MPE — Volume 1, pp. 139-177
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ADAPTIVE BAND-LIMITED DISTURBANCE REJECTION IN LINEAR DISCRETE-TIME SYSTEMS

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(Received 14 October 1994)

The problem of adaptively rejecting a disturbance consisting of a linear combination of sinusoids with unknown and/or time varying frequencies for SISO LTI discrete-time systems is considered. The rejection of the disturbance input is achieved by constructing the set of stabilizing controllers using the Youla parametrization and adjusting the Youla parameter to achieve asymptotic disturbance rejection. The first main result in this paper concerns *off-line controller design* where a controller that achieves regulation is *numerically designed* off-line based on the assumption that only the sequence of discrete disturbance input values (as opposed to a model of the disturbance) is available. A least squares based optimization algorithm is used in the controller design. As expected, it is shown, under some mild assumptions, that if the off-line designed controller achieves regulation, then it must include a model of the disturbance input. The second main result concerns *on-line controller design* where recursive versions of the off-line algorithm used above for controller design are presented and their convergence properties analyzed. Conditions under which the on-line algorithms yield an asymptotic controller that achieves regulation are presented. Conditions both for the case where the disturbance input properties are constant but unknown and for the case where they are unknown and time-varying are given. The on-line controller construction amounts to an adaptive implementation of the Internal Model Principle. The performance robustness of the off-line designed controller in the face of plant model uncertainties is investigated. It is shown, under some mild assumptions, that performance robustness is realized provided internal stability is maintained. The performance of the adaptation algorithms is illustrated through a simulation example.

AMS Nos.: 93C40, 93B35
KEYWORDS: Adaptive control, disturbance rejection, interpolation conditions, robustness

1. INTRODUCTION AND MOTIVATION

Consider a single-input single-output (SISO) linear time-invariant (LTI) system subject to a band-limited discrete disturbance input of the form:

$$w(k) = \sum_{n=0}^{k_0} c_n(k)cos(\omega_n(k)k + \phi_n(k)) \tag{1}$$

with unknown and possibly time-varying frequencies $\omega_n(k)$, amplitudes $c_n(k)$, and phases $\phi_n(k)$, $0 \leq n \leq k_0$. For the plant and class of disturbance inputs considered above, it is desired to design a controller that yields internal stability and asymptotic disturbance rejection which is robust in the face of variations in the disturbance amplitudes, frequencies, and phases. Henceforth, asymptotic disturbance rejection is also referred to as regulation.

The primary motivation for considering the class of disturbance inputs given in (1) comes from repetitive control problems ([1], [2], [3], [4]) with applications to machine tools, robotics, and so on. In such problems, the plant is subject to periodic continuous-time disturbance inputs. A digital controller is required for the asymptotic rejection of the periodic disturbance input. Since the digital control system includes an anti-aliasing filter, the effects of only the first few harmonics of the disturbance input are fed-back to the controller. This makes it practical to consider disturbance models which include only the first few harmonics of the periodic disturbance signal. A discrete model for such disturbance would be as in (1) where ω_n, $0 \le n \le k_0$, are integer multiples of the discrete disturbance frequency. It turns out that in some situations, such as flutter attenuation problems in aircrafts [5], the disturbance period is not known *a priori* and may be time-varying, which prompted the consideration of disturbance representations with time-varying frequencies. Finally, in order to make the disturbance rejection problem even more general, arbitrary (*i.e.*, not rationally related) frequencies are considered in (1). It should be noted that, in the context of repetitive control, *periodic* disturbance models similar to (1) have been considered in [6] and [7]. The disturbance model in (1) is considered to represent *almost periodic* signals.

When the disturbance model is completely known, two approaches can be used to achieve asymptotic disturbance rejection. The first approach is based on the Internal Model Principle which can be traced back to the paper [8]. Roughly speaking, the Internal Model Principle states that asymptotic disturbance rejection can be achieved only if a model of the exosystem generating the disturbance is included in the stable closed-loop system. This approach has been extensively used in many engineering applications such as repetitive control problems. The second approach consists of casting the regulation requirement in the form of a set of interpolation conditions as discussed in [9] and [10, p. 187]. The controller can then be designed by minimizing the l_2 norm of the disturbance response, subject to some interpolation constraints, as briefly mentioned in [9].

It is not always the case that the disturbance model is completely known or that the disturbance properties are constant over time. Johnson, 1982 [11] presented a disturbance accommodating control approach based on estimating the states of the disturbance model and using them in the design of a disturbance rejection controller. Adaptive techniques can also be used to achieve regulation. Depending on the type of unknowns or time varying parameters in the disturbance model, we can distinguish three different approaches discussed in the literature to solve the adaptive disturbance rejection problem. The first approach concerns repetitive control problems where the period of the disturbance is either unknown or slowly time varying. The approach consists of *explicitly* estimating the period of the disturbance input and using the period estimate in a repetitive controller. Different methods have been suggested to estimate the disturbance period. Hu, 1992 [12] used a set of identifiers based on Kalman filtering and working in parallel to estimate the period of the disturbance. Each of the identifiers estimates the single parameter in the disturbance model of the form ($d(k) = ad(k - N)$). In the nominal disturbance model, the parameter a is 1 which corresponds to a periodic signal of period N. The value of N is different in each identifier. During on-line period identification, the period of the disturbance is taken to be that used in the identifier whose parameter estimate is closest to 1. Tsao and Nemani, 1992 [13] presented a period identification algorithm based on convex optimization and which relies on the periodic nature of the signal. The identification algorithm may converge to an integer multiple of the true disturbance period. The disturbance period estimates are used either directly in the repetitive controller discussed in [14] or to adjust the sampling period so that the estimated signal period is as close as possible to an integer multiple of the adjusted sampling period. Tsao and Qian, 1993 [15] improved the algorithm in [13] to allow the estimation of the period of a periodic signal with a resolution better than the sampling period. The modified algorithm has better performance than that of the algorithm with fixed sampling period in [13]. Hillerstrom *et al.*, 1994 [6] considered the case of band-limited periodic discrete-time disturbance inputs with unknown period. A gradient descent algorithm is used to minimize a cost function and update the period estimates on-line.

