13}$$

We compute $\mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2, \mathbb{Z}/2)$ thereby determining $\mathbb{E}_2$ of the ASS for $MSp^{\widehat{\Sigma}_n}$. Recall from [**5**, Theorem 3.7] that $\mathrm{Cotor}_B(\mathbb{Z}/2, \mathbb{Z}/2)$ can be described as the algebra generated by h_0 and by seven families $F(k_1, \dots, k_t)$ of generators with forty familites of relations. In particular, F is one of the following symbols: $q\,(t = 1)$, $Q\,(t \geq 1)$, $R\,(t = 1)$, $P\,(t = 2)$, $P_2\,(t \geq 3)$, $Y\,(t \geq 7)$ *or* $Z_s\,(t \geq s + 2 \geq 4)$.

Proposition 4.2. *For $n \geq 2$, let $\mathfrak{C}_n$ denote the subalgebra of* $\mathrm{Cotor}_B(\mathbb{Z}/2, \mathbb{Z}/2)$ *generated by*

h_0	R_{k_1}
q_{k_1}	$Q(k_1, \dots, k_t)\quad (t \geq 1)$
$P(k_1, k_2)$	$P_2(k_1, \dots, k_t)\quad (t \geq 3)$
$Y(k_1, \dots, k_t)\ (t \geq 7)$	$Z_s(k_1, \dots, k_t)\quad (t \geq s + 2 \geq 4)$

where each of the k_i is either zero or greater than or equal to n. Then $\mathbb{E}_2$ of the ASS for $MSp^{\widehat{\Sigma}_n}$ is given by

$$\mathbb{E}_2 \cong \mathfrak{C}_n \otimes \mathbb{Z}/2\left[Q_1, \dots, Q_{n-1}\right] \otimes S.$$

Proof: By (12), (13), $\mathbb{E}_2 = \mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2,\mathbb{Z}/2)\otimes\mathbb{Z}/2[Q_1,\ldots,Q_{n-1}]\otimes S$. Let $\iota_n : \widehat{C}(n) \to B$ denote the inclusion map. Define a map $\sigma_n : B \to \widehat{C}(n)$ of Hopf algebras which splits ι_n by

$$\sigma_n(\xi_k) = \begin{cases} \xi_k & \text{if } k = 1 \text{ or } k > n \\ 0 & \text{if } 2 \le k \le n \end{cases}.$$

Then σ_{n*} is a splitting of the inclusion

$$\iota_{n*} : \mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2,\mathbb{Z}/2) \hookrightarrow \mathrm{Cotor}_B(\mathbb{Z}/2,\mathbb{Z}/2).$$

Thus, we view $\mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2,\mathbb{Z}/2)$ as a subalgebra of $\mathrm{Cotor}_B(\mathbb{Z}/2,\mathbb{Z}/2)$. The effect of σ_{n*} on the algebra generators $F(k_1,\ldots,k_t)$ of $\mathrm{Cotor}_B(\mathbb{Z}/2,\mathbb{Z}/2)$ is given by $\sigma_{n*}(h_0) = h_0$ and

(14)
$$\sigma_{n*}(F(k_1,\ldots,k_t)) = \begin{cases} F(k_1,\ldots,k_t) & \text{if } \{k_1,\ldots,k_t\}\cap\{1,\ldots,n-1\} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for F one of q, Q, R, P, P_2, Y or Z_s. Observe that $\mathfrak{C}_n = \mathrm{Image}\,\iota_{n*}$ is the subalgebra of $\mathrm{Cotor}_B(\mathbb{Z}/2,\mathbb{Z}/2)$ spanned by all $F(k_1,\ldots,k_t)$ with $\{k_1,\ldots,k_t\} \subset \{1,\ldots,n-1\}$. Thus by (14), $\sigma_{n*} : \mathfrak{C}_n \to \mathrm{Image}\,\sigma_{n*}$ is an isomorphism. Therefore, $\mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathrm{Image}\,\sigma_{n*} \cong \mathfrak{C}_n$. ■

Note 4.1. The map π_r, $r \ge 2$, of ASS induced by the canonical map of spectra $\pi : MSp \to MSp^{\widehat{\Sigma}_n}$ does not induce the projection map

$$\sigma_{n*}\otimes 1 : \mathbb{E}_2 \cong \mathrm{Cotor}_B(\mathbb{Z}/2,\mathbb{Z}/2)\otimes S \to \mathbb{E}_2 = \mathrm{Cotor}_{\widehat{C}(n)}(\mathbb{Z}/2,\mathbb{Z}/2)\otimes S.$$

For example, $\pi_2(P(1,n)) = Q_1R_n$ while $(\sigma_{n*}\otimes 1)(P(1,n)) = 0$.

We conclude with an alternate description of $\mathbb{E}_2$ in terms of the projections Φ_n of the Ray elements ϕ_n to the ASS. If

$$\Phi_k = \sum_{j\ge 0} R_j V_{I(k,j)} \in \mathbb{E}_2^{1,8k-3}$$

in $\mathbb{E}_2$ of the ASS for MSp and $F(k_1,\ldots,k_t)$ is one of the above seven families of algebra generators of $\mathbb{E}_2$ of the ASS for $MSp^{\widehat{\Sigma}_n}$ then define

$$\underline{F}(k_1,\ldots,k_t) = \sum_{j_1\ge 0}\cdots\sum_{j_t\ge 0} F(j_1,\ldots,j_t)\, V^{e_F}_{I(k_1,j_1)}\cdots V^{e_F}_{I(k_t,j_t)}$$

where e_F equals 1, 4, 2, 1, 2, 1, 2 if F equals R, q, Q, P, P_2, Y, Z_s, respectively. We denote $\underline{Q}_k$ as Ψ_k. We do not explicitly specify the forty relations in $\mathbb{E}_2$ of the ASS for $MSp^{\widehat{\Sigma}_n}$ induced from $\mathbb{E}_2$ of the ASS for MSp because we do not use them in this paper.

Corollary 4.3. *For $n \geq 2$, $\mathbb{E}_2$ of the ASS for $MSp^{\widehat{\Sigma}_n}$ is the $\mathbb{Z}/2$-algebra generated by*

$$\begin{array}{ll} \Psi_k \ (1 \leq k < 2^{n-1}) & V_a, \ a \neq 2^r - 1 \\ h_0 & \underline{R}_k \\ \underline{q}_k & \underline{Q}\,(k_1, \dots, k_t) \quad (t \geq 1) \\ \underline{P}\,(k_1, k_2) & \underline{P}_2\,(k_1, \dots, k_t) \ (t \geq 3) \\ \underline{Y}\,(k_1, \dots, k_t) \ (t \geq 7) & \underline{Z}_s\,(k_1, \dots, k_t) \ (t \geq s+2 \geq 4). \end{array}$$

A complete set of relations for $\mathbb{E}_2$ is given by the forty relations listed in [**5**, Theorem 3.7] *as well as the following relations:*

(a) *if $0 < k_t < 2^{n-1}$ and F is Q, P_2, Y or Z_s then*

$$\underline{F}\,(k_1, \dots, k_t) = \underline{F}\,(k_1, \dots, k_{t-1})\,\Psi_{k_t}^{e_F};$$

(b) *if $0 < k_1 < 2^{n-1}$ then*

$$\underline{R}_{k_1} = 0; \ \underline{q}_{k_1} = \Psi_{k_1}^4; \ \underline{P}\,(k_1, k_2) = \Psi_{k_1}\underline{R}_{k_2};$$

(c) *if F is any of the above ten families except h_0 or V_a then*

$$\underline{F}\,(k_1, \dots, k_t) = \sum_{j_1 \geq 0} \cdots \sum_{j_t \geq 0} \underline{F}\left(2^{j_1 - 1}, \dots, 2^{j_t - 1}\right) V_{I(k_1, j_1)}^{e_F} \cdots V_{I(k_t, j_t)}^{e_F}.$$

From now on we only use the description of $\mathbb{E}_2$ in terms of the $\underline{F}\,(k_1, \dots, k_t)$, and we abuse notation by denoting them as $F\,(k_1, \dots, k_t)$.

5. Construction of Higher Torsion Elements

In this section we prove Theorem 1 and use it to construct elements of higher torsion. The vanishing of the Toda brackets $\langle \phi_1, 2, \phi_n \rangle$ of Theorem 1 allows us to construct specific Sp-manifolds V_n *with no singularities* in Lemma 5.3 such that ∂V_n is the canonical element in this Toda bracket. Let $J = [j_1, \dots, j_s]$ with $s \geq 1$ and $J' = [j_1, \dots, j_{s-1}]$ with $s \geq 2$ throughout this section. In Proposition 5.4 we use the V_n to generalize the constructions of Section 5 of [**2**] to construct the elements $t[J] \in MSp_*^{\widehat{\Sigma}_3}$ which define the elements $g[J] = \widehat{\beta}_3\,(t[J]) \in MSp_*^{\widehat{\Sigma}_2}$ and $a[J] = \widehat{\beta}_2(\widehat{\beta}_3(t[J])) \in MSp_*$ described in the Introduction for $J = [j_1, \dots, j_s]$. We give the basic properties of the $t[J]$ and $g[J]$ including their Toda bracket decompositions and their projection in the

ASS. We abbreviate those constructions which are analogous to those of [**2**].

