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# Adaptation is Unnecessary in $\mathcal{L}_1$ –“Adaptive” Control

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## I. WHAT MAKES “ADAPTIVE” AN ADAPTIVE CONTROLLER?

The basic premise upon which adaptive control is based is the existence of a parameterized controller that achieves the control objective. It is, moreover, assumed that these parameters are not known but that they can be estimated on–line from measurements of the plant signals. Towards this end, an identifier is added to generate the parameter estimates. Then, applying in an *ad–hoc* manner a certainty equivalence principle, these estimates are directly applied in the aforementioned control law.

Let us illustrate the discussion above with the simplest example of direct, adaptive, state–feedback stabilization of single–input, linear time–invariant (LTI) system of the form

$$\dot{x} = Ax + bu \quad (1)$$

where the state  $x \in \mathbb{R}^n$  is assumed to be *measurable*,  $u \in \mathbb{R}$  is the control signal,  $A \in \mathbb{R}^{n \times n}$  is the system matrix and  $b \in \mathbb{R}^n$  the input vector. It is assumed that there exists a vector  $\theta \in \mathbb{R}^n$  such that

$$A + b\theta^\top =: A_m$$

is a Hurwitz matrix, but this vector is *unknown*. In this case, the ideal control law takes the form

$$u = \theta^\top x, \quad (2)$$

that, as mentioned above, is made adaptive adding an identifier that generates the estimated parameters  $\hat{\theta} \in \mathbb{R}^n$ . In this way, we obtain the adaptive control law

$$u = \hat{\theta}^\top x. \quad (3)$$

Defining the parameter error

$$\tilde{\theta} := \hat{\theta} - \theta, \quad (4)$$

the control law may be written as

$$u = \theta^\top x + \tilde{\theta}^\top x.$$

If the parameter estimates converge to the desired value  $\theta$  the control signal converges to the ideal control law (2) and asymptotic stabilization is achieved—provided  $x$  remains bounded.<sup>1</sup>

A key observation is that the ideal control signal (2) *cannot be implemented* without knowledge of the unknown parameters. If this were not the case adaptation would be unnecessary and we simply would plug in the controller that results when  $\tilde{\theta} = 0$ !

In a (long) series of recent papers—see, *e.g.*, [5] and the extensive list of references therein—it has been proposed to replace (3) by

$$\dot{u} = -k(u - \hat{\theta}^\top x), \quad (5)$$

<sup>1</sup>Actually, to achieve stabilization it is enough that  $\hat{\theta}$  converges to the set  $\{K \in \mathbb{R}^n \mid A + bK^\top \text{ is Hurwitz}\}$ . This is the fundamental self–tuning property of direct adaptive control.

where  $k > 0$  is a design parameter. Combining (5) with a standard state prediction–based estimator is called in [5]  $\mathcal{L}_1$ –adaptive control, which in the sequel we refer to as  $\mathcal{L}_1$ –AC.

The purpose of this paper is to prove, via a proposition given below, that adaptation is *unnecessary* in  $\mathcal{L}_1$ –AC in the following precise sense.

F1 For any parameter estimation law, the control signal (5) *exactly* coincides with the output of the LTI, full–state feedback, perturbed, PI controller

$$\begin{aligned} \dot{v} &= -K_I^\top x + k\tilde{\theta}^\top x \\ u &= v - K_P^\top x, \end{aligned} \quad (6)$$

with the gains  $K_P, K_I \in \mathbb{R}^n$  are *independent* of the parameters  $\theta$ .

F2 The term  $\tilde{\theta}^\top x$  converges to zero. Hence, the  $\mathcal{L}_1$ –AC *always* converges to a controller that can be obtained *without* knowledge of the unknown parameters.

F3 If the implementable PI controller<sup>2</sup>

$$\begin{aligned} \dot{v} &= -K_I^\top x \\ u &= v - K_P^\top x, \end{aligned} \quad (7)$$

*does not stabilize* the plant (19) then the  $\mathcal{L}_1$ –AC does not stabilize it either.