The second approach to adaptive regulation consists of either totally or partially estimating the disturbance model on-line and using the estimated model in the controller design. We can distinguish two main methods. The first method consists of implementing an adaptive version of the Internal Model Principle. Feng *et al.*, 1992 [16] and Palaniswami, 1993 [17] used an indirect discrete-time adaptive pole placement approach to asymptotically reject unknown disturbance inputs with known model structure. The disturbance model is explicitly included in the controller in the feedback system. A modified plant model is used to identify the disturbance model parameters using variants of the least squares algorithm. The disturbance model identification does not require knowledge of the disturbance values. Convergence of the adaptive algorithm given in [17] relies on a persistent excitation assumption whereas no such assumption has been made in showing the stability and global convergence of the adaptive algorithm in [16]. Errors in the adaptive algorithm that result from the use of the certainty equivalence principle in [16] have been treated as unmodeled dynamics with known bounds. An improved algorithm that does not require knowledge of these bounds is given in [18]. The second method, discussed in Yang and Tomizuka, 1994 [19], consists of implementing and adaptive version of the External Model Principle. The External Model Principle consists of using the disturbance model outside the loop to provide values of the disturbance input. The latter are used in a feedforward disturbance cancellation algorithm. In [19], both partially known non-minimum phase plant and partially known disturbance input model have been considered. It is assumed that only the orders of the numerator and denominator polynomials and delay in the plant model are known. The modes of the disturbance are assumed known but the coefficients of the disturbance model numerator polynomial are unknown. A two-stage identification algorithm is used to identify the unknown plant and disturbance model parameters.

The third and last approach, presented in Wang *et al.*, 1991 [20] is based on augmenting a stabilizing controller with an adaptation mechanism to improve the overall system tracking and disturbance rejection performance. Although not explicitly stated, the adaptation is based on writing a parametrization of stabilizing controllers in terms of a stable transfer function and adjusting the parameters of the latter. The gradient descent algorithm is used in the parameter adjustment to minimize a performance index represented by the mean square tracking error. An averaging analysis is used to derive qualitative properties of the adaptation algorithm. It was shown that in the absence of unmodeled plant dynamics the performance index to be minimized gets close to its absolute minimum. Although the asymptotic performance of the adaptive control system can be better than that of the nonadaptive system, there is no guarantee that the disturbance is completely rejected. This is due to the fact that no specifications on the output disturbance were given. The main result of the paper concerns the boundedness of the different signals and estimated parameters in the adaptive control system.

In this paper we consider the specific problem of adaptively rejecting band-limited discrete disturbance inputs in (1) with unknown and time-varying frequencies $w_n(k)$, amplitudes $c_n(k)$, and phases $\phi_n(k)$, $0 \le n \le k_0$. More precisely, it is assumed that the frequencies, amplitudes, and phases are piece-wise constant functions of time. For a given nominal plant model, we present the following original results:

- *Off-line controller design*: where a controller that achieves regulation is *numerically designed* off-line based on the assumption that only the sequence of discrete disturbance input values (as opposed to a model of the disturbance) is available. A least squares based optimization algorithm is used in the controller design. It is shown that if the off-line designed controller achieves regulation, then it must include a model of the disturbance input.
- *On-line controller design*: Recursive versions of the off-line algorithm used above for controller design are presented and their convergence properties analyzed. Conditions under which the on-line algorithms yield an asymptotic controller that achieves regulation are presented. Conditions both for the case where the disturbance input properties are constant but unknown and for the case where they are unknown and time varying are given. The on-line controller construction amounts to an adaptive implementation of the internal model principle.

The class of disturbance inputs under consideration is more general than the class of band-limited periodic signals treated in [6]. The regulation problem is solved within the set of stabilizing controllers constructed using the Youla parametrization [21], [22]. The purpose of the adaptation is to tune the Youla parameter in the stabilizing controller in order to asymptotically satisfy a set of interpolation conditions that are equivalent to disturbance rejection. Hence, the adaptation algorithm does not have to deal, in the case of periodic signals, with the explicit estimation of the period of the disturbance as is done in [6], [13], [15]. The adaptation approach is the same as that used in [20], [23]. The purpose of the adaptation in [23] is to improve the performance of a nominal optimal disturbance rejection controller in the face of plant model uncertainties by adjusting the Youla parameter. In this work, the primary objective of the adaptation is to construct, on-line and for a given nominal plant, an asymptotic controller capable of performing asymptotic disturbance rejection. In fact, the asymptotic controller represents an adaptive implementation of the internal model principle. Such result is lacking in [20] since no assumptions were made on the disturbance input. The adaptation approach under consideration requires knowledge of the disturbance input–plant output transfer function which is different from the cases treated in [6], [13], [15], [20] where output disturbances have been considered.

The parameter adaptation is performed using two variants of the recursive least squares (RLS) algorithm. The first RLS algorithm uses a dead zone with exponentially decreasing width. The algorithm is used in the case where the disturbance properties are unknown but constant. It is shown that asymptotic disturbance rejection can be achieved under the assumption of boundedness of the regression vector. The algorithm requires knowledge of a bound on the disturbance response of the optimal control system. In order to deal with disturbance inputs with time varying frequencies, a RLS algorithm with time-varying weighting is considered. In this case, the persistent excitation assumption is invoked to show asymptotic rejection of disturbance inputs with time varying properties.

The rest of the paper is organized as follows. Section 2 summarizes standard results on stabilization and construction of the set of stabilizing controllers using the Youla parametrization. Section 3 presents the formulation of regulation requirement as a set of interpolation conditions. The off-line design of a controller that achieves regulation is discussed in section 4. Recursive versions of the off-line design algorithm, to be used in adaptation, are presented and their properties analyzed in section 5. The performance robustness of the off-line designed controller in the face of plant model uncertainties is discussed in section 6. The performance of the adaptation algorithms is illustrated through an example in section 7.

2. PRELIMINARIES

In the following, standard results regarding the use of coprime fraction representation in the stability analysis of feedback systems are summarized. Detailed discussions of the coprime fraction representations can be found in [24], [25], [26].

2.1. Parametrization of the Set of All Stabilizing Controllers

Let R_p denote the set of proper real rational transfer matrices and RH_∞ the subset of asymptotically stable real rational transfer matrices. Let $G_0 \in R_p$ denote the SISO plant transfer function. Consider a coprime factorization of G_0 given by:

$$G_0 = N_0 M_0^{-1} \tag{2}$$

where N_0 and M_0 are in RH_∞. Let U and V in RH_∞ be such that the following Bezout identity is satisfied:

$$M_0 V - U N_0 = 1 \tag{3}$$

A stabilizing controller K_0 for the plant G is then given by:

$$K_0 = U V^{-1} \tag{4}$$

A block diagram of the closed loop system is shown in Fig. 1. Using the base controller K_0, the set of all stabilizing controllers can be constructed using the Youla parametrization [21], [22]. In fact, for any $Q \in RH_\infty$, the controller K given by:

$$K = (U + M_0 Q)(V + N_0 Q)^{-1} \tag{5}$$

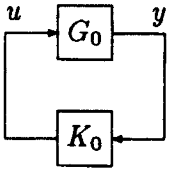

Figure 1 The closed loop system.

is a stabilizing controller for the plant G_0. Moreover, every rational stabilizing controller K has the form (5) for some $Q \in RH_\infty$.