We begin with the proof of Theorem 1. Its proof relies on decomposing ϕ_1 as a triple Toda bracket based upon the smash product. Recall from [**8**] the definition of this type of Toda bracket. We are given three maps of spectra $\alpha : S \to E$, $\beta : S \to F$, $\gamma : S \to G$ and associative pairings of spectra

$$\omega_{EF} : E \wedge F \to M,\ \omega_{FG} : F \wedge G \to N,\ \omega_{MG} : M \wedge G \to P,\ \omega_{EN} : E \wedge N \to P$$

such that $\omega_{EF}(\alpha \wedge \beta) = 0$ and $\omega_{FG}(\beta \wedge \gamma) = 0$. Let $\xi : D \to M$ be an extension of $\omega_{EF} \circ (\alpha \wedge \beta)$ to a disc and let $\zeta : D' \to N$ be an extension of $\omega_{FG} \circ (\beta \wedge \gamma)$ to a disc. Then $\langle \alpha, \beta, \gamma \rangle$ is defined as the set of homotopy classes of all maps

$$(\omega_{MG} \circ (\xi \wedge \gamma)) \cup (\omega_{EN} \circ (\alpha \wedge \zeta)) : S = (D \wedge S) \cup (S \wedge D') \to P$$

for all choices of ξ and ζ. We identify such a Toda bracket in the case $E = F = S$ and $G = MSp$ which decomposes ϕ_1. We also give a similar decomposition of ϕ_2 in terms of a four-fold Toda bracket. Recall that $MSp_8 = \mathbb{Z}$ with the generator q_0.

Lemma 5.1. *Let $\mu : S \to MSp$ denote the unit of the spectrum MSp. Then*

(a) $\phi_1 = \langle \eta, \nu, \mu \rangle$;
(b) $\phi_2 \in \langle \eta, \nu, \sigma, \mu \rangle = \{\phi_2, \phi_2 + \phi_1 q_0\}$.

Proof: The proof of this lemma is based upon the analysis of the following Atiyah-Hirzebruch spectral sequence.

$$E^2_{*,*} = H_* MSp \otimes \pi^S_* \Longrightarrow MSp_*. \tag{15}$$

This spectral sequence was analyzed through degree 50 in [**9**]. Fortunately, we only require its structure through degree 5 which is depicted in Figure 1. We use the notation $H_* MSp = \mathbb{Z}[b_1, \ldots, b_n, \ldots]$ where $H_* HP^\infty$ has the $\mathbb{Z}$-basis $\{b_1, \ldots, b_n, \ldots\}$. The only differential in our range is $d^4(b_1) = \nu$.

(a) Since $MSp_5 = \mathbb{Z}/2\phi_1$ and the only infinite cycle in $E^2_{*,*}$ of degree 5 is ηb_1, the only possibility for the projection of ϕ_1 to $E^\infty_{*,*}$ is ηb_1. The fact that ηb_1 is an infinite cycle of (15) means that if $B_1 : D' \to MSp$ represents b_1 such that $B_1 \mid S' = \nu$ and $\xi : D \to S$ such that $\xi \mid S = \eta \wedge \nu$ then ϕ_1 is represented by:

$$\eta \wedge B_1 \cup \xi \wedge \mu \in \langle \eta, \nu, \mu \rangle.$$

Note that we have suppressed the canonical pairings of spectra involved in the previous statement. Since $\mu_*\left(\pi_5^S\right) = 0$ and $\eta \cdot MSp_4 = 0$, the indeterminacy of $\langle \eta, \nu, \mu \rangle$ is zero.

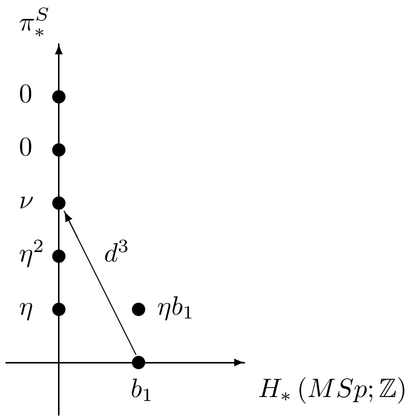


Figure 1: The Atiyah-Hirzebruch Spectral Sequence for MSp_*

(b) Observe that $\langle \eta, \nu, \sigma \rangle \subset \pi_{12}^S = 0$ and $\langle \nu, \sigma, \mu \rangle \subset MSp_{11} = 0$. Thus, the Toda bracket $\langle \eta, \nu, \sigma, \mu \rangle \subset MSp_{13}$ is defined. Consider any defining system of $\lambda \in \langle \eta, \nu, \sigma, \mu \rangle$ and let ξ be the element of this defining system whose boundary is an element of $\langle \nu, \sigma, \mu \rangle$. Then ξ projects to a nonzero element $X \in H_{12}\left(MSp; \mathbb{Z}\right)$ in the zero row of the Atiyah-Hirzebruch spectral sequence (15) which is not divisible by two. If b_ω is a monomial summand of X with a coefficient that is nonzero modulo two then $\eta\xi$ is a unionand of λ and $s_\omega\left(\lambda\right) = \eta$. Since $MSp_{13} = \mathbb{Z}_2\phi_2 \oplus \mathbb{Z}_2\phi_1 q_0$ and $s_\omega\left(\phi_1 q_0\right) = 0$, it follows that $\lambda = \phi_2 + k\phi_1 q_0$ for some $k \in \mathbb{Z}/2$. Note that the indeterminacy of $\langle \eta, \nu, \sigma, \mu \rangle$ contains $\langle \eta, \nu, MSp_8 \rangle$ which contains $\phi_1 q_0$ by (a). Thus, $\langle \eta, \nu, \sigma, \mu \rangle = \{\phi_2, \phi_2 + \phi_1 q_0\}$ as asserted. ■

Our representative of ϕ_1 can be described in terms of (Sp, fr)-manifolds as

$$\eta \times Y^4 \cup W^5$$

where Y^4 is an Sp-manifold with $\partial Y^4 = \nu$ and W^5 is a framed manifold with $\partial W^5 = \eta \times \nu$.

Recall from [**12**] that the Ray elements ϕ_n are closed under the action of the Landweber-Novikov operations. In particular, $s_{\Delta_{2k}}\left(\phi_m\right) = \phi_{m-k}$ if $1 \leq k < m$. By [**6**, Theorem 11.4], the action of the Landweber-Novikov operations on the Toda brackets $\langle \phi_m, 2, \phi_n \rangle$ satisfies the Cartan formula:

$$s_\omega\langle \phi_m, 2, \phi_n\rangle \subset \sum_{\omega=\omega_1+\omega_2} \langle s_{\omega_1}(\phi_m), 2, s_{\omega_2}(\phi_n)\rangle.$$

We thus have the following formula for the action of the $s_{\Delta_{2k}}$ on our Toda brackets.

Lemma 5.2. *For $m > k \geq 1$, $s_{\Delta_{2k}}\langle \phi_1, 2, \phi_m\rangle \subset \langle \phi_1, 2, \phi_{m-k}\rangle$.*

We use the action of the $s_{\Delta_{2k}}$ on our Toda brackets and the decomposition of ϕ_1 to prove Theorem 1.

Proof of Theorem 1: By Lemma 5.1, $\langle \phi_n, 2, \phi_1\rangle = \langle \phi_n, 2, \langle \eta, \nu, \mu\rangle\rangle$ which contains an element which is also an element of

$$\langle \phi_n, \langle 2, \eta, \nu\rangle, \mu\rangle + \langle\langle \phi_n, 2, \eta\rangle, \nu, \mu\rangle = \langle \phi_n, 0, \mu\rangle + \langle \eta A, \nu, \mu\rangle.$$

The last equality uses Gorbunov's Theorem [**1**, Theorem 4.3.5] which says that $0 \in \langle \phi_n, 2, \eta\rangle$. Therefore, any element of $\langle \phi_n, 2, \eta\rangle$ is of the form ηA. By Lemma 5.1 and the observation that $\eta A \cdot MSp_4 = 0$, we see that $\langle \phi_n, 2, \phi_1\rangle$ contains an element which is also an element of

$$\phi_n \cdot MSp_6 + A\phi_1 \textit{ modulo } \operatorname{Image} \mu_*.$$

This sum is contained in the ideal spanned by ϕ_1 modulo Image μ_*. Thus, for all n, we conclude that $\langle \phi_n, 2, \phi_1\rangle$ contains an element which is in Image μ_*. By Lemma 5.2,

$$s_{\Delta_{2n}}\langle \phi_{2n}, 2, \phi_1\rangle \subset \langle \phi_n, 2, \phi_1\rangle.$$

Recall that an element in the image of the unit μ_* of MSp is annihilated by all Landweber-Novikov operations. It follows that $\langle \phi_n, 2, \phi_1\rangle$ contains zero. ■

The main technique which we use in constructing the $t[J]$ is the existence of Sp-manifolds V_j as in the following lemma. The proof of this lemma is based upon Theorem 1. We abuse notation below by denoting a cobordism class ϕ_n and an Sp-manifold representing ϕ_n by the same symbol ϕ_n.