## II. IS ADAPTATION NECESSARY IN $\mathcal{L}_1$ –ADAPTIVE CONTROL?

We analyze in this paper the  $\mathcal{L}_1$ –AC proposed in [5] to address the basic problem of stabilization of single–input, LTI systems discussed in the previous section. In  $\mathcal{L}_1$ –AC, besides the (overly restrictive) assumption of measurable state, it is assumed that the system can be represented in canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & -a_3 & \dots & -a_n \end{bmatrix}, \quad (8)$$

where  $a_i \in \mathbb{R}$ ,  $i \in \bar{n} := \{1, \dots, n\}$  are unknown coefficients, and that the input vector  $b$  is *known*. In the sequel we set  $b = e_n$ , the  $n$ –th vector of the Euclidean basis, which is done without loss of generality in view of the assumption of known  $b$ . The system (19) can also be expressed in the form

$$\dot{x} = A_m x - b(\theta^\top x - u) \quad (9)$$

<sup>2</sup>In the sequel we will say that a controller is “implementable” if its gains are independent of the unknown plant parameters.

with

$$A_m = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_1^m & -a_2^m & -a_3^m & \dots & -a_n^m \end{bmatrix} \quad (10)$$

where  $a_i^m > 0$ ,  $i \in \bar{n}$ , are designer chosen coefficients and  $\theta \in \mathbb{R}^n$  is a vector of *unknown* parameters, given by

$$\theta = \text{col}(a_1 - a_1^m, a_2 - a_2^m, \dots, a_n - a_n^m), \quad (11)$$

where  $\text{col}(\cdot)$  denotes column vector. In the  $\mathcal{L}_1$ -AC proposed in [5] the control law is computed via (5). The parameters are updated using the classical state predictor-based estimator

$$\begin{aligned} \dot{\hat{x}} &= A_m \hat{x} - b(\hat{\theta}^\top x - u) \\ \dot{\hat{\theta}} &= \gamma x(\hat{x} - x)^\top P b \end{aligned} \quad (12)$$

where  $\gamma > 0$  is the adaptation gain and  $P > 0$  is a Lyapunov matrix for  $A_m$ , that is,

$$P A_m + A_m^\top P = -Q,$$

where  $Q \in \mathbb{R}^{n \times n}$  is a positive definite matrix.

The proposition below formally establishes the facts F1–F3 stated in the previous section.

*Proposition 1:* Consider the plant (19) with  $A$  given by (8) and  $b = e_n$ .

P1 Independently of the parameter estimation, the signal  $u$  generated by the  $\mathcal{L}_1$ -AC control law (5) exactly coincides with the output of the perturbed, full-state feedback, LTI, implementable PI controller (6) with

$$\begin{aligned} K_I &= k \text{col}(a_1^m, a_2^m, a_3^m, \dots, a_n^m) \\ K_P &= k e_n. \end{aligned} \quad (13)$$

P2 If the  $\mathcal{L}_1$ -AC controller (5), (12) ensures boundedness of trajectories then the perturbation term verifies

$$\lim_{t \rightarrow \infty} |\hat{\theta}^\top(t)x(t)| = 0. \quad (14)$$

Consequently, the (bounded state)  $\mathcal{L}_1$ -AC *always* converges to the PI controller.

P3 If the PI (7), (13) *does not* ensure stability of the closed-loop system then the  $\mathcal{L}_1$ -AC (5), (12) does not ensure boundedness of trajectories.

*Proof:* To establish P1 we use the definition of the parameter error (4) to write the control signal (5) as

$$\dot{u} = -k(u - \theta^\top x) + k\hat{\theta}^\top x. \quad (15)$$

Now, pre-multiplying the plant dynamics (9)—that is equivalent to (19)—by  $e_n^\top$ , and rearranging terms, we get

$$u - \theta^\top x = e_n^\top (\dot{x} - A_m x),$$

that, upon replacement in (15), yields

$$\dot{u} = -k e_n^\top (\dot{x} - A_m x) + k\hat{\theta}^\top x.$$

The proof is completed defining the signal

$$v = u + k e_n^\top x,$$

and using the definition of  $A_m$  given in (10).