2.2. The Closed Loop System

Given a stabilizing controller as in (5), then the closed loop system can be reconfigured as shown in Fig. 2 where **J** is given by:

$$\mathbf{J} = \begin{bmatrix} K_0 & V^{-1} \\ V^{-1} & -V^{-1}N_0 \end{bmatrix} \tag{6}$$

Let w denote the disturbance input to the plant and e an error signal. Let **P** be the transfer matrix of the augmented plant with inputs $\begin{bmatrix} w \\ u \end{bmatrix}$ and outputs $\begin{bmatrix} e \\ y \end{bmatrix}$. We have:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{7}$$

where $P_{22} = G_0$ (Fig. 3). The two blocks **P** and **J** can be lumped together in a single block

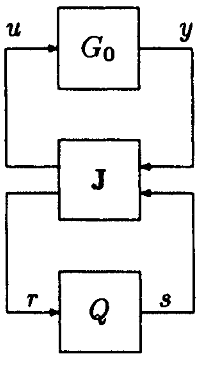

Figure 2 Block diagram of the closed loop system with a parametrized controller.

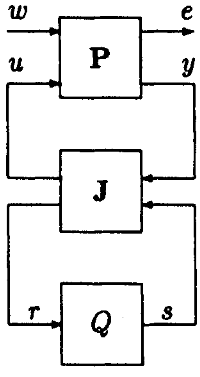

Figure 3 Block diagram of the closed loop system with augmented plant.

(Fig. 4) with transfer matrix **T** given by:

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix} \tag{8}$$

where

$$T_{11} = P_{11}+P_{12}UM_0P_{21} \tag{9}$$

$$T_{12} = P_{12}M_0 \tag{10}$$

$$T_{21} = M_0P_{21} \tag{11}$$

The stability properties of the closed loop system are given as follows:

LEMMA 1 *[25] Assume G_0 is strictly proper and has no hidden unstable modes. Then a controller $K \in R_p$ stabilizes* **P** *if and only if K stabilizes G. Moreover, the resulting closed-loop system transfer functions T_{11}, T_{12}, and T_{21} are all in RH_∞.*

In the following, we assume the error signal {e(•)} *represents the disturbance response. Let* W (z) *and* E(z) *denote the* $\mathcal{Z}$ transforms of the disturbance input {w(•)} *and the*

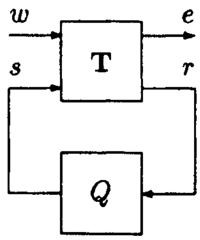

Figure 4 Block diagram of the closed loop system with lumped P-J blocks.

disturbance response $\{e(\bullet)\}$ *respectively. We have:*

$$E(z) = F_{T,Q}(z)W(z) \tag{12}$$

where

$$F_{T,Q}(z) = [T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)] \tag{13}$$

Based on Lemma 1 and the fact that $Q \in RH_\infty$, we have $F_{T,Q} \in RH_\infty$.

3. CONDITIONS FOR REGULATION

In this section, the disturbance input $\{w(\bullet)\}$ *in (1) is assumed to be known with time invariant properties. In other words* k_0 *and* $\{w_n(k), c_n(k), \phi_n(k)\}$, $n = 0,\ldots, k_0$, *are assumed known with* $\{w_n(k), c_n(k), \phi_n(k)\}$, $n = 0,\ldots, k_0$, being constant over time. In order to accommodate any DC offset in the disturbance input, one of the frequencies, chosen to be w_0, is zero. The remaining frequencies satisfy $\omega_n \neq 0$, $n = 1,\ldots, k_0$. Also, the amplitudes are such that $c_n \neq 0$, $n = 0,\ldots, k_0$.

Conditions for regulation (asymptotic disturbance rejection) are given in the form of interpolation conditions. The main reason for casting the regulation requirement in the form of interpolation conditions is that such formulation lends it self, in a straight forward manner, to optimization.

3.1. Interpolation Conditions for Regulation

Let p_i; $i = 1,\ldots, n_p$; denote the poles of the $W(z)$. According to (1), all the poles of $W(z)$ are simple and located on the unit circle.

LEMMA 2 *Consider the closed loop system transfer function* $F_{T,Q}$ *(13) and define the following interpolation conditions:*

$$[T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)]|_{z = pi} = 0, \qquad i = 1, \cdots, n_p \tag{14}$$

Then regulation (asymptotic disturbance rejection) is achieved if and only if the interpolation conditions (14) are satisfied.

Proof See Appendix A.
The above conditions imply that the poles of $W(z)$ are also zeros of $F_{T,Q}(z)$.

Remark 1 The expression of the regulation requirement in the form of interpolation conditions can be extended to the case where the disturbance model does not have only simple poles. Let n_{dp} denote the number of distinct poles of $W(z)$ and n_i the multiplicity of a pole p_i, $1 \leq i \leq n_{dp}$. The poles of $W(z)$ are assumed located either on or outside the unit circle. The interpolation conditions for regulation can then be given as follows:

$$\frac{d^j}{dz^j}[T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)]|_{z=pi} = 0; \quad i = 1, \cdots, n_{dp}; j = 0, \cdots, n_i - 1$$

Since the space RH_∞ is infinite dimensional, it is impractical to search over the whole space for functions Q that satisfy the interpolation conditions (14). Rather, a special form of the Youla parameter is considered which restricts the domain of search. More precisely, we will use a Ritz-type parametrization in the form [10]:

$$Q(z) = \sum_{i=1}^{nq} q_i \psi_i(z) \tag{15}$$

where $\psi_i(z) = z^{1-i}$. As $n_q \to \infty$, the representation (15) can be used to represent any function in RH_∞.