Lemma 5.3. *There are Sp-manifolds ψ_n for $n \geq 0$ and V_n for $n \geq 2$ such that*

$$\begin{aligned} \partial\psi_1 &= \phi_1 \times 2, \\ \partial\psi_n &= 2 \times \phi_n \textit{ for } n \neq 1, \\ \partial V_n &= \psi_1 \times \phi_n \cup \phi_1 \times \psi_n. \end{aligned}$$

In particular, ψ_1 does not depend on n.

Proof: By Theorem 1, there are Sp-manifolds $\psi_1^{(n)}$, ψ_n for $n \geq 1$ and V_n' for $n \geq 2$ such that $\partial\psi_1^{(n)} = \phi_1 \times 2$, $\partial\psi_n = 2 \times \phi_n$ and $\partial V_n' = \psi_1^{(n)} \times \phi_n \cup \phi_1 \times \psi_n$. Let $\psi_1 = \psi_1^{(2)}$. Since $MSp_6 = \mathbb{Z}/2\eta\phi_1$, $\psi_1 \cup -\psi_1^{(n)}$ is bordant to $k_n\eta\phi_1$ for some $k_n \in \mathbb{Z}/2$. By Theorem 1,

$$0 = \langle \phi_n, 2, \phi_1 \rangle 2 = \phi_n \langle 2, \phi_1, 2 \rangle = \phi_n\phi_1\eta$$

in MSp_* noting that $\langle 2, \phi_1, 2 \rangle = \phi_1 [\Delta(2)] = \phi_1\eta$ by [**2**, Lemma 3.3 and Note 3.1]. Thus, there exists an Sp-manifold Y_n with

$$\partial Y_n = \psi_1 \times \phi_n \cup -\psi_1^{(n)} \times \phi_n.$$

Define $V_n = V_n' \cup Y_n$. Then $\partial V_n = \phi_1 \times \psi_n \cup \psi_1 \times \phi_n$ as required. ■

We are now ready to construct $t[J] \in MSp_*^{\widehat{\Sigma}_3}$. We denote the product construction of $\widehat{\Sigma}_3$-manifolds by $\widehat{\mathfrak{m}}_3$, the associativity construction by $\widehat{\mathfrak{A}}_3$ and the commutativity construction by $\widehat{\mathfrak{K}}_3$.

Proposition 5.4. *For each $J = [j_1, \dots, j_s]$, there exists an element $t[J] \in MSp_*^{\widehat{\Sigma}_3}$ with the following properties.*

(a) $t[j_1] = \phi_{j_1}$.
(b) $\psi_1 t[J] = 0$.
(c) $t[j_1, \dots, j_s] \in \langle \phi_{j_s}, \psi_1, t[j_1, \dots, j_{s-1}] \rangle$ *for* $s \geq 2$.
(d) *If* $j_k = 2^{i_k - 2}$ *for* $1 \leq k \leq s$ *and* $\mathfrak{i} = (i_1, \dots, i_s)$ *then* $\lambda_{3*}(t[J]) = \tau_3(\mathfrak{i})$ *under the canonical map* $\lambda_3 : MSp^{\widehat{\Sigma}_3} \to MSp^{\Sigma_3}$.
(e) $t[J]$ *projects to*

$$\mathfrak{t}[J] = \sum_{k=1}^{s} \Phi_{j_k} V_{1,j_1} \dots \widehat{V}_{1,j_k} \dots V_{1,j_s}.$$

in both $\mathbb{E}_2^{1,4*+1}$ *of the ASS for* $MSp_*^{\Sigma_3}$ *and* $\mathbb{E}_2^{1,4*+1}$ *of the ASS for* $MSp_*^{\widehat{\Sigma}_3}$.

(f) *There are* $\nu_0(j) \in MSp_*^{\widehat{\Sigma}_2}$ *which project to the infinite cycles* $h_0 V_{1,j}$ *in* $\mathbb{E}_2^{1,4*}$ *of the ASS for* $MSp^{\widehat{\Sigma}_2}$ *such that for* $s \geq 2$,

$$2t[J] = \widehat{\mathfrak{m}}_3(\nu_0(j_s), t[J']).$$

Proof: (a)-(c) We construct the $t\,[j_1,\dots,j_s]$ by induction on $s\geq 1$ to satisfy (a)-(c) as in the proof of [**2**, Lemma 5.3].

(d) To insure that the $t[J]$ map to the $\tau_3(\mathfrak{i})$ under λ_{3*} we must be careful how we choose $t[J]$ in the Toda bracket of (c). In particular, for each sequence $[j_1,\dots,j_s]$ we proceed as in the proof of [**2**, Lemma 5.4] to use induction on $s\geq 1$ to define $\widehat{\Sigma}_3$-manifolds H_s and T_s such that:

(1) $T_1=\phi_{j_1}$ and $H_1=V_{j_1}$;

(2) $\delta H_s=\widehat{\mathfrak{m}}_3\,(\psi_1,T_s)$;

(3) For $s\geq 2$,

$$\begin{aligned}T_s&=\phi_{j_s}\times H_{s-1}\cup\widehat{\mathfrak{m}}_3\left(V'_{j_s},T_{s-1}\right),\\ H_s&=\widehat{\mathfrak{m}}_3\,(V_{j_s},H_{s-1})\cup-\widehat{\mathfrak{A}}_3\,(V_{j_s},\psi_1,T_{s-1})\cup-\widehat{\mathfrak{m}}_3(\widehat{\mathfrak{K}}_3(V_{j_s},\psi_1),T_{s-1})\\ &\qquad\cup\widehat{\mathfrak{m}}_3(B\times\phi_{j_s},T_{s-1})\cup\widehat{\mathfrak{A}}_3(\psi_1,V'_{j_s},T_{s-1})\end{aligned}$$

where $V'_{j_s}=V_{j_s}\cup\widehat{\mathfrak{K}}_3(\phi_{j_s},\psi_1)$ and B is a $\widehat{\Sigma}_3$-manifold with $\delta(B)=\widehat{\mathfrak{K}}_3(\psi_1,\psi_1)$. Such a $\widehat{\Sigma}_3$-manifold B exists because $MSp_{13}^{\widehat{\Sigma}_3}=0$. By [**2**, Lemma 5.4], $t[J]$ defined as the $\widehat{\Sigma}_3$-cobordism class of T_s maps under λ_{3*} to $\tau_3(\mathfrak{i})$.

(e) By induction on $s\geq 1$, we prove that T_s projects to $t\,[j_1,\dots,j_s]\in\mathbb{E}_2^{1,4*+1}$ and H_s projects to $V_{1,j_1}\dots V_{1,j_s}\in\mathbb{E}_2^{0,4*}$ in the ASS for $MSp^{\widehat{\Sigma}_3}$. The case $s=1$ follows from (1). If $s\geq 2$, the induction hypothesis and (3) show that the projection of T_s to the one line of the ASS equals

$$\phi_{j_s}V_{1,j_1}\dots V_{1,j_{s-1}}+V_{1,j_s}\mathfrak{t}\,[j_1,\dots,j_{s-1}]=\mathfrak{t}\,[j_1,\dots,j_s]\,.$$

Since ψ_1, T_{s-1} and ϕ_{j_s} have Adams filtration degree one, the projections of the manifolds $\widehat{\mathfrak{A}}_3\,(V_{1,j_s},\psi_1,T_{s-1})$, $\widehat{\mathfrak{m}}_3(\widehat{\mathfrak{K}}_3(V_{j_s},\psi_1),T_{s-1})$, $\widehat{\mathfrak{m}}_3(B\times\phi_{j_s},T_{s-1})$ and $\widehat{\mathfrak{A}}_3(\psi_1,V'_{1,j_s},T_{s-1})$ to the zero line of the ASS are trivial. Thus by (3), the projection of H_s to the zero line of the ASS equals the projection of $\widehat{\mathfrak{m}}_3(V_{j_s},H_{s-1})$ which by the induction hypothesis is $V_{1,j_s}\cdot V_{1,j_1}\dots V_{1,j_{s-1}}$.

(f) The element $2t[J]$ is represented by the manifold

$$2\phi_{j_s}\times H_{s-1}\cup 2\widehat{\mathfrak{m}}_3(V'_{j_s},T_{s-1})\cup-\delta\,(\psi_{j_s}\times H_{s-1})$$

which is bordant to

$$\widehat{\mathfrak{m}}_3\left(2V'_{j_s}\cup\psi_{j_s}\times\psi_1,T_{s-1}\right)=\widehat{\mathfrak{m}}_3(\nu_0\,(j_s)\,,T_{s-1})$$

where $\nu_0\,(j_s)$ is defined as the $\widehat{\Sigma}_2$-cobordism class of $2V'_{j_s} \cup \psi_{j_s} \times \psi_1$ which projects to $h_0 V_{1,j_s}$ in $\mathbb{E}_2^{1,4*}$ of the ASS for $MSp^{\widehat{\Sigma}_2}$. ■

Consider the $T[J]$, $H[J]$ constructed above as a $\widehat{\Sigma}_2$-manifold $\widetilde{T}[J]$, $\widetilde{H}[J]$, respectively. Then

$$\delta\widetilde{H}[J] = \widehat{\mathfrak{m}}_2\left(\psi_1, \widetilde{T}[J]\right) \cup \phi_2 \times E[J],$$
$$\delta\widetilde{T}[J] = \phi_2 \times G[J],$$
$$\delta E[J] = \widehat{\mathfrak{m}}_2\,(\psi_1, G[J])$$

where $G[J] = \widehat{\beta}_3(T[J])$ represents the $\widehat{\Sigma}_2$-cobordism class $g[J]$. To identify the projection of $g[J]$ into the ASS we need to know the projection of $E[J]$ into the ASS.