To prove P2 we first write the dynamics of the system (9) in closed-loop with the  $\mathcal{L}_1$ -AC (12), (15),

$$\begin{aligned} \dot{\tilde{x}} &= A_m \tilde{x} - b\tilde{\theta}^\top x \\ \dot{\tilde{\theta}} &= \gamma x \tilde{x}^\top P b \\ \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}} \end{bmatrix} &= \mathcal{A}_0 \begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ k\tilde{\theta}^\top x \end{bmatrix}, \end{aligned} \quad (16)$$

where<sup>3</sup>

$$\mathcal{A}_0 = \begin{bmatrix} A & b \\ k\theta^\top & -k \end{bmatrix},$$

and  $\tilde{x} = \hat{x} - x$  is the prediction error. Consider the function

$$V(\tilde{x}, \tilde{\theta}) = \frac{1}{2} \tilde{x}^\top P \tilde{x} + \frac{1}{2\gamma} |\tilde{\theta}|^2,$$

whose derivative along the trajectories of (16) is

$$\dot{V} = -\frac{1}{2} \tilde{x}^\top Q \tilde{x}.$$

Since it has been assumed that all trajectories are bounded we can invoke LaSalle's invariance principle to conclude that all trajectories converge to the largest invariant set contained in  $\{\tilde{x} = 0\}$ . The proof is completed analyzing the first equation of (16).

The proof of P3 is established proving the converse implication, *i.e.*, that the trajectories of the  $\mathcal{L}_1$ -AC are bounded implies stability of the plant in closed-loop with the PI. In point P2 we proved that if the trajectories of (16) are bounded (14) holds true. Now, the system in the third equation of (16) is an LTI system whose input, *i.e.*,  $\tilde{\theta}^\top x$  converges to zero and whose output  $\text{col}(x, u)$  is bounded, for all initial conditions  $\text{col}(x(0), u(0))$ , consequently the matrix  $\mathcal{A}_0$  is stable. ■

### III. SOME FURTHER REMARKS

**R1** The property P1 in Proposition 1 underscores that the stabilization mechanism of  $\mathcal{L}_1$ -AC is independent of the parameter adaptation, instead it is an elementary linear systems principle. As shown in the proposition, the effect of the adaptation appears as a perturbation term  $k\tilde{\theta}^\top x$  to the implementable PI controller that, if trajectories are bounded, asymptotically converges to zero. This explains why in  $\mathcal{L}_1$ -AC it is suggested to increase the adaptation gain—hoping that this term will die-out quickly. Moreover,  $\mathcal{L}_1$ -AC includes a parameter projection that, due to the use of utterly high adaptation gains, induces a bang-bang-like behavior in the estimate that, in average, behaves like a constant value. See [2] for some conclusive simulated evidence.

**R2** The qualifier “implementable” is essential to appreciate the significance of our results. Of course, all (linearly parameterized) adaptive controllers can be implemented as an LTI system perturbed by the parameter error but the resulting LTI system depends on *unknown plant* parameters. Due to the inclusion of the input filter, this is not the case in  $\mathcal{L}_1$ -AC—rendering irrelevant the use of adaptation. In [10] this deleterious effect of the input filter has been shown to be pervasive for all model reference controller structures, not just the state-feedback, canonical system representation treated in this paper.

**R3** In [11] it has been shown that there exists  $k_c > 0$  such that the PI controller (7), (13) ensures global asymptotic stability of the closed-loop system for all  $k > k_c$ , all unknown parameters  $a_i, i \in \bar{n}$ , and all Hurwitz matrices  $A_m$  of the form (10). On the other hand, to the best of the authors' knowledge, it is not known

<sup>3</sup>This matrix is reported in equation (12) of [3].

whether there exists suitable values of  $\gamma$  and  $k$  such that the origin of (16) is (asymptotically) stable for all unknown parameters  $a_i, i \in \bar{n}$ , and all Hurwitz matrices  $A_m$  of the form (10).

