Adaptive Analysis of Sparse Signals

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Abstract: We introduce an adaptive method for analysis of sparse signals using bandpass filters obtained by modulated Slepian sequences. Similar to the recently introduced empirical wavelet transform, the proposed method decomposes a signal into different modes which corresponds to segmenting the Fourier spectrum and filtering the existing support. The simulations illustrate the correct signal decomposition for a multiband signal which has a sparse spectrum. The proposed method can be used as an alternative to empirical wavelet transform.

Key–Words: Slepian sequences, multiband signals, filter banks, orthogonal bases, empirical wavelet transform

1 Introduction

Methods that analyze signals in an adaptive manner have important applications such as decomposition, denoising, compression, etc. Adaptive methods aim to construct a basis dependent on the information contained in the signal. Empirical Mode Decomposition (EMD) has gained a lot of interest as an adaptive representation method [1]. EMD detects the principal "modes" to represent the signal such that each mode has a compactly supported Fourier spectrum. However, EMD approach is more of an algorithmic approach and lacks of a mathematical theory.

On the other hand, wavelet based analysis techniques can be viewed as the application of a filter bank where each filter corresponds to a scale. An adaptive representation called Malvar-Wilson wavelet is based on segmenting a signal in the temporal domain where the time intervals contain the different spectral information [2]. However, it turns out that temporal segmentation is difficult to implement efficiently. In [3], the authors use the idea in [2] to built an adaptive filter bank directly in the Fourier domain. However, the reconstruction is quite complicated and based on prescribed subdivisions. "Synchrosqueezed" wavelets method combines a classical wavelet analysis and time-frequency information for the location detection to obtain more accurate mode extraction [4]. Based on wavelet transform, Empirical Wavelet Transform (EWT) is another recently developed adaptive method [5]. EWT builds adaptive wavelets to extract amplitude modulated-frequency modulated (AM-FM) components based on the idea that AM-FM components have a compact Fourier spectrum. EWT method adapts the wavelet transform approach for signal decomposition by considering the distinct Fourier supports and then building a set of functions which form an orthonormal basis. The empirical wavelets are defined as bandpass filters on each Fourier support. In this paper, we take the Fourier point of view to develop an analysis method for sparse signals whose Fourier transform is concentrated on a small number of continuous bands, i.e., multiband signals. Indeed, any signal with a Fourier transform supported on a finite range of frequencies cannot also be supported on a finite range of time. We will show that it is possible to decompose a multiband signal into its bandlimited modes of finite-length samples. Similar to EWT, our approach is based on building a set of bandpass filters using discrete prolate spheroidal sequences [6]. Use of DPSS as an orthogonal basis can reduce the sampling rate and reconstruction error [7]. The representation of bandpass signals, as the modulation of baseband components, can be obtained using modulated Slepian basis. The method is illustrated in analysis of a multiband signal of which coefficients are complex Gaussian random variables. We compare the EWT method to the proposed method for a multi band signal and see that the proposed method can be used as an alternative to EWT.

1.1 Discrete Prolate Spheroidal Sequences

In this section, first we provide a brief overview of the discrete prolate spheroidal sequences (DPSSs) also known as Slepian sequences. DPSSs resulted from the work by Slepian, Landau, and Pollack on the effects of timelimiting and bandlimiting operations [6]. Consider first timelimiting and then bandlimiting a sequence, the DPSSs are defined to be the eigenvectors of this two step procedure. Given N and $0 < W < 1/2$, the DPSSs are a collection of N real valued, strictly bandlimited $|f| \leq W$ discrete time sequences

$$\boldsymbol{S}_{N,W} = \left[s^{(1)}_{N,W}, s^{(2)}_{N,W}, \cdots, s^{(N)}_{N,W} \right]$$

with their corresponding eigenvalues

$$1 > \lambda^{(1)}_{N,W} > \lambda^{(2)}_{N,W} \cdots \lambda^{(N)}_{N,W} > 0.$$

Let τ_N denote an operator that takes an infinite length discrete time signal and zeros out all entries outside the index range $\{0, 1, \cdots, N-1\}$ but still the resulting signal is infinite length.