Lemma 5.5.
(a) *$E[j_1] = \emptyset$ and $E[j_1, j_2] = \phi_{j_1}\phi_{j_2}$.*
(b) *For $s \geq 2$, $E[J]$ projects in $E_2^{2,4*+2}$ of the ASS for MSp^{Σ_2} and in $E_2^{2,4*+2}$ of the ASS for $MSp^{\widehat{\Sigma}_2}$ to*

$$\mathfrak{e}\,[j_1, \dots, j_s] = \sum_{1 \leq t_1 < t_2 \leq s} \Phi_{j_{t_1}} \Phi_{j_{t_2}} V_{1,j_1} \dots \widehat{V}_{1,j_{t_1}} \dots \widehat{V}_{1,j_{t_2}} \dots V_{1,j_s}.$$

Proof: (a) We can take $T[j_1] = \phi_{j_1}$ and $H[j_1] = V_{j_1}$ as a $\widehat{\Sigma}_2$-manifold with $\delta V_{j_1} = \psi_1 \times \phi_{j_1}$. Thus, $E[j_1] = \emptyset$. It will follow from (iii) below that $E[j_1, j_2] = \phi_{j_2}\phi_{j_1}$.

(b) Observe that just as in the proof of [**2**, Lemma 6.2(b)], we can use induction on $s \geq 2$ to construct $\widehat{\Sigma}_2$-manifolds $\widetilde{T}_s$, $\widetilde{H}_s$, E_s and $\widetilde{L}_s$ such that:

(i) $\widetilde{T}_s$ represents $t[J]$;
(ii) $\delta\widetilde{H}_s = \widehat{\mathfrak{m}}_2\left(W_2, \widetilde{T}_s\right) \cup \phi_2 \times E_s \cup \widetilde{L}_s$;
(iii) $E[J] = \widehat{\mathfrak{m}}_2\left(V'_{j_s}, E[J']\right) \cup \phi_{j_s} \times \widetilde{T}(j_1, j_2)$;
(iv) $\widetilde{T}_s$ projects in the one line of the ASS for $MSp^{\widehat{\Sigma}_2}$ and in the one line of the ASS for MSp^{Σ_2} to $\mathfrak{t}[J] \in \mathbb{E}_2^{1,4*+1}$;
(v) $\widetilde{H}_s$ projects in the zero line of the ASS for $MSp^{\widehat{\Sigma}_2}$ and in the zero line of the ASS for MSp^{Σ_2} to $V_{1,j_1} \dots V_{1,j_s} \in \mathbb{E}_2^{0,4*}$;
(vi) E_s projects in the two line of the ASS for $MSp^{\widehat{\Sigma}_2}$ and in the two line of the ASS for MSp^{Σ_2} to

$$\sum_{1 \leq t_1 < t_2 \leq s} \Phi_{j_{t_1}} \Phi_{j_{t_2}} V_{1,j_1} \dots \widehat{V}_{1,j_{t_1}} \dots \widehat{V}_{1,j_{t_2}} \dots V_{1,j_s} \in \mathbb{E}_2^{2,4*+2};$$

(vii) $\widetilde{L}_s$ has Adams filtration degree four. ■

Using this lemma, we determine the basic properties of the $g[J]$.

Proposition 5.6. *The elements* $g[J] = \widehat{\beta}_3(t[J]) \in MSp_*^{\widehat{\Sigma}_2}$ *satisfy the following conditions.*

(a) $g[j_1] = g[j_1, j_2] = 0$.
(b) $g[j_1, j_2, j_3] = \phi_{j_1}\phi_{j_2}\phi_{j_3}$.
(c) $\psi_1 g[J] = 0$.
(d) $g[J] \in \langle \phi_{j_s}, \psi_1, g[J'] \rangle$ *for* $s \geq 4$.
(e) *For* $s \geq 3$, $g[J]$ *projects in* $\mathbb{E}_2^{3,4*+3}$ *of the ASS for* MSp^{Σ_2} *and in* $\mathbb{E}_2^{3,4*+3}$ *of the ASS for* $MSp^{\widehat{\Sigma}_2}$ *to*

$$\mathfrak{g}[J] = \sum_{1 \leq t_1 < t_2 < t_3 \leq s} \Phi_{j_{t_1}}\Phi_{j_{t_2}}\Phi_{j_{t_3}} V_{1,j_1} \ldots \widehat{V}_{1,j_{t_1}} \ldots \widehat{V}_{1,j_{t_2}} \ldots \widehat{V}_{1,j_{t_3}} \ldots V_{1,j_s}.$$

(f) $2g[J] = \widehat{\mathfrak{m}}_2(\nu_0(j_s), g[J'])$ *for* $s \geq 2$.

Proof: (a)-(d) These statements are proved in the same way as the analogous statements in [**2**, Proposition 6.3(a)-(c)]. In particular, $g[J] = \widehat{\beta}_3(t[J])$ is represented by the $\widehat{\Sigma}_2$-manifold

$$G[J] = \widehat{\mathfrak{m}}_2\left(V'_{j_s}, G[J']\right) \cup \phi_{j_s} \times E[J']. \tag{16}$$

(e) We use induction on $s \geq 3$. The case $s = 3$ follows from (b). Assume the case $s - 1$. By (16), $g[J]$ projects in the three line of the ASS to

$$\mathfrak{g}[J] = V_{1,j_s}\mathfrak{g}[J'] + \Phi_{j_s}\mathfrak{e}[J'].$$

By the induction hypothesis and the previous lemma,

$$\begin{aligned}\mathfrak{g}[J] = V_{1,j_s} &\sum_{1 \leq t_1 < t_2 < t_3 \leq s-1} \Phi_{j_{t_1}}\Phi_{j_{t_2}}\Phi_{j_{t_3}} V_{1,j_1} \ldots \widehat{V}_{1,j_{t_1}} \ldots \widehat{V}_{1,j_{t_2}} \ldots \widehat{V}_{1,j_{t_3}} \ldots V_{1,j_{s-1}} \\ + \Phi_{j_s} &\sum_{1 \leq t_1 < t_2 \leq s-1} \Phi_{j_{t_1}}\Phi_{j_{t_2}} V_{1,j_1} \ldots \widehat{V}_{1,j_{t_1}} \ldots \widehat{V}_{1,j_{t_2}} \ldots V_{1,j_{s-1}}.\end{aligned}$$

This is the asserted value of $\mathfrak{g}[J]$ in (e).

(f) By (16), $2g[J]$ is represented by the manifold

$$2G[J] = 2\widehat{\mathfrak{m}}_2\left(V'_{j_s}, G[J']\right) \cup 2\phi_{j_s} \times E[J'] \cup -\delta\left(\psi_{j_s} \times E[J']\right)$$

which is bordant to $\widehat{\mathfrak{m}}_2\left(2V'_{j_s} \cup \psi_{j_s} \times \psi_1, G[J']\right) = \widehat{\mathfrak{m}}_2(\nu_0(j_s), G[J'])$. ■

6. Elements of Higher Torsion and the ASS

In this section we analyze the elements

$$a[J] = \widehat{\beta}_2(g[J]) = \widehat{\beta}_2(\widehat{\beta}_3(t[J])) \in MSp_*.$$

In particular, we give their decomposition in terms of four-fold Toda brackets and identify their projections to the ASS. These results are summarized by Theorem 2. In addition, we shall see that the projections of the $2^k a[J]$, $k \geq 0$, to $\mathbb{E}_2$ of the ASS for MSp determine towers whose top halves are zero in $\mathbb{E}_\infty$. Throughout this section $J = [j_1, \dots, j_s]$ with $s \geq 1$ and $J' = [j_1, \dots, j_{s-1}]$ with $s \geq 2$.

We begin by determining the projection of the $a[J]$ to the ASS for MSp. We will use the following notation from [**5**, Definition 7.12(19a)]. Let $H = (h_1, \dots, h_k)$. Assume that $r \geq k$, $s \geq 2r - k + 3$ and $s - k$ is even. Then the following elements of $\mathbb{E}_2$ are d_2-cycles in the ASS for MSp:

$$\zeta^r\,(H)\,Y\,(j_1, \dots, j_s) = \sum Y(j_1, \dots, \widehat{j}_{t_1}, \dots, \widehat{j}_{t_{2r-k}}, \dots, j_s) \\ V_{j_{t_1}, h_1} \dots V_{j_{t_k}, h_k} V_{j_{t_{k+1}}, j_{t_{k+2}}} \dots V_{j_{t_{2r-k-1}}, j_{t_{2r-k}}}$$

where this sum is taken over all sequences $(t_1, \dots, t_{2r-k})$ of distinct integers between 1 and s such that $t_1 < \dots < t_k$, $t_{k+1} < t_{k+3} < \dots < t_{2r-k-1}$ and $t_{k+2q-1} < t_{k+2q}$ for $1 \leq q \leq r - k$. We introduce the following notation for the particular elements of this family which we will be studying.

$$\begin{aligned} \mathfrak{a}[J] &= \zeta^{s-4}\left(1^{s-4}\right) Y\,(j_1, \dots, j_s) \\ &= \sum_{1 \leq t_1 < t_2 < t_3 < t_4 \leq s} V_{1,j_1} \dots \widehat{V}_{1,j_{t_1}} \dots \widehat{V}_{1,j_{t_4}} \dots V_{1,j_s} Y\,(j_{t_1}, j_{t_2}, j_{t_3}, j_{t_4})\,. \end{aligned}$$

To describe the projections of the $2^k a[J]$ in $\mathbb{E}_2$ of the ASS for MSp we introduce the following notation. For $0 \leq k \leq s-4$, let

$$\mathfrak{a}_k[J] = \zeta^{s-k-4}\left(1^{s-k-4}\right) Y\left(1^k, j_1, \dots, j_s\right).$$

Note that $\mathfrak{a}_0[J] = \mathfrak{a}[J]$.