**R4** We have assumed for simplicity the case of regulation to zero and taken the input filter used in  $\mathcal{L}_1$ -AC as  $D(s) = \frac{k}{s+k}$ . The proposition extends *verbatim* to the case of nonconstant reference and general (stable, strictly proper) LTI filters  $D(s)$ . See [10] and [11].

**R5** As shown in [11] the characteristic polynomial of  $\mathcal{A}_0$  satisfies<sup>4</sup>

$$\det(sI_n - \mathcal{A}_0) = s \det(sI_n - A) + k \det(sI_n - A_m). \quad (17)$$

From which it is clear that, if the plant is unstable, it is necessary to take “large” values of  $k$  to stabilize the  $\mathcal{L}_1$ -AC, see (16). This is in contradiction with the main promotional argument of  $\mathcal{L}_1$ -AC, namely that “it compensates for the mismatch between the ideal system and the plant within the frequency range of the lowpass filter  $D(s)$ ”. Moreover, it is recognized in [5] that “the allowed bandwidth of the filter  $D(s)$  is limited by robustness considerations”—contradicting, again, the need for large  $k$ . It is interesting to note that in the limit, as  $k \rightarrow \infty$ , from (5) we recover the good old model reference adaptive controller  $u = \hat{\theta}^\top x!$

**R6** In [5] it is argued that the inclusion of the input LTI filter and the use of large adaptation gains “decouples the estimation and control loops”—a notion that is never explained mathematically. In adaptive control “decoupling” between the adaptation and the control loops is (partially) achieved using small adaptation gains that ensure the estimated parameters vary slowly—with respect to the variation of the plant states. Leaving aside the numerical problems and unpredictable transient behavior generated with large adaptation gains, Point P2 of Proposition 1 clarifies this decoupling effect, namely, making the  $\mathcal{L}_1$ -AC converge faster to the implementable PI controller.

**R7** In [5] the condition<sup>5</sup>

$$\|(pI_m - A_m)^{-1} b \theta^\top \frac{p}{p+k}\|_\infty < 1 \quad (18)$$

where  $p := \frac{d}{dt}$  and  $\|\cdot\|_\infty$  is the  $\mathcal{L}_\infty$ -induced operator norm,<sup>6</sup> is imposed to derive some  $\mathcal{L}_\infty$  bounds on some suitably chosen signals—that, interestingly, do not include the tracking error. Clearly, this condition is far more restrictive than the condition  $k > k_c$  discussed in Remark R3 above, which ensures stability of the PI. As a matter of fact, in [9] it is shown that, for scalar systems, (18) cannot be satisfied for all systems and reference models.

**R8** The present paper extends the results of [9], where we treat only scalar systems. It is similar in spirit to the proof of [6] that output feedback  $\mathcal{L}_1$ -AC is, actually, nonadaptive. It also complements the recent report [7] where the claims of robustness and performance improvement of  $\mathcal{L}_1$ -AC are scrutinized via theoretical analysis and a series of numerical examples. The interested reader is also referred to [2], [7] where the issues of numerical instability due to high-gain adaptation and bang-bang behavior of the control due to parameter projection  $\mathcal{L}_1$ -AC, are discussed. The inability of  $\mathcal{L}_1$ -AC to track non-constant references is widely acknowledged, see [9] for a particular example. A freezing property of high-gain estimators,

<sup>4</sup>From (17) it follows that  $\mathcal{A}_0$  may not have an eigenvalue at zero, but it may have eigenvalues in the  $j\omega$  axis. Hence, the stability statement in P3 of Proposition 1 cannot be strengthened to asymptotic stability. The authors thank Denis Efimov for this insightful remark.

<sup>5</sup>Actually, the condition (85) given in [5] is far more restrictive than (18).

<sup>6</sup>For convolution operators  $\|H\|_\infty = \|h(t)\|_1 = \int_0^\infty |h(t)| dt$ , where  $h(t)$  is the impulse response [4].

that puts a question mark on the interest of using it, is proven in [1], see also [9].