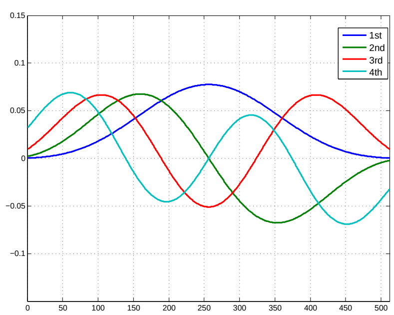

Figure 1: First 4 discrete prolate spheroidal sequences (DPSSs) for $N = 512$ and $NW = 2.5$.

Define a bandlimit operator B_W that takes a discrete signal and bandlimits its Discrete Time Fourier Transform (DTFT) to the frequency range $|f| \leq W$ and returns the corresponding signal in time domain. The DPSSs satisfy $B_W(\tau_N(s^{(\ell)}_{N,W})) = \lambda^{(\ell)}_{N,W} s^{(\ell)}_{N,W}$, for all $\ell \in \{1, 2, \cdots, N\}$. The first $2NW$ eigenvalues are very close to 1 and the rest tend to be close to 0 which is a very distinct behaviour. For a given integer $K \leq N$, we can get $N \times K$ matrix formed by taking the first K columns of $\boldsymbol{S}_{N,W}$. When $K = 2NW$, it is a highly efficient basis that captures most of the signal energy. In numerical computations each of the DPSSs has infinite support in time but if we timelimit the DPSSs we can obtain finite length DPSSs which are approximately bandlimited to the digital frequency range $|f| \leq W$. Any DPSS is bandlim-

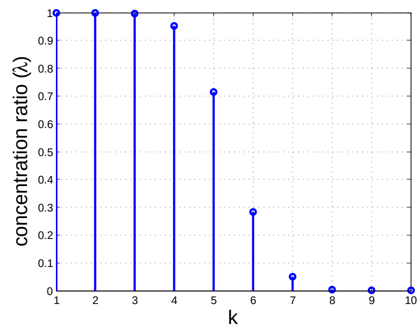

Figure 2: Eigenvalues of the DPSSs as shown in Figure 1.

ited but once timelimited they will not be strictly bandlimited but only be concentrated in the bandwidth of interest for the first $2NW$ DPSS. The timelimited DPSSs are orthogonal;

$$\langle \tau_N(s^{(\ell)}_{N,W}), \tau_N(s^{(\ell')}_{N,W}) \rangle = 0 \text{ for } \ell \neq \ell'.$$

Timelimited infinite length DPSSs can be restricted to the index range $\{n = 1, 2 \cdots, N\}$ which are DPSS vectors and defined as

$$s^{(\ell)}_{N,W}[n] := \tau_N(s^{(\ell)}_{N,W})[n], \ \ \forall \ell, n \in \{1, 2, \cdots, N\}.$$

These vectors form an orthonormal basis for $\Re^N$ and using just $\approx 2NW$ DPSS vectors, the energy of a signal can be captured effectively.

The bandlimiting operator B_W can be constructed as an $N \times N$ matrix with entries $B_{N,W}[m, n] := 2W \text{sinc}(2W(m-n))$ of which eigenvectors are DPSS vectors. We can obtain the eigendecomposition of $B_{N,W}$ as

$$B_{N,W} = S_{N,W} \Lambda_{N,W} S^H_{N,W}$$

where $N \times N$ matrix $\boldsymbol{S}_{N,W} := [s^{(1)}_{N,W} \ s^{(2}_{N,W} \ \cdots \ s^{(N)}_{N,W}] \in \Re^{N \times N}$ results from concatenating DPSS vectors into an $N \times N$ matrix and $\Lambda_{N,W}$ denote an $N \times N$ diagonal matrix with the DPSS eigenvalues along the main diagonal. Since the first $2NW$ eigenvalues of DPSS vectors cluster

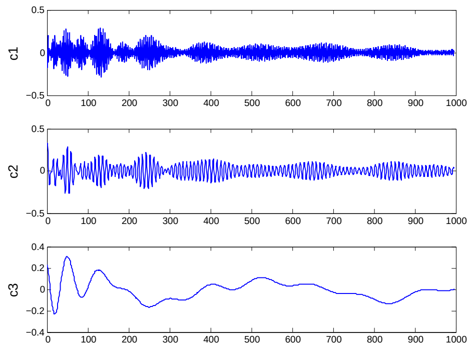

Figure 3: Original components of the signal.