Define the following parameter vector:

$$\Theta = [q_1, \cdots, q_{nq}]^T \tag{16}$$

There are two main reasons for considering the above representation for Q, the first being that $Q \in RH_\infty$ for any $\Omega \in R^{n_q}$. Such property is important especially in adaptation discussed in section 5. In fact, during adaptation, the parameter vector Θ is to be adjusted on line to minimize a given performance index. Since $Q \in RH_\infty$ for any $\Theta \in R^{n_q}$, then, at any time instant, the frozen time closed loop system transfer function $F_{T,Q}$ is stable. Therefore, if adaptation is slow enough, the closed loop system can be guaranteed to be stable during the adaptation process [27, p. 125]. The second reason for considering the representation (15) for Q is related to the fact that the closed loop system transfer functions satisfying the interpolation conditions (14) form a convex set. With the particular representation of Q given in (15), the convexity of the set of closed loop system transfer functions satisfying (14) translates into convex conditions on the parameter vector Θ as given by the following Lemma:

LEMMA 3 *The interpolation conditions (14) are equivalent to the following linear-affine constraint on the parameter vector* Θ:

$$\mathbf{A}\,\Theta + B = 0 \tag{17}$$

where $\mathbf{A}$ *and* B *are real matrices given by (94) and (95) and* $\mathbf{A}$ *is* $n_p \times n_q$.

Proof *See* Appendix B.

The above constraint (17) can be used in Boyd's formalism for the design of linear controllers based on convex optimization [10]. In fact, many controller design specifications are closed-loop convex (*i.e.*, the set of closed-loop system transfer functions that satisfy the given specifications is convex). Numerical optimization algorithms can be used to search within the convex set for the optimal closed-loop system transfer function (or controller). The interpolation condition (17) being convex can be added to the design objectives and the optimization problem becomes that of convex optimization subject to a convex constraint.

The condition for existence of a solution to (17) can be given as follows:

A1 The vector B is in the column space of $\mathbf{A}$.

Assuming **A1** is satisfied, we can have two types of solutions:

- *Case 1* Unique solution, which happens when rank of $\mathbf{A}$ is n_q.
- *Case 2* Infinitely many solutions, which results from the situation where the rank of $\mathbf{A}<n_q$.

The adaptation process is based on adjusting the parameter vector Θ on-line to minimize a time domain performance index. Since the regulation requirement is expressed in the frequency domain using interpolation conditions (14), it is necessary to relate the latter to the minimization of a time domain criterion. The following lemma provides such result:

LEMMA 4 *Assume the signal* $\{w(.)\}$ *is quasi-stationary, the matrix* $\mathbf{A}$ *in (17) has rank* n_p, *and* **A1** *is satisfied. Then solving the minimization problem*

$$\min_{\Theta} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} e^2(k, \Theta) \tag{18}$$

is equivalent to satisfying the interpolation conditions in (17).

Proof See Appendix C.

Under the assumptions of Lemma 4, minimizing the mean square error is equivalent to satisfying the interpolation conditions only when the number of parameters in (15) is greater than or equal to the number of interpolation conditions (14).

4. OFF-LINE CONTROLLER DESIGN

The purpose of this section is to discuss the *numerical* off-line design of a controller capable of achieving regulation. The off-line design is based on the assumption that a sequence of discrete values of the disturbance input is available *a priori*. The controller is numerically designed based only on the existing discrete disturbance values and not on a model of the disturbance input. The off-line controller design algorithm represents a starting point for the derivation of on-line recursive controller design algorithms to be used in adaptation. The conditions under which the off-line design algorithm yields a controller that achieves regulation are given and the structure of the controller is analyzed.

In order to present a simple controller design procedure, we assume in (15) the following:

$$\psi_i(z) = z^{1-i}, \qquad i=1, \cdots, n_q \tag{19}$$

The Youla parameter $Q(z)$ in (15) is a Finite Impulse Response (FIR) filter with transfer function $q_1 + q_2^{z-1} + \ldots + q_{nq}z^{-nq+1}$. Define the following signals:

$$\begin{aligned} \{\upsilon_0(\cdot)\} &= Z^{-1}(T_{11}(z)W(z)) \\ \{\upsilon_i(\cdot)\} &= Z^{-1}(T_{12}(z)\psi_i(z)T_{21}(z)W(z)) \qquad i = 1, \cdots, n_q \end{aligned} \tag{20}$$

The disturbance response (12) can then be expressed as:

$$e(k, \Theta) = v_0(k) - \phi(k)^T\Theta \tag{21}$$

where

$$\phi(k) = [-v_1(k), \ldots, -v_1(k - n_q + 1)]^T \tag{22}$$

Since the sequence of discrete disturbance input values is available a priori, then the sequence $\{v_1(.)\}$ can be easily computed and used to determine the sequence of vectors $\{\phi(.)\}$. The latter can then be used in a least squares based controller design algorithm as described in the following lemma:

LEMMA 5 *Assume the matrix* **A** *in (17) is square* ($n_p = n_q$) *and invertible and that* $\{v_1(\cdot)\}$ *is persistently exciting of order* n_q. *Then there exists a unique minimizer,* Θ_{min}, *for the mean square error and we have*:

$$\Theta_{min} = [\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \phi(k)\,\phi(k)^T]^{-1}[\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \phi(k)^T v_0(k)] \tag{23}$$

Moreover, Θ_{min} *satisfies the interpolation conditions.*

Proof. See Appendix D.

If the numerator of $T_{12}(z)\psi_1(z)T_{21}(z)$ and the denominator of $W(z)$ are coprime, then the condition that $\{v_1(.)\}$ be persistently exciting of order n_q is equivalent to that of $\{w(\cdot)\}$ being persistently exciting of order n_q.

In the following, it is desired to study the structure of the controller that results from the use of the parameter vector (23) in (15) and (5). More precisely, since such controller achieves regulation, it is desired to see if this numerically designed controller contains a model of the disturbance input. Consider the coprime fraction representations:

$$W(z) = N_w(z)M_w^{-1}(z) \tag{24}$$

$$T_{21}(z)W(z) = N_d(z)M_d^{-1}(z) \tag{25}$$

where $M_d^{-1}(z)$ contains only unstable poles. Hence the poles of $M_d^{-1}(z)$ are also poles of $W(z)$. The assumption from Lemma 5 that the matrix **A** is nonsingular implies that the pairs (T_{21}, M_w) and (T_{12}, M_d) are coprime.

Remark 2: The simplest interpretation of the above coprimeness results can be given in the case where $e = y$ (*i.e.*, $P_{11} = P_{21} = P_1$ and $P_{12} = P_{22} = P_2$) and $P_1 = 1$ (*i.e.*, case of output disturbance). In that case, the pair (T_{21}, M_w) being coprime implies that the plant does not contain any of the disturbance modes and the pair (T_{12}, M_d) being coprime implies that no disturbance modes are zeros of the plant.