Proposition 6.1. *Let $s \geq 4$ and $0 \leq k \leq s-4$. Then*

(a) *$a[J]$ projects to the infinite cycle $\mathfrak{a}[J]$ in $\mathbb{E}_2^{4,4*+1}$ of the ASS for MSp;*

(b) *$2^k a[J]$ projects to the infinite cycle $\mathfrak{a}_k[J]$ in $\mathbb{E}_2^{2k+4,4*+1}$ of the ASS for MSp.*

Proof: (a) Let $G_0[J]$ denote $G[J]$ viewed as an *Sp*-manifold. Since $G_0[J]$ is a representative manifold of $g[J]$ and $a[J] = \widehat{\beta}_2(g[J])$,

$$\partial G_0[J] = \phi_1 \times A[J]$$

where $A[J]$ is a representative manifold of $a[J]$. By Proposition 5.6(e), $G[J]$ projects in $E_2^{3,4*+3}$ of the ASS for MSp to

$$\mathfrak{g}_0 = \sum_{1\le t_1<t_2<t_3\le s} \Phi_{j_{t_1}}\Phi_{j_{t_2}}\Phi_{j_{t_3}} V_{1,j_1}\dots\widehat{V}_{1,j_{t_1}}\dots\widehat{V}_{1,j_{t_3}}\dots V_{1,j_s} + \Phi_1 X.$$

Therefore, $d_2(\mathfrak{g}_0)$ is equal to

$$\begin{aligned}&\sum_{1\le t_1<t_2<t_3<t_4\le s} \Phi_{j_{t_1}}\Phi_{j_{t_2}}\Phi_{j_{t_3}} P(1,j_{t_4}) V_{1,j_1}\dots\widehat{V}_{1,j_{t_1}}\dots\widehat{V}_{1,j_{t_4}}\dots V_{1,j_s} + \Phi_1 d_2(X)\\ &= \Phi_1\left(\sum_{1\le t_1<t_2<t_3<t_4\le s} V_{1,j_1}\dots\widehat{V}_{1,j_{t_1}}\dots\widehat{V}_{1,j_{t_4}}\dots V_{1,j_s} Y(j_{t_1},j_{t_2},j_{t_3},j_{t_4}) + d_2(X)\right)\\ &= \Phi_1\left(\mathfrak{a}[J] + d_2(X)\right).\end{aligned}$$

Since multiplication by Φ_1 is a monomorphism on $\mathbb{E}_2^{4,4*+1}$ of the ASS for MSp and d_2-boundaries project to zero in $\mathbb{E}_3$, $a[J]$ projects to $\mathfrak{a}[J]$.

(b) We prove (b) by induction on k. (a) gives the case $k = 0$. Assume that (b) is true for some k with $0 \le k \le s-5$. We show that in $\mathbb{E}_\infty$ of the ASS of MSp twice $\mathfrak{a}_k[J]$ is equal to $\mathfrak{a}_{k+1}[J]$ by a nontrivial extension of degree one. We apply [**6**, Theorem 12.2] to

$$\begin{aligned}Z = &\sum_{1\le t_1<\dots<t_{s-k-4}\le s} V_{1,j_{t_1}}\dots V_{1,j_{t_{s-k-4}}} Y\left(N, 1^k, j_1,\dots,\widehat{j}_{t_1},\dots,\widehat{j}_{t_{s-k-4}},\dots,j_s\right)\\ &\in \left\langle \zeta^{s-k-4}\left(1^{s-k-4}\right) Y\left(1^k, j_1,\dots,j_s\right), h_0, \Phi_N\right\rangle\end{aligned}$$

in $\mathbb{E}_2$ of the ASS for MSp. Then

$$d_2(Z) = \Phi_N \zeta^{s-k-5}\left(1^{s-k-5}\right) Y\left(1^{k+1}, j_1,\dots,j_s\right).$$

The annihilator ideal of $\{\Phi_N \mid N \ge 0\}$ in $\mathbb{E}_2$ of the ASS for MSp is the ideal spanned by h_0, and the latter ideal is zero in $\mathbb{E}_2^{2*,4*+1}$. Thus, twice $\mathfrak{a}_k[J]$, the projection of $2^{k+1}a[J]$, equals $\mathfrak{a}_{k+1}[J]$ by a nontrivial extension of degree one. ■

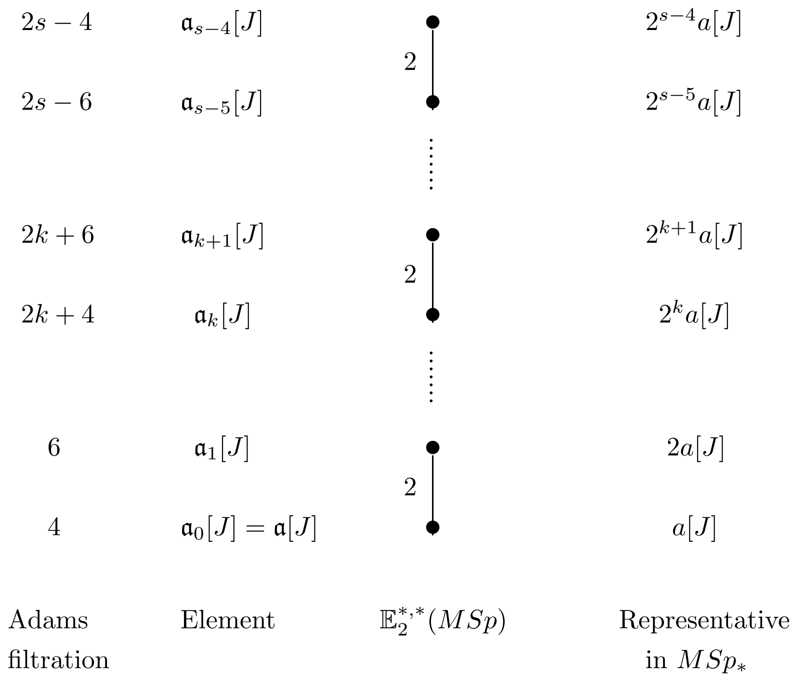


Figure 2: Higher Torsion in $\mathbb{E}_2$ of the Adams Spectral Sequence

Figure 2 illustrates how the $2^k a[J]$, $0 \leq k \leq s-4$, $5 \leq s$, project to the tower of elements $\mathfrak{a}_k[J]$ in $\mathbb{E}_2^{*,4*+1}$ of the ASS for MSp.

We use our understanding of the $g[J]$ from Section 5 and our identification of the projection of the $a[J]$ to the ASS to prove Theorem 2.

Proof of Theorem 2:

(a) It follows from Proposition 5.6(a),(b) that $a\,[j_1, \ldots, j_s] = 0$ for $s \leq 3$.

(c) By Proposition 6.1(a), $a\,[2j_1, \ldots, 2j_s]$ projects to

$$\mathfrak{a}\,[2j_1, \ldots, 2j_s] = \zeta^{s-4}\left(1^{s-4}\right) Y\left(2j_1, \ldots, 2j_s\right)$$

in $\mathbb{E}_2^{4,4*+1}$ of the ASS for MSp. Since the $V_{1,2j_1}, \ldots, V_{1,2j_s}$ are special choices of the distinct polynomial generators $V_{4j_1+1}, \ldots, V_{4j_s+1}$ of S, $\mathfrak{a}\,[2j_1, \ldots, 2j_s]$ is indecomposable in $\mathbb{E}_3^{4,4*+1}$ for $s \geq 5$. Since $\mathbb{E}_2^{1,4*+2} = \mathbb{E}_2^{0,4*+2} = 0$, no d_r-boundary, $r \geq 3$, can land in $\mathbb{E}_r^{4,4*+1}$. Therefore, $a\,[2j_1, \ldots, 2j_s]$ projects to an indecomposable element of $\mathbb{E}_\infty^{4,4*+1}$ and must be indecomposable in MSp_*.