**R9** The motivation to crank up the adaptation gain in  $\mathcal{L}_1$ -AC is related with some transient performance bounds claimed by the authors. Indeed, it is easy to show [5] that, if the initial conditions of the predictor coincide with the initial conditions of the plant, the prediction error is upper-bounded by a constant that is inversely proportional to the adaptation gain. The intrinsic fragility of results relying on particular initial conditions is a key issue often overlooked by (mathematically oriented) researchers in our community. See Sidebar 2.

**R10** It is not surprising that  $\mathcal{L}_1$ -AC has been successful in some applications. As shown above, it (essentially) coincides with a full state feedback PI controller that, as is well-known, is robust and can reject constant disturbances and track constant references, a scenario that seems to fit the realm of applications reported for this controller.

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## SIDEBAR 1: COMMENTS REGARDING [8]

In the abstract of [8] one finds the following unusually candid sentence: “the L1 adaptive controllers approximate an implementable non-adaptive linear controller.” In this sidebar we derive, in a mathematically rigorous way, the calculations done in [8] that motivated the previous sentence and place it in the context of our work.

Towards this end, consider the LTI system

$$y(s) = G(s)u(s),$$

where  $G(s) \in \mathbb{R}(s)$  and is strictly proper. This corresponds to equation<sup>7</sup> {1} in [8], where the symbol  $A(s)$  is used instead of  $G(s)$ , and a much more general plant model is treated. We consider here the simplest scenario needed to convey our message. The plant model is rewritten in {2} as

$$y(s) = M(s)[u(s) + \sigma(s)], \quad (19)$$

where

$$\sigma(s) = \left[ \frac{G(s)}{M(s) - 1} \right] u(s), \quad (20)$$

with  $M(s) \in \mathbb{R}(s)$ , stable of relative degree smaller than the relative degree of  $G(s)$ . The  $\mathcal{L}_1$ -AC {7} is given in this case by

$$u(s) = C(s)[r(s) - \hat{\sigma}(s)], \quad (21)$$

where  $r(t)$  is some reference signal,  $C(s) \in \mathbb{R}(s)$ , is strictly proper, stable and verifies  $C(0) = 1$ , and  $\hat{\sigma}(t)$  is a signal that represents an "estimate" of  $\sigma(t)$  generated with the estimator {4}–{6}.

In {8} and {9} of [8] the following signals, called reference signals, are introduced

$$\begin{aligned} y_r(s) &= M(s)[u_r(s) + \sigma_r(s)] \\ u_r(s) &= C(s)[r(s) - \sigma_r(s)] \\ \sigma_r(s) &= \frac{G(s)}{M(s) - 1} u_r(s). \end{aligned} \quad (22)$$

It important to note that (22) exactly coincides with (19)–(21) when

$$\hat{\sigma}(t) \equiv \sigma(t),$$

which corresponds to the case *without adaptation and known plant parameters*.

It is then claimed, without proof, that the  $\mathcal{L}_\infty$  norm of the errors  $y(t) - y_r(t)$  and  $u(t) - u_r(t)$  can be made arbitrarily small "reducing the sampling time" of the estimator {4}–{6}. This leads the authors to affirm that "the  $\mathcal{L}_1$ -AC system *approximates* the reference system (22)".

The authors then proceed with some transfer function manipulations to establish {13}, that is,

$$u_r(s) = \frac{C(s)}{[1 - C(s)]M(s)} [M(s)r(s) - y_r(s)]. \quad (23)$$

This is referred to as *limiting controller* and it is stated that "it is equivalent to the  $\mathcal{L}_1$ -AC under *fast adaptation*". Notice that the controller (23)—for a plant with output  $y_r$ —can be implemented without knowledge of the original plant parameters. This is the justification given to the sentence in the abstract of [8] mentioned above.