The following lemma is an extension of Theorem 4.1 in [28]:

LEMMA 6 *Assume the conditions in* Lemma 5 *are satisfied and* $e = y$. *Then the controller obtained by using* Θ_{min} *from (23) in (15) and (5) contains a model of the disturbance input, that is, the poles of* $W(z)$ *are also poles of* $K(z)$ *in (5).*

Proof See Appendix D.

5. ADAPTIVE REGULATION

In this section, recursive versions of a variant of the least squares algorithm used in the off-line controller design are presented. The recursive algorithms are to be used in adaptive regulation where it is desired to construct, on-line, an asymptotic controller that rejects disturbance inputs of the form (1) with unknown and/or time varying properties. The weighted least squares algorithm, of which the standard least squares algorithm used in section 4 is a particular case, is used in the derivation of the recursive algorithms. Two types of recursive algorithms are considered. The first algorithm represents a recursive least squares (RLS) algorithm with dead zone where the width of the dead zone is exponentially decaying. This algorithm is to be used in the case where the disturbance input properties are unknown but constant. In the case where the disturbance input properties are unknown and time varying, an RLS algorithm with a time varying forgetting factor is considered. Although the second algorithm is considered for use in the case of time varying disturbance input properties, it can also deal with the case of constant and unknown disturbance input properties. The reason why the first algorithm is considered is that its convergence properties can be given under milder assumptions than those required for the convergence of the second algorithm. In fact, the RLS algorithm with dead zone requires only boundedness of the regression vector (22) to show asymptotic regulation whereas the RLS algorithm with time varying forgetting factor requires a persistent excitation assumption.

The regression vector (22) used in (21) requires knowledge of $\{v_1(.)\}$ which was computed based on the known disturbance input values. During on-line controller design, the disturbance input is assumed to be unmeasurable. Therefore, it is desired to determine values of the sequence $\{v_1(.)\}$ without knowing the disturbance input values. Using (9), (10), and (11), the disturbance response in (12) is given by:

$$\begin{aligned} E(z) &= [T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)]W(z) \\ &= P_{11}(z)W(z) + [P_{12}(z)U(z)M_0(z)P_{21}(z) + P_{12}(z)M_0(z)Q(z)M_0(z)P_{21}(z)]W(z) \\ &= P_{11}(z)W(z) + [P_{12}(z)U(z) + P_{12}(z)M_0(z)Q(z)]M_0(z)P_{21}(z)W(z) \qquad (26) \end{aligned}$$

From (2) and (7), we also have:

$$\begin{aligned} y(z) &= P_{21}(z)W(z) + P_{22}(z)u(z) \\ &= P_{21}(z)W(z) + G_0(z)u(z) \\ &= P_{21}(z)W(z) + M^{-1}_0(z)N_0(z)u(z) \qquad (27) \end{aligned}$$

Let $r(z) = M_0(z)P_{21}(z)W(z)$. From (27) we get:

$$r(z) = M_0(z)y(z) - N_0(z)u(z) \tag{28}$$

The disturbance response can then be given as follows:

$$E(z) = [P_{11}(z)W(z) + P_{12}(z)U(z)r(z)] + [P_{12}(z)M_0(z)Q(z)r(z)] \tag{29}$$

Assuming Q is given as in (15), then:

$$E(z) = V_0(z) + \sum_{i=1}^{n_q} q_i V_i(z) \tag{30}$$

where

$$V_0(z) = P_{11}(z)W(z)+P_{12}(z)U(z)r(z) \tag{31}$$

$$V_i(z) = P_{12}(z)M_0(z)\psi_i(z)r(z) \qquad i = 1,\cdots, n_q \tag{32}$$

If Q is given as in (15) and (19), then we have:

$$E(z) = V_0(z) + \sum_{i=1}^{n_q} q_i z^{1-i} V_1(z) \tag{33}$$

Hence, we have

$$e(k) = \upsilon_0(k) - \phi(k)^T\Theta \tag{34}$$

where $\{\upsilon_i(.)\} = Z^{-1}(V_i(z))$, $i = 0, 1$, and where $\phi(.)$ and Θ are given by (22) and (16), respectively. It can be seen from the derivations given above that the regression vector does not require knowledge of the disturbance input values. In fact, according to (28) it is only necessary to know $u(.)$ and $y(.)$ in order to compute $\upsilon_1(.)$ which is the only variable present in the regression vector. In the following, the two RLS algorithms are presented and their convergence properties discussed.

5.1. Case of Unknown but Time-Invariant Disturbance Properties

In this case, the disturbance input is of the form:

$$w(k) = \Sigma_{n=0}^{k_0} c_n \, cos(\omega_n k + \phi_n) \tag{35}$$

where the amplitudes c_n, frequencies ω_n, and phases ϕ_n, $n = 0,\cdots, k_0$ are unknown. The regulation problem is equivalent in this case to the estimation of the parameters of a system subject to a bounded disturbance input. In such situations, a RLS algorithm with dead zone can be used to estimate the unknown parameters. In the following, we introduce a bound on the optimal disturbance response which will be used in the adaptation algorithm. First consider the following assumption:

A3 The matrix **A** in (17) is square and nonsingular.

Assuming **A3** is satisfied, then there exists a unique parameter vector Θ_{min} that satisfies the interpolation conditions. Let $\{e_{min}(.)\}$ denote the exponentially decaying disturbance response that results from the use of Θ_{min} in (21). The disturbance response rate of decay, denoted β_{min}, can be determined by examining the poles of $T_{11} + T_{12}QT_{21}$, regardless of the disturbance input. The disturbance response maximum magnitude, denoted α_{max}, is assumed known based on the physical laws governing the plant dynamics. Therefore, we have

$$|e_{min}(k)| \leq \alpha_{max} e^{-\beta_{min} k} \tag{36}$$

The following assumption is invoked since it will be used in subsequent proofs.

A4 The constants α_{max} and β_{min} in (36) are assumed known *a priori*.