(b), (d) We have Sp-manifolds ψ'_1, ψ'_{j_s} and V'_{j_s} such that

$$\begin{aligned}\partial\psi'_1 &= 2 \times \phi_1;\\ \partial\psi'_{j_s} &= \phi_{j_s} \times 2;\\ \partial V'_{j_s} &= \psi'_{j_s} \times \phi_1 \cup \phi_{j_s} \times \psi'_1;\\ \partial G_0[J] &= \phi_1 \times A[J]\end{aligned}$$

where $A[J]$ is an Sp-manifold which represents $a[J]$. Let $F[J] = \widehat{\beta}_2\left(E[J]\right)$. Let $s \geq 5$. Since $\delta E[J] = \widehat{\mathfrak{m}}_2\left(\psi_1, G[J]\right)$,

$$\partial F[J] = 2 \times G_0[J] \cup -\psi'_1 \times A[J].$$

Thus, $\phi_1 a[j] = 0$ and we have the following defining system for $\langle \phi_{j_s}, 2, \phi_1, a[J']\rangle$:

$$\begin{array}{ccccccc}\phi_{j_s} & & 2 & & \phi_1 & & A[J'] \\ & \psi'_{j_s} & & -\psi'_1 & & G_0[J'] & \\ & & -V'_{j_s} & & -F[J'] & & \end{array}$$

By Proposition 5.6(d), $G[J] = -\phi_{j_s} \times E[J'] \cup -\widehat{\mathfrak{m}}_2(V'_{j_s}, G[J'])$. Therefore, $a[J]$ is represented by

$$\begin{aligned}A[J] &= \widehat{\beta}_2\left(G[J]\right)\\ &= -\phi_{j_s} \times F[J'] \cup -\psi'_{j_s} \times G_0[J'] \cup V'_{j_s} \times A[J'] \in \langle \phi_{j_s}, 2, \phi_1, a[J']\rangle.\end{aligned}$$

When $s = 4$ we have $A[J'] = \emptyset$ and $G_0[J'] = \phi_{j_3} \times \phi_{j_2} \times \phi_{j_1}$. Since $\widetilde{T}(j_1, j_2) = (\phi_{j_2} \times V_{j_1}) \cup \widehat{\mathfrak{m}}_3(V'_{j_2}, \phi_{j_1})$, we have $\beta_2(\widetilde{T}(j_1, j_2)) = p(j_2, j_1)$ where

$$p(m, n) = (\psi'_m \times \phi_n) \cup (\phi_m \times \psi_n) \in \langle \phi_m, 2, \phi_n\rangle.$$

Using (iii) from the proof of Lemma 5.5, we have

$$\begin{aligned}F[J'] = \beta_2(E[J']) &= \beta_2\left(\mathfrak{m}_2(V'_{j_3}, \phi_{j_2} \times \phi_{j_1}) \cup \phi_{j_3} \times \widetilde{T}(j_1, j_2)\right)\\ &= (\psi_{j_3} \times \phi_{j_2} \times \phi_{j_1}) \cup (\phi_{j_3} \times p(j_2, j_1)).\end{aligned}$$

Thus,

(17)
$$\begin{aligned}A[J] &= -\phi_{j_4} \times [\psi_{j_3} \times \phi_{j_2} \times \phi_{j_1} \cup \phi_{j_3} \times p(j_2, j_1)] \cup -(\psi'_{j_4} \times \phi_{j_3} \times \phi_{j_2} \times \phi_{j_1})\\ &= -(p(j_4, j_3) \times \phi_{j_2} \times \phi_{j_1}) \cup -(\phi_{j_4} \times \phi_{j_3} \times p(j_2, j_1))\\ &\sim (\phi_{j_1} \times \phi_{j_2} \times p(j_3, j_4)) \cup (p(j_1, j_2) \times \phi_{j_3} \times \phi_{j_4}).\end{aligned}$$

(e) Let $j_k = 2^{i_k - 2}$ for $1 \le k \le s$, let $\mathfrak{i} = (i_1, \dots, i_s)$ and let $J = [j_1, \dots, j_s]$. Consider the following commutative diagram.

$$\begin{array}{ccccc}
MSp_*^{\widehat{\Sigma}_3} & \xrightarrow{\widehat{\beta}_3} & MSp_*^{\widehat{\Sigma}_2} & \xrightarrow{\widehat{\beta}_2} & MSp_* \\
\downarrow{\scriptstyle \lambda_{3*}} & & \downarrow{\scriptstyle \lambda_{2*}} & & \downarrow{\scriptstyle \pi_{1*}} \\
MSp_*^{\Sigma_3} & \xrightarrow{\beta_3} & MSp_*^{\Sigma_2} & \xrightarrow{\beta_2} & MSp_*^{\Sigma_1}
\end{array}$$

By Proposition 5.4, the definition of $\gamma(\mathfrak{i})$ in [**2**, Proposition 6.3] and the definition of $\alpha'(\mathfrak{i})$ in [**2**, Proposition 6.4], it follows that

$$\begin{aligned}
\pi_{1*}(a[J]) &= \pi_{1*}(\widehat{\beta}_2(\widehat{\beta}_3(t[J]))) = \beta_2(\beta_3(\lambda_{3*}(t[J]))) = \beta_2(\beta_3(\tau_3(\mathfrak{i}))) \\
&= \beta_2(\gamma(\mathfrak{i})) = \alpha'(\mathfrak{i})
\end{aligned}$$

By [**2**, Proposition 7.1(ii)], $\alpha'(\mathrm{i})$ has order at least $2^{[(s+1)/2]-2}$, and therefore $a[J]$ also has order at least $2^{[(s+1)/2]-2}$. ■

Our canonical representative $A[j_1, j_2, j_3, j_4]$ of $a[j_1, j_2, j_3, j_4]$ projects to $\mathfrak{a}[j_1, j_2, j_3, j_4] \in \mathbb{E}_2^{4,4*+1}$ which is a d_2-boundary. In fact, $a[j_1, j_2, j_3, j_4]$ has a representative which projects to a nonbounding infinite cycle in $\mathbb{E}_2^{5,4*+1}$. To describe this element let $\epsilon(m,n)$ denote the projection of $\langle \phi_m, 2, \phi_n \rangle$ to $\mathbb{E}_2^{3,4*-1}$ of the Adams spectral sequence. By [**6**, Thm. 8.13(c)], these $\epsilon(m,n)$, for $(m,n) \ge (3,5)$ in the lexicographical order, are nonbounding infinite cycles which are represented in $\mathbb{E}_2$ by S-linear combinations of the elements $\Phi_a \Phi_b \Phi_c$.

Corollary 6.2. *The element $a[j_1, j_2, j_3, j_4]$ has a representative in $F^5 MSp_*$ which projects to the infinite cycle*

$$\Phi_{j_1} \Phi_{j_2} \epsilon(j_3, j_4) + \Phi_{j_3} \Phi_{j_4} \epsilon(j_1, j_2)$$

in $\mathbb{E}_2^{5,4*+1}$ of the Adams spectral sequence.

Proof: $p(m,n)$ projects to $d_2(V_{m,n}) \in E_2^{2,8m+8n-5}$. Consider the d_2-cycle

$$\Sigma(1,m,n) = \Phi_1 V_{m,n} + \Phi_m V_{1,n} + \Phi_n V_{1,m}.$$

By [**6**, Thm. 8.13], $d_3(\Sigma(1,m,n)) = \Phi_1 \epsilon(m,n)$ where $\epsilon(m,n) \in E_3^{3,8m+8n-5}$ is an infinite cycle. It follows that $V_{m,n}$ is represented by a symplectic manifold $\nu_{m,n}$ such that

$$\partial(\nu_{m,n}) = p(m,n) \cup \epsilon'(m,n)$$

where $\epsilon'(m,n)$ is a closed symplectic manifold of Adams filtration degree three which projects to $\epsilon(m,n)$ in the Adams spectral sequence. By (17),

$$A[j_1,j_2,j_3,j_4] \cup \partial\left((\phi_{j_4} \times \phi_{j_3} \times \nu(j_2,j_1)) \cup (\nu(j_4,j_3) \times \phi_{j_2} \times \phi_{j_1})\right)$$

is a closed symplectic manifold of Adams filtration degree five which projects to the infinite cycle $\Phi_{j_1}\Phi_{j_2}\epsilon(j_3,j_4) + \Phi_{j_3}\Phi_{j_4}\epsilon(j_1,j_2)$ in $E_3^{5,4*+1}$ of the Adams spectral sequence. ■

When we multiply the Toda brackets for $a[J]$ by two, their length decreases.

Corollary 6.3. *For $s \geq 5$,*

$$2a[J] \in \langle \eta\phi_{j_s}, \phi_1, a[J'] \rangle, \quad 4a[J] = \nu\,(j_s)\,a[J']$$

where $\nu\,(j_s)$ projects to the infinite cycle $h_0^2 V_{1,j_s}$ in $\mathbb{E}_2^{2,8j_s+4}$ of the ASS for MSp.