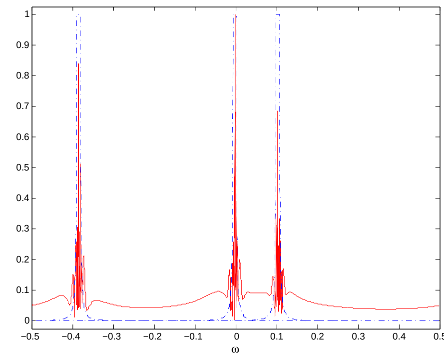

Figure 4: Fourier Transform of signal (solid line) vs. Fourier transform of squared gain function of Slepian bases (dash line) for the signal components.

around 1 and the rest around 0 (see. Fig. 1 and Fig. 2), we can obtain efficient bases using small number of DPSS vectors.

From a filter design point of view, the Slepian sequences can be viewed as the coefficients of a set of finite impulse response (FIR) filters which are designed to satisfy some optimality conditions. The designed filters are lowpass filters whose main lobes are within a five range and have minimum stop band energy. Moreover, the designed filters are selected so that their coefficients form a set of orthogonal vectors.

1.2 Multiband Signals

We are interested in the signals of which Fourier spectrum can be segmented into intervals. Let us consider a model based on defining a set of frequencies for a continuous time signal $x(t)$, from $\frac{-B_{Nyq}}{2}$ to $\frac{B_{Nyq}}{2}$ into bands of width B_{band} where the support of the i^{th} band would be defined as $\Delta_i = [\frac{-B_{Nyq}}{2} + iB_{band}, \frac{B_{Nyq}}{2} + (i+1)B_{band}$. The above setting is for the infinite length $x(t)$ as any signal with a Fourier transform supported on a finite range of frequencies cannot also be supported on a finite range of time. In the discrete domain, where we have a finite set of samples due to Nyquist sampling rate, we focus on representing sampled multiband signals.

1.3 Bandpass Modulated DPSS

As an efficient basis of signal representation, we describe modulated DPSSs as the basis for decomposition of multiband signals. In our previous work [7], we showed how to represent sampled low pass and band pass signals by means of modulated Slepian projection. Assume that $\boldsymbol{x} = [x(0), x(T_s), ..., x(N-1)T_s]$ is a vector of N samples of $x(t)$ where T_s is the sampling period. Let $\boldsymbol{S}_{N,W}$ denote the $N \times N$ DPSS matrix (baseband DPSS basis used to capture each band). Modulated DPSSs are obtained by taking the first K columns of the matrix $[\boldsymbol{E}_{fc}\boldsymbol{S}_{N,W}]$ where $K = 2NW$ and $\boldsymbol{E}_{fc}$ is the diagonal matrix with entries $e^{j2\pi f_c m}$ where $f_c = FcT_s$ and Fc is the center frequency as a results modulated DPSSs, which we will represent by $\boldsymbol{S_M}$ from now on.

1.4 Decomposition of Multiband Signals

Decomposition of a multi band signal using Slepian sequences will require projection of the sampled signal $\boldsymbol{x}$ onto the modulated Slepian matrix $[\boldsymbol{E}_{fc}\boldsymbol{S}_{N,W}]$ (complete basis for identified space) which provides projection coefficients. In this paper we assume that the locations of the K center frequencies Fc are known a priori using energy calculations in the Fourier domain and applying some threshold to detect the locations corresponding to maximum energy values. These center frequencies are then used to design

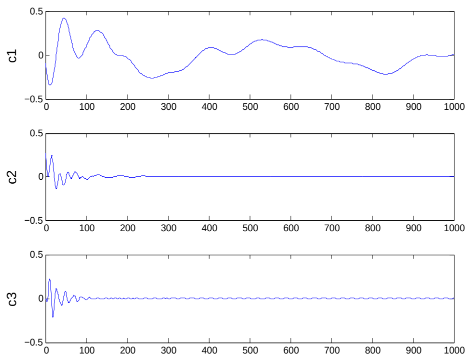

Figure 5: First 3 components of the 6 components obtained by EWT method.

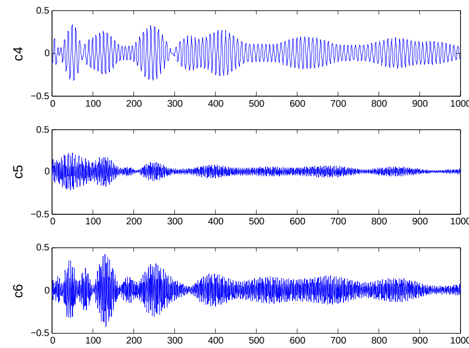

Figure 6: Last 3 components of the 6 components obtained by EWT method.