The adaptation algorithm is given as follows:

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) + \lambda(k+1) \frac{P(k)\,\phi(k+1)}{1 + \lambda(k+1)\,\phi(k+1)^T P(k)\,\phi(k+1)} e(k+1) \tag{37}$$

$$P(k+1) = P(k) - \lambda(k+1) \frac{P(k)\phi(k+1)\phi(k+1)^T P(k)}{1 + \lambda(k+1)\phi(k+1)^T P(k)\phi(k+1)} \tag{38}$$

with $\hat{\Theta}(0) = \hat{\Theta}_0$ and $P(0) = P_0 > 0$ and where:

$$\lambda(k) = \begin{cases} 1 \text{ if } \dfrac{e^2(k)}{1+\phi(k)^T P(k-1)\phi(k)} > \alpha_{max}^2 e^{-2\beta_{min} k} \\ 0 \qquad \textit{otherwise,} \end{cases} \tag{39}$$

The convergence properties of the algorithm are given in the following Theorem:

THEOREM 1 *Assume **A3** and **A4** are satisfied. Then the algorithm given by (37), (38), and (39) yields:*

(*a*)

$$\lim_{k\to\infty} \frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} = 0 \tag{40}$$

(*b*)

$$\lim_{k\to\infty} \|\hat{\Theta}(k) - \hat{\Theta}(k-l)\| = 0 \qquad \forall\, l > 0 \tag{41}$$

Moreover, if the sequence $\{\phi(.)\}$ in (22) is bounded, then we have:

(*c*)

$$\lim_{k\to\infty} e(k) = 0 \tag{42}$$

Proof *See* Appendix E.

Remark 3 The estimated parameter vector converges to a fixed parameter vector. Since regulation is achieved, and since there is a unique parameter vector that achieves regulation, we have $\lim_{k\to\infty} \hat{\Theta}(k) = \Theta_{min}$.

Remark 4. The adaptation algorithm works well only in the case where the nominal parameter vector Θ_{min} is constant. When Θ_{min} changes with time, the performance of the adaptation algorithm deteriorates due to the fact that the adaptation gain, given by $\frac{P(k-1)\phi(k)}{1+\lambda(k)\phi(k)^T P(k-1)\phi(k)}$ in (37), becomes very small a few time steps after the algorithm is started. Therefore, the estimation algorithm is not capable of tracking any changes in the nominal parameter vector. A second adaptation algorithm, capable of tracking piece-wise constant parameter vector Θ_{min} is given in the next section.

5.2. Case of Unknown and Time-Varying Disturbance Properties

In this section, it is assumed that the coefficients c_n, frequencies ω_n, and phases ϕ_n; $n = 0,\cdots, k_0$; in (1) are unknown and possibly time-varying. In order to be able to reject such disturbance inputs, it is necessary to use an adaptation algorithm capable of tracking time varying parameters. The recursive least squares algorithm with time varying forgetting factor is considered for this purpose. Some of the properties of such algorithm are discussed in [29], [30] and the references therein. The algorithm is given as follows:

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) + \lambda(k+1)P(k+1)\phi(k+1)e(k+1) \tag{43}$$

$$P^{-1}(k+1) = \lambda(k+1)[P^{-1}(k) + \phi(k+1)\phi^T(k+1)] \tag{44}$$

with $\hat{\Theta}(0) = \hat{\Theta}_0$, $P(0) = P_0 > 0$, and where $\lambda(k)$ is the time varying forgetting factor satisfying $0 < \lambda_{min} \le \lambda(k) \le \lambda_{max} < 1$. Using the Matrix Inversion Lemma [31], the above equations can be rewritten as follows:

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) + \frac{P(k)\phi(k+1)}{1+\phi(k+1)^T P(k)\phi(k+1)} e(k+1) \tag{45}$$

$$P(k+1) = \frac{1}{\lambda(k+1)}[P(k) - \frac{P(k)\phi(k+1)\phi(k+1)^T P(k)}{1+\phi(k+1)^T P(k)\phi(k+1)}] \tag{46}$$

where $\hat{\Theta}(0) = \hat{\Theta}_0$ and $P(0) > 0$. In the following, it is assumed that the disturbance parameters in (1) are piecewise constant and that changes in the parameters are sufficiently spaced in time to allow parameter convergence. Let $\Theta_{min}(k)$ denote the parameter vector satisfying the interpolation conditions corresponding to the disturbance input properties at time k. We can then define the parameter error at time k:

$$\tilde{\Theta}(k) = \Theta_{min}(k) - \hat{\Theta}(k) \tag{47}$$

and the change $\Delta\Theta_{min}(k)$ in the parameter vector satisfying the interpolation conditions:

$$\Delta\Theta_{min}(k) = \Theta_{min}(k) - \Theta_{min}(k-1) \tag{48}$$

and the following disturbance response signals:

$$e(k+1) = \upsilon_0(k+1) - \phi^T(k+1)\hat{\Theta}(k) \tag{49}$$

$$e_{min}(k+1) = \upsilon_0(k+1) - \phi^T(k+1)\Theta_{min}(k) \tag{50}$$

Using eq. (43), we have,

$$\begin{aligned}\Theta_{min}(k+1) - \hat{\Theta}(k+1) = {} & [\Theta_{min}(k+1) - \Theta_{min}(k)] + [\Theta_{min}(k) - \hat{\Theta}(k)] \\ & -\lambda(k+1)P(k+1)\phi(k+1)[e(k+1) - e_{min}(k+1) \\ & + e_{min}(k+1)]\end{aligned} \tag{51}$$

which can be rewritten as

$$\begin{aligned}\tilde{\Theta}(k+1) = {} & [I - \lambda(k+1)P(k+1)\phi(k+1)\phi^T(k+1)]\tilde{\Theta}(k) + \Delta\Theta_{min}(k) \\ & - \lambda(k+1)P(k+1)\phi(k+1)e_{min}(k+1)\end{aligned} \tag{52}$$

From eq. (44), we have

$$[I - \lambda(k+1)P(k+1)\phi(k+1)\phi^T(k+1)] = \lambda(k+1)P(k+1)P^{-1}(k) \tag{53}$$

Therefore, we have

$$\begin{aligned}\tilde{\Theta}(k+1) = {} & [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\Theta}(k) + \Delta\Theta_{min}(k) - \\ & \lambda(k+1)P(k+1)\phi(k+1)e_{min}(k+1)\end{aligned} \tag{54}$$