Proof: Using the manifold $A[J]$ which we constructed in Theorem 2(d) to represent $a[J]$, we represent $2a[J]$ by

$$2A[J] = -2\phi_{j_s} \times F[J'] \cup -2\psi'_{j_s} \times G_0[J'] \cup 2V'_{j_s} \times A[J'] \cup \partial(\psi_{j_s} \times F[J']).$$

Thus, $2A[J]$ is bordant to

$$\begin{aligned} A_2[J] &= \left\{\psi_{j_s} \times 2 \cup -2\psi'_{j_s}\right\} \times G_0[J'] \cup \left\{-\psi_{j_s} \times \psi'_1 \cup 2V'_{j_s}\right\} \times A[J'] \\ &= B_{j_s} \times G_0[J'] \cup C_{j_s} \times A[J']. \end{aligned}$$

Since $\psi_{j_s} = \psi'_{j_s} \cup -\mathfrak{K}(\phi_{j_s},2)$, the Hirsch formula shows that $\mathfrak{K}(\psi'_{j_s},2)$ gives a cobordism between B_{j_s} and $-\phi_{j_s} \times \mathfrak{K}(2,2) = -\phi_{j_s} \times \eta$. In addition, $\partial\,(C_{j_s}) = -B_{j_s} \times \phi_1$. Thus, $2a[J]$ is represented by $A_2[J]$ which is in $\langle \phi_{j_s}\eta, \phi_1, a[J'] \rangle$. Then we can represent $4a[J]$ by the manifold

$$2A_2[J] \cup \left\{-2\mathfrak{K}(\psi'_{j_s},2) \cup \psi_{j_s} \times \eta\right\} \times G_0[J']$$

which is bordant to

$$A_4[J] = \left\{2C_{j_s} \cup -2\mathfrak{K}(\psi'_{j_s},2) \times \phi_1 \cup \psi_{j_s} \times \eta \times \phi_1\right\} \times A[J'] = D_{j_s} \times A[J'].$$

From the definition of C_{j_s}, we see that the cobordism class $\nu\,(j_s)$ of D_{j_s} projects to $h_0^2 V_{1,j_s} \in \mathbb{E}_2^{2,8j_s+4}$ of the ASS for MSp. ■

We conclude this section by showing that certain $\mathfrak{a}_k[J] = 0$ in $\mathbb{E}_\infty$ of the ASS for MSp. Note that all the $\mathfrak{a}_k\,[j_1,\ldots,j_s]$ are nonzero in $\mathbb{E}_2$ for $0 \leq k \leq s-4$. We begin by determining when $\mathfrak{a}[J] = 0$ in $\mathbb{E}_\infty^{4,4*+1}$. Since $\mathbb{E}_2^{0,4*+2} = \mathbb{E}_2^{1,4*+2} = 0$, $\mathfrak{a}[J]$ can only bound as a d_2-boundary.

Proposition 6.4. *(a) For $s \geq 3$,*

$$\mathfrak{a}\,[1, j_1, \ldots, j_s] = d_2 \left(\sum_{1 \leq h < k \leq s} \Phi_{j_h} \Phi_{j_k} V_{1,j_1} \ldots \widehat{V}_{1,j_h} \ldots \widehat{V}_{1,j_k} \ldots V_{1,j_s} \right).$$

(b) Let $j_1, \ldots, j_s$ be distinct even natural numbers with $s \geq 4$. Then the element $\mathfrak{a}\,[j_1, \ldots, j_s] = 0$ in $\mathbb{E}_\infty^{4,4*+1}$ if and only if $s = 4$.

Proof: (a) Let $\sum_D$ denote the sum of all distinct elements of the given form. Observe that

$$\begin{aligned}\mathfrak{a}\,[1, j_1, \ldots, j_s] &= \sum_D \Phi_{j_1} \Phi_{j_2} P\,(1, j_3)\, V_{1,j_4} \ldots V_{1,j_s} \\ &= d_2 \left(\sum_D \Phi_{j_1} \Phi_{j_2} V_{1,j_3} \ldots V_{1,j_s} \right).\end{aligned}$$

(b) When $s = 4$, $\mathfrak{a}\,[j_1, j_2, j_3, j_4] = d_2\,(P\,(j_1, j_2)\, V_{j_3,j_4} + P\,(j_3, j_4)\, V_{j_1,j_2})$. If $s \geq 5$ write $j_r = 2j'_r$. Then

$$\mathfrak{a}[j_1,\ldots,j_s] = \sum_{1 \leq t_1 < t_2 < t_3 < t_4 \leq s} Y(j_{t_1}, j_{t_2}, j_{t_3}, j_{t_4}) V_{1,j_1} \ldots \widehat{V}_{1,j_{t_1}} \ldots \widehat{V}_{1,j_{t_4}} \ldots V_{1,j_s}$$

which can not be a d_2-boundary because $V_{1,j_1} = V_{1,2j'_1} = V_{2j_1+1}, \ldots, V_{1,j_s} = V_{1,2j'_s} = V_{2j_s+1}$ are distinct indecomposable elements of $\mathbb{E}_2^{0,4*}$. ■

Theorem 2 implies that when the entries of J are distinct powers of two then the elements in the bottom half of the tower in Figure 2 represent nonzero elements of MSp_*. We will show that all of the remaining elements in the top half of the tower in Figure 2 are boundaries in the ASS. We begin by introducing notation that we will need to describe specific elements in the ASS for MSp. Recall the d_2-cycles $\Sigma(a, b, c) \in \mathbb{E}_2^{1,4*+1}$ which were defined in (9).

Let $A_1, B_1, \ldots, A_n, B_n$ be a sequence such that each (A_k, B_k) equals either (1) a pair of non-negative integers, (2) $(1, \Sigma\,(1, x, y))$ or (3) $(0, \Sigma\,(1, x, y))$. Let $V_{1,\Sigma(1,x,y)} = V_{1,x} V_{1,y}$ and $V_{0,\Sigma(1,x,y)} = V_{0,1} V_{x,y} + V_{0,x} V_{1,y} + V_{0,y} V_{1,x} + V_{1,x,y}$. Thus, in all three cases we have elements V_{A_k,B_k} in $\mathbb{E}_2$ of the ASS for MSp such that:

$$d_2\,(V_{A_k,B_k}) \in \langle A_k, h_0, B_k \rangle$$

where this Toda bracket is defined in $\mathbb{E}_2 = H_*(\mathfrak{P} \otimes S)$. Thus, as in [**5**, Definition 7.12(19a)], we can define the following elements of $\mathbb{E}_2^{2n-2k,4*+1}$:

$$\begin{aligned}&\zeta^k Y(A_1, B_1, \ldots, A_n, B_n)\\ &= \sum_{1\le j_1<\cdots<j_k\le n} Y\left(A_1, B_1, \ldots, \widehat{A}_{j_1}, \widehat{B}_{j_1}, \ldots, \widehat{A}_{j_k}, \widehat{B}_{j_k}, \ldots, A_n, B_n\right)\\ &\qquad\qquad V_{A_{j_1},B_{j_1}} \ldots V_{A_{j_k},B_{j_k}}\end{aligned}$$

where $2 \le n$ and $0 \le k \le n-2$.

Lemma 6.5. *All of the elements $\zeta^k Y(A_1, B_1, \ldots, A_n, B_n)$ are infinite cycles which are zero in $\mathbb{E}_\infty$ of the ASS for MSp where $2 \le n$ and $0 \le k \le n-2$.*

Proof: A proof analogous to that of Proposition 6.1(b), shows that for $1 \le k \le n-2$, twice $\zeta^k Y(A_1, B_1, \ldots, A_n, B_n)$ equals $\zeta^{k-1} Y(A_1, B_1, \ldots, A_n, B_n)$ by a nontrivial extension of degree one. Observe that

$$\zeta^{n-2} Y(A_1, B_1, \ldots, A_n, B_n) = d_2\left(\sum_{1\le i\le n} \Phi_{A_i}\Phi_{B_i} V_{A_1,B_1} \ldots \widehat{V}_{A_i,B_i} \ldots V_{A_n,B_n}\right).$$

By [**6**, Theorems 12.1,12.4], all the $\zeta^k Y(A_1, B_1, \ldots, A_n, B_n)$, $0 \le k \le n-2$, are boundaries in the ASS. Thus, they are infinite cycles which are zero in $\mathbb{E}_\infty$. ■

In the next two lemmas we identify the elements in the top half of the tower in Figure 2 in terms of various $\zeta^k Y(A_1, B_1, \ldots, A_n, B_n)$. We will use the following notation. Let $\mathfrak{i} = [i_1, \ldots, i_p]$ and $\mathfrak{k} = [k_1, \ldots, k_{2q}]$. Define

$$\mathfrak{i}' = [1, i_1, \ldots, 1, i_p] \text{ and } \mathfrak{k}'' = [1, \Sigma(1, k_1, k_2), \ldots, 1, \Sigma(1, k_{2q-1}, k_{2q})].$$

Lemma 6.6. *Let $J = [j_1, \ldots, j_{2t+\epsilon}]$ with $3 \le t$, $0 \le s \le t-2$ and $\epsilon = 0, 1$. Then*

$$\begin{aligned}&\mathfrak{a}_{2t-s+\epsilon-4}[J]\\ &\quad= \sum_{\alpha=0}^{s} \sum_{\beta=t-\alpha}^{t} \zeta^{s-\alpha} Y(\mathfrak{i}', j_{2t-2\alpha+\epsilon-3}, j_{2t-2\alpha+\epsilon-2}, \mathfrak{k}'', j_{2\beta+\epsilon-1}, j_{2\beta+\epsilon})\end{aligned}$$

where

$$\mathfrak{i} = [j_1, \dots, j_{2t-2\alpha+\epsilon-4}] \textit{ and}$$
$$\mathfrak{k} = \left[j_{2t-2\alpha+\epsilon-1}, \dots, \widehat{j}_{2\beta+\epsilon-1}, \widehat{j}_{2\beta+\epsilon}, \dots, j_{2t+\epsilon}\right].$$