It is, furthermore, argued that the "architectures of both controller are fundamentally different". The motivation for this statement is that, in contrast with  $\mathcal{L}_1$ -AC, the implementation of the limiting controller involves the inversion of  $M(s)$  while "the estimation loop in  $\mathcal{L}_1$ -AC computes the approximate desired systems inverse".

For the case treated in our paper

$$C(s) = \frac{k}{s + k}, \quad \sigma(t) = \theta^\top x(t), \quad \hat{\sigma}(t) = \hat{\theta}^\top(t)x(t).$$

For scalar plants the controller (23), with

$$y_r(t) \equiv y(t), \quad M(s) = \frac{1}{s + a_m}, \quad r(t) \equiv 0,$$

exactly coincides with the PI controller reported in our paper.

As usual in the manipulation of stable transfer functions the calculations of [8] neglect the exponentially decaying terms due to initial conditions. This kind of assumption is untenable in nonlinear systems, like adaptive controllers, since it is well-known that even for globally exponentially stable systems the trajectories can be driven to infinity when perturbed by exponentially decaying disturbances [12].

In Proposition 1 we consider general  $n$ -th order plants with *arbitrary* initial conditions, and prove that the output of the  $\mathcal{L}_1$ -AC—including the parameter estimator with *arbitrary* adaptation gain—exactly coincides with the output of an implementable PI perturbed by a term coming from the adaptation. It is also established that this term converges to zero, therefore the  $\mathcal{L}_1$ -AC always converges to the PI. It is easy to prove that the latter result cannot be recovered with the transfer function manipulations of [8]. Indeed, defining in the standard way an estimation error

$$\tilde{\sigma}(t) := \hat{\sigma}(t) - \sigma(t),$$

and doing the calculations above with the original plant it follows that the control signal (21) can be written as

$$u(s) = \frac{C(s)}{[1 - C(s)]M(s)} [M(s)r(s) - y(s)] + \frac{C(s)}{1 - C(s)} \tilde{\sigma}(s).$$

Unfortunately, from this equation we cannot conclude that the  $\mathcal{L}_1$ -AC converges to the implementable LTI controller. Indeed, because of the constraint  $C(0) = 1$ , the perturbing term  $\tilde{\sigma}(t)$  passes through an integrator. Since we do not know if this signal is integrable we cannot even claim that its contribution to the control signal is bounded—let alone converging to zero.

In the light of this discussion, the interest of the "approximation" and "equivalence" statements of [8] is questionable and they are certainly far from substantiating the claim made in the abstract of the paper—a fact that Proposition 1 presents in a crystal-clear manner.

#### SIDEBAR 2: COMMENTS REGARDING THE ANALYSIS DONE IN THE $\mathcal{L}_1$ -AC LITERATURE

As indicated in Remark R9 trajectory-dependent claims in control theory are intrinsically fragile. Indeed, the whole body of control theory has been developed to design controllers whose performance is guaranteed *independently of the initial conditions*—for instance, ensure stability (in the sense of Lyapunov) of a desired equilibrium point or finite gain of an operator. When the result is valid for a specific initial condition, this property is valid only for that specific trajectory, and cannot be extrapolated to any other one. Hence, in the presence of unknown and unpredictable disturbances, or measurement errors, that will drive away the state from that trajectory, nothing can be said about the new trajectory.

To illustrate this point consider the simple case of the LTI plant

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = x_2 - x_1.$$

Clearly, for all initial conditions in the set  $\{x \in \mathbb{R}^2 \mid x_1 = x_2\}$ , the corresponding trajectory is bounded and converges to zero. But, obviously, all trajectories starting outside this set grow unbounded.

Unfortunately, the analysis of  $\mathcal{L}_1$ -AC reported in the literature is trajectory dependent since it relies on the assumption that the initial state of the estimator (12) *coincides* with the initial state of the plant, that is  $\hat{x}(0) = x(0)$ . Since the plant state cannot be exactly measured, or maybe subject to disturbances that would require an unpractical resetting of the estimator, this condition is always violated in practice.

<sup>7</sup>Numbers in brackets refer to the equations of [8].