Using the approach in [29] for the convergence analysis of the adaptation algorithm, the effects of the initial conditions, changes in the parameter vector Θ_{min}, and the signal $\{e_{min}(.)\}$ on the parameter estimation error $\tilde{\Theta}\ (k)$ can be studied separately using the following equations:

$$\tilde{\Theta}_1(k+1) = [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\Theta}_1(k),\ \tilde{\Theta}_1(0) = \tilde{\Theta}(0) \tag{55}$$

$$\tilde{\Theta}_2(k+1) = [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\Theta}_2(k) + \Delta\Theta_{min}(k),\ \tilde{\Theta}_2(0) = 0 \tag{56}$$

$$\begin{aligned}\tilde{\Theta}_3(k+1) = {} & [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\Theta}_3(k) - \\ & \lambda(k+1)P(k+1)\phi(k+1)e_{min}(k+1),\ \tilde{\Theta}_3(0) = 0\end{aligned} \tag{57}$$

In order to show convergence of the algorithm, the following persistent excitation assumption is invoked [32]:

A5 The signal $\{v_1(.)\}$ in the regression vector $\phi(.)$ is such that

$$\lim_{N\to\infty} \lambda_{min}[\sum_{i=1}^{N} \phi(i)\phi^T(i)] = \infty \tag{58}$$

THEOREM 2 *Assume* **A3** *and* **A5** *are satisfied. Then the algorithm given by (45), (46) yields*:

(*a*)

$$\lim_{k \to \infty} \tilde{\Theta}_1(k) = 0 \tag{59}$$

(*b*)

$$\lim_{k \to \infty} \tilde{\Theta}_2(k) = 0 \tag{60}$$

for a single change in the parameter vector Θ_{min}.

(*c*)

$$\lim_{k \to \infty} \tilde{\Theta}_3(k) = 0 \tag{61}$$

Proof *See* Appendix F.

6. ROBUSTNESS OF THE OFF-LINE DESIGNED CONTROLLER

The purpose of this section is to study the performance robustness of the off-line designed controller in the face of uncertainties in the plant model coprime factor representation. The off-line controller is assumed to contain a model of the disturbance input. It is shown that, if the stability of the closed loop system involving the true plant is realized, then under some mild assumptions, the disturbance rejection performance is preserved. Different approaches to the study of robustness and which led to similar conclusions are given in [33] and [28].

Consider a nominal plant model G_0 with coprime factors satisfying (3) and a stabilizing controller K_0 as in (4). Let G denote the actual plant model with a coprime factorization $G = NM^{-1}$ where:

$$M = (M_0 + \Delta M) \in RH_\infty \tag{62}$$

$$N = (N_0 + \Delta N) \in RH_\infty \tag{63}$$

The type of model uncertainties considered above is very general. The coprime factor uncertainties ΔM and ΔN are both in RH_∞.

A6 The controller K_0 stabilizes both G_0 and G.

The above assumption is not very restrictive since, in most engineering applications, a stabilizing controller is always designed based on a nominal plant model and then used with the actual plant whose model is always different from the nominal model.

LEMMA 7 *Assume* **A6** *is satisfied. There exists* $R \in RH_\infty$ *such that* G *is given by*:

$$G = (N_0 + RV)(M_0 + RU)^{-1} \tag{64}$$

Proof The result in the Lemma is the dual of the parametrization of the set of all stabilizing controllers and can be obtained by interchanging the role of the plant and controller.

Based on the above representation of the plant, the plant factor uncertainties are given by $\Delta M = RU$ and $\Delta N = RV$. Moreover, it can be easily verified that the following Bezout identity holds in this case:

$$MV - UN = 1 \tag{65}$$

Consider a parametrized stabilizing controller $K = (U + QM_0)(V + QN_0)^{-1}$ for the plant G_0. The stability of the closed loop system involving K and G is given by the following lemma [23]:

LEMMA 8 *The closed-loop system is stable if and only if Q stabilizes R, that is* $(1 - QR)^{-1} \in RH_\infty$.

Assume the closed-loop system is stable. Let the controller K be used with the true plant P. The resulting closed loop system transfer matrix relating $\begin{bmatrix} w \\ s \end{bmatrix}$ and outputs $\begin{bmatrix} e \\ r \end{bmatrix}$ is given by:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{66}$$

where

$$T_{11} = P_{11} + P_{12} U M P_{21} \tag{67}$$

$$T_{12} = P_{12} M \tag{68}$$

$$T_{21} = M P_{21} \tag{69}$$

$$T_{22} = R \tag{70}$$

We have

$$E(z) = F_{T,Q}(z) W(z) \tag{71}$$

where

$$F_{T,Q}(z) = [T_{11}(z) + T_{12}(z) \frac{Q(z)}{1 - Q(z)\,R(z)} T_{21}(z)] \tag{72}$$

It is desired to study the performance robustness of the off-line designed disturbance rejection controller in the face of the plant model uncertainties. Consider the coprime fraction representations:

$$W(z) = N_{\omega}(z)M_{\omega}^{-1}(z) \tag{73}$$

$$T_{21}(z)W(z) = N_d(z)M_d^{-1}(z) \tag{74}$$

where $M_d^{-1}(z)$ contains only unstable poles. Hence, assuming $T_{21} \in RH_{\infty}$, the poles of $M_d^{-1}(z)$ are also poles of $W(z)$.

LEMMA 9 *Assume the closed-loop system involving the controller K designed as in Lemma 6 and the true plant given by (64) is stable (i.e. $(1 - QR^{-1} \in RH_{\infty})$). Moreover, assume, $P_{21} \in RH_{\infty}$, and $e = y$. Then the closed loop system disturbance rejection performance is robust in the face of the uncertainties $\Delta M = RU$ and $\Delta N = RV$ in the plant coprime factor representation.*

Proof See Appendix H.

Remark 5 If in addition, the pair (T_{21}, M_w) is coprime, then the interpolation conditions obtained with the true plant are equivalent to the interpolation conditions obtained with the nominal plant.

7. EXAMPLES

Consider the SISO plant given by the following state space representations:

$$\begin{aligned} x(k+1) &= .8x(k) + u(k) + .5w(k) \\ y(k) &= x(k) \\ x(0) &= 0 \end{aligned}$$

The disturbance input $\{w(.)\}$ is a sine wave given by:

$$w(k) = sin(\omega_1 kT_s) + sin(\omega_2 kT_s) \tag{75}$$

where ω_1 and ω_2 are the frequencies of the continuous time sinusoids and $T_s = 1$ *sec* the sampling period.