Slepian bandpass filters which are modulated Slepian matrices at the frequency location Fc to extract each component.

The coefficient vector $\boldsymbol{c}$ of the projection is given as

$$\boldsymbol{c} = \overline{\boldsymbol{S_M}}\boldsymbol{x}^T$$

where $\overline{\boldsymbol{S_M}}$ represents conjugate transpose of $\boldsymbol{S_M}$ and $\boldsymbol{x}^T$ is the transpose of $\boldsymbol{x}$. Once we have the information of Fc, we can obtain each basis which is a modulated Slepian matrix $\boldsymbol{S_{Mi}}$ at that particular center frequency. The components $\boldsymbol{x_i}$ of the multi band signal then can be found as

$$\boldsymbol{x_i} = \boldsymbol{S_{Mi}}\boldsymbol{c}$$

where $i = 1, \cdots, K$.

2 Experimental Results

In our performance analysis experiment, we compare the proposed method to EWT which builds adaptive wavelet bases to decompose signals into components. We want to analyze a complex signal composed of three exponential components each with Gaussian random variables as coefficients. Figure 3 shows the components that the signal is composed of. Fig. 5 is the result of decomposition using EWT method showing first 3 components. Although signal has 3 distinct bands the EWT method extracts 6 components. The rest of the components are shown in Fig. 6. Using the modulated Slepian bases, we obtain 3 components that correspond to the actual components in the signal as shown in Fig. 7. The original signal is compared to the signals obtained by adding the components shown in the Figures [5-7]. The results of the reconstructions are shown in Fig. 8 and Fig. 9. As the figures illustrates, the performance of modulated DPSS basis is very similar to EWT. However, EWT extracted more components than the actual signal had which created components that did not exist in the original signal.

3 Conclusions

In this paper, we proposed to use modulated Slepian sequences as an efficient basis of signal representation for decomposition of a sparse signal modeled as a multiband signal. Similar to EWT method, the proposed method can be viewed as filterbank method. In simulations, we kept the bandwidth of each component equal to each other so our bases had the same bandwidth for each component. It is possible to extent the method to signals with components having different bandwidths.

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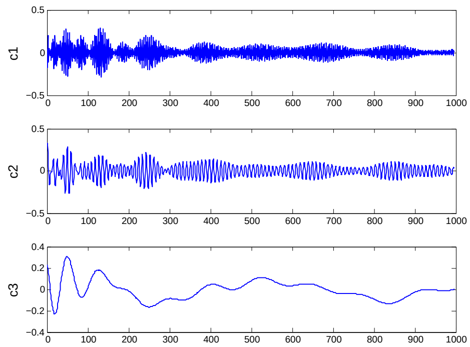

Figure 7: Components obtained by modulated Slepian basis.

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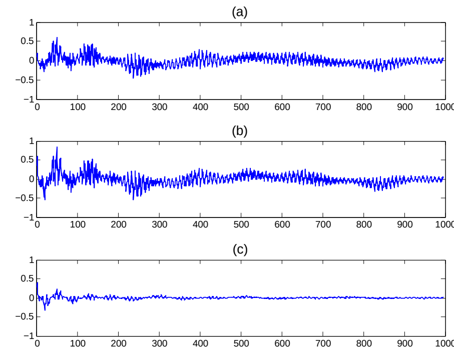

Figure 8: original signal compared (a) to the reconstructed signal (b) by addition of components obtained by proposed method and reconstruction error (c).

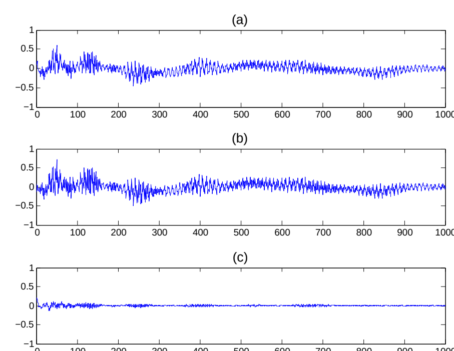

Figure 9: original signal compared (a) to reconstructed signal (b) by addition of components obtained by EWT method and reconstruction error (c).