Proof: Let Λ_k denote either Φ_k or Ψ_k. Consider the above double sum as an element $\mathfrak{D}$ of $\mathfrak{P} \otimes S$. Write $\mathfrak{D}$ as a polynomial in the canonical generators h_0, Φ_k, Ψ_k, $V_{a,b}$ and V_{2n} of $\mathfrak{P} \otimes S$. Then each monomial summand of $\mathfrak{D}$ in $\mathfrak{P} \otimes S$ has a factor of maximal length of the following form:

(18)
$$\Lambda_{j_{2i_1+\epsilon-1}} \Lambda_{j_{2i_1+\epsilon}} \cdots \Lambda_{j_{2i_p+\epsilon-1}} \Lambda_{j_{2i_p+\epsilon}} V_{j_{2k_1+\epsilon-1}, j_{2k_1+\epsilon}} \cdots V_{j_{2k_q+\epsilon-1}, j_{2k_q+\epsilon}}$$

where $i_1 < \cdots < i_p$, $k_1 < \cdots < k_q$. Each of the factors $\Lambda_{j_{2i_r+\epsilon-1}} \Lambda_{j_{2i_r+\epsilon}}$ in (18) comes from either $\mathfrak{i}'$, $j_{2t-2\alpha+\epsilon-3}$, $j_{2t-2\alpha+\epsilon-2}$ or $j_{2\beta+\epsilon-1}$, $j_{2\beta+\epsilon}$. Thus, $p \le t-\alpha$. The remaining $t-\alpha-p$ possible sources for factors $\Lambda_{j_{2i_r+\epsilon-1}} \Lambda_{j_{2i_r+\epsilon}}$ in (18) must be producing factors $V_{1,j_{2i_r+\epsilon-1}} V_{1,j_{2i_r+\epsilon}}$. The total number of such factors V_{A_h,B_h} in the summand with the factor (18) is $s-\alpha$. Thus, $t-\alpha-p \le s-\alpha \le t-\alpha-2$ and $2 \le p$. Each of the factors $V_{j_{2k_r+\epsilon-1}, j_{2k_r+\epsilon}}$ in (18) comes from either $j_{2t-2\alpha+\epsilon-3}$, $j_{2t-2\alpha+\epsilon-2}$, $\mathfrak{k}''$ or $j_{2\beta+\epsilon-1}, j_{2\beta+\epsilon}$. Thus,

(1) $k_1 \ge t-\alpha-1$;
(2) $i_{p-1} \le t-\alpha-1$;
(3) if $i_{p-1} = t-\alpha-1$ then $i_p = \beta$ and $i_{p-1} < k_1$.

Therefore, there are three types of factors (18):

(I) $q \ge 1$ and $i_{p-1} < k_1 < i_p$;
(II) $q \ge 1$ and $i_p < k_1$;
(III) $q = 0$.

Observe that each factor of type I occurs twice:

$$t-\alpha-1 = i_{p-1},\ \beta = i_p \text{ and } t-\alpha-1 = k_1,\ \beta = i_p.$$

Observe that each factor of type II occurs $2q$ times:

$$\begin{aligned} t-\alpha-1 &= i_p, & \beta &= k_r\ (1 \le r \le q) \\ t-\alpha-1 &= i_{p-1}, & \beta &= i_p \\ t-\alpha-1 &= k_1, & \beta &= k_r\ (2 \le r \le q). \end{aligned}$$

Observe that each factor of type III occurs once:

$$t-\alpha-1 = i_{p-1},\ \beta = i_p.$$

The sum of the summands with factors of type III give exactly

$$\sum_{1\le h_1<\cdots<h_s\le 2t-s+\epsilon} Y\Big(1^{2t-s+\epsilon-4}, j_1, \dots, \widehat{j}_{h_1}, \dots, \widehat{j}_{h_s}, \dots, j_{2t-s+\epsilon}\Big) V_{1,j_{h_1}} \dots V_{1,j_{h_s}}$$

$= \mathfrak{a}_{2t-s+\epsilon-4}\,(j_1, \dots, j_{2t+\epsilon})$. ∎

The next lemma gives the obstruction to extending Lemma 6.6 to $s = t-1$ and thereby bounding one more element of the tower in Figure 2.

Lemma 6.7. *Let* $J = [j_1, \dots, j_{2t+\epsilon}]$ *with* $4 \le t$ *and* $\epsilon = 0, 1$. *Then*

$$\mathfrak{a}_{t+\epsilon-3}[J] = \sum_{\alpha=0}^{t-2}\sum_{\beta=t-\alpha}^{t} \zeta^{t-\alpha-1} Y(\mathfrak{i}', j_{2t-2\alpha+\epsilon-3}, j_{2t-2\alpha+\epsilon-2}, \mathfrak{k}'', j_{2\beta+\epsilon-1}, j_{2\beta+\epsilon})$$
$$+\sum_{k=1}^{t} Y\Big(1^{t+\epsilon-3}, j_\epsilon, \Sigma\,(1, j_{\epsilon+1}, j_{\epsilon+2}), \dots, \widehat{\Sigma}\,(1, j_{2k+\epsilon-1}, j_{2k+\epsilon}),$$
$$\dots, \Sigma\,(1, j_{2t-1+\epsilon}, j_{2t+\epsilon}), j_{2k+\epsilon-1}, j_{2k+\epsilon}\Big)$$

where

$$\mathfrak{i} = [j_1, \dots, j_{2t-2\alpha+\epsilon-4}],$$
$$\mathfrak{k} = \Big[j_{2t-2\alpha+\epsilon-1}, \dots, \widehat{j}_{2\beta+\epsilon-1}, \widehat{j}_{2\beta+\epsilon}, \dots, j_{2t+\epsilon}\Big]$$

and the "j_0" should be deleted from the last sum when $\epsilon = 0$.

Proof: We apply Lemma 6.6 to the element $\mathfrak{a}_{t+\epsilon-1}\,[j_1, \dots, j_{2t+\epsilon+2}]$ where $j_{2t+\epsilon+1}$ and $j_{2t+\epsilon+2}$ are large powers of two. Applying the Landweber-Novikov operation $s_{\Delta_{2j_{2t+\epsilon+1}-2}+\Delta_{2j_{2t+\epsilon+2}-2}}$ and dividing by q_1, we obtain this lemma. ∎

We combine the previous three lemmas to show that the elements of the top half of the tower of Figure 2 are boundaries in the ASS for MSp.

Proposition 6.8. *(a) For* $\epsilon = 0, 1$, $3 \le t$ *and* $0 \le s \le t-2$, *the element*

$$\mathfrak{a}_{s+t+\epsilon-2}\,[j_1, \dots, j_{2t+\epsilon}]$$

is zero in $\mathbb{E}_\infty$ *of the ASS for* MSp.

(b) For $t \ge 3$, *the following element is also zero in* $\mathbb{E}_\infty$ *of the ASS for* MSp:

$$\mathfrak{a}_{t-2}\,[0, j_2, \dots, j_{2t+1}].$$

Proof: (a) Each summand of the decomposition of $\mathfrak{a}_{s+t+\epsilon-2}[j_1,\ldots,j_{2t+\epsilon}]$ in Lemma 6.6 is a boundary in the ASS by Lemma 6.5.

(b) When $\epsilon = 1$ and $j_1 = 0$ in Lemma 6.7, the last sum in the decomposition of $\mathfrak{a}_{t-2}\,[0, j_2, \ldots, j_{2t+1}]$ equals

$$\sum_{k=1}^{t} Y\Big(0, \Sigma\,(1, j_2, j_3)\,, 1, \Sigma\,(1, j_4, j_5)\,, \ldots\,, \widehat{1}, \widehat{\Sigma}\,(1, j_{2k}, j_{2k+1})\,, \ldots\,, 1, \Sigma(1, j_{2t}, j_{2t+1}), j_{2k}, j_{2k+1}\Big)\,.$$

Thus, by Lemma 6.5 each summand of the decomposition of $\mathfrak{a}_t\,[0, j_2, \ldots, j_{2t+1}]$ in Lemma 6.7 is a boundary in the ASS. ■

Note 6.1. If $j_1, \ldots, j_{2t+\epsilon}$ is an increasing sequence of non-negative even integers then the only apparent way that the last sum in Lemma 6.7 can bound is as in (b) when $\epsilon = 1$ and $j_1 = 0$.

It follows from this proposition that the order of $a[J]$ given in Theorem 2 is exact after projection into $\mathbb{E}_\infty$ of the ASS.

Corollary 6.9. *For $s \geq 7$ and $0 \leq j_1 < \cdots < j_s$, the projection of the element $2^{[(s+1)/2]-2}a\,[j_1, \ldots, j_s]$ into $\mathbb{E}_\infty^{2[(s+3)/2],*}$ of the ASS for MSp is zero.*

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Boris I. Botvinnik:
Department of Mathematics
University of Oregon
Eugene, OR 97403
U.S.A.

e-mail: botvinnik@poincare.uoregon.edu

Stanley O. Kochman:
Department of Mathematics
and Statistics
York University
4700 Keele Street
North York, Ontario M3J 1P3
Canada

e-mail: kochman@atop.yorku.ca

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