In the example given above, the matrix **P** in (7) is such that $P_{11} = P_{21}$ and $P_{12} = P_{22}$. Hence the disturbance response e is the same as the plant output y. The following stabilizing controller is considered:

$$K_0(z) = -\frac{.06}{z - .1} \tag{76}$$

In order to construct a set of stabilizing controllers, the Youla parameter Q in (5) is chosen to be of the form:

$$Q(z) = q_1 + q_2 z^{-1} + q_3 z^{-2} + q_4 z^{-3} \tag{77}$$

Notice that the number of parameters n_q is the same as the number of poles n_p of $W(z)$. The resulting parametrized controller transfer function (5) is:

$$K(z) = \frac{q_1 z^4 + (-.06 + q_2 - .8q_1)z^3 + (q_3 - .8q_2)z^2 + (q_4 - .8q_3)z - .8q_1}{z^4 + (q_1 - .1)z^3 + q_2 z^2 + q_3 z + q_4} \quad (78)$$

and the transfer function $F_{T,Q}(z)$ relating the disturbance input w to the disturbance response e is given by:

$$F_{T,Q}(z) = \frac{E(z)}{W(z)} = \frac{z^4 + (q_1 - .1)z^3 + q_2 z^2 + q_3 z + q_4}{z^3(z^2 - .9z + .14)} \quad (79)$$

For the disturbance input considered in (75), the disturbance response is given by:

$$E(z) = \left[\frac{z^4 + (q_1 - .1)z^3 + q_2 z^2 + q_3 z + q_4}{z^3(z^2 - .9z + .14)}\right]\left[\frac{sin(\omega_1 T_s)z}{z^2 - 2cos(\omega_1 T_s)z + 1} + \frac{sin(\omega_2 T_s)z}{z^2 - 2cos(\omega_2 T_s)z + 1}\right] \quad (80)$$

In order to have $E(z) \in RH_\infty$, we must have:

$$q_1 = 0.1 - 2(\cos(\omega_1 T_s) + \cos(\omega_2 T_s)) \quad (81)$$

$$q_2 = 2(1 + 2\cos(\omega_1 T_s)\cos(\omega_2 T_s)) \quad (82)$$

$$q_3 = -2(\cos(\omega_1 T_s) + \cos(\omega_2 T_s)) \quad (83)$$

$$q_4 = 1 \quad (84)$$

It is important to notice that if, for a given ω_1 and ω_1, the values of q_1, q_2, q_3, and q_4 given by (81), (82), (83), and (84), respectively, are used in the controller K in (78), the controller would contain a model of the disturbance input (75) (the poles of $W(z)$ are also poles of $K(z)$). Hence, if during adaptation, the adjusted Q parameters converge to the nominal parameters given in (81), (82), (83), and (84), then the controller design would represent an adaptive implementation of the Internal Model Principle.

The frequencies of the disturbance are $\omega_1(k) = .5$ *rad/sec* and $\omega_2(k) = 2$ *rad/sec* for $0 \leq k < 400$ and then change to $\omega_1(k) = 1.5$ *rad/sec* and $\omega_2(k) = 3$ *rad/sec* for $400 \leq k$. Therefore, the parameter vector $\Theta = [q_1, q_2, q_3, q_4]^T$ that should be used to achieve regulation is as follows: for $0 \leq k < 400$, $\Theta = [-.8227, .539, -.923, 1]^T$ and for $400 \leq k$, $\Theta = [1.94, 1.72, 1.84, 1]^T$. The performance of the two adaptation algorithms is discussed below.

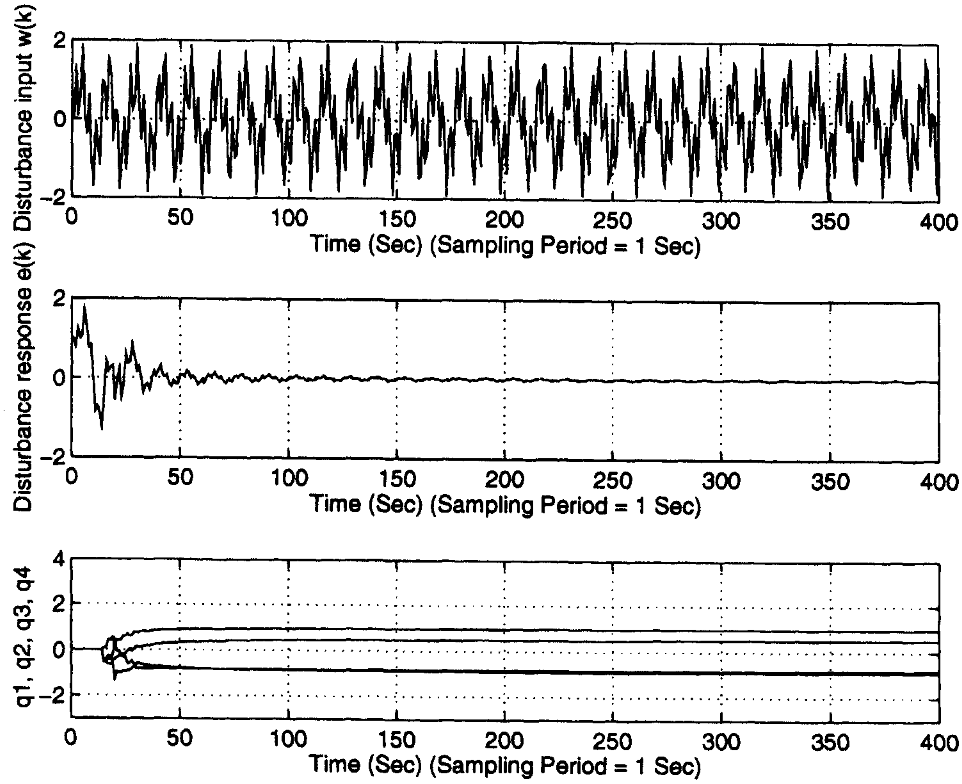

Figure 5 Response of the adaptive control sysem using the RLS algorithm with dead zone. Top: Disturbance input w(k). Middle: Response of the adaptive control system to the disturbance input w(k). Bottom: Parameters of the controller parametrizing mapping Q.

7.1. The RLS Algorithm with Dead Zone

In order to use the RLS algorithm with dead zone, it is necessary to determine β_{min} and α_{max} in (36). A conservative value for β_{min} can be given by examining the poles of $F_{T,Q}(z)$ in (79) which are located at $z = 0, .2$, and .7. The time constant corresponding to the slower pole at $z = .7$ is 2.8 sec. Therefore, we can take $\beta_{min} = .3 < \frac{1}{2.8}$. The value of α_{max} is set equal to 1. The performance of the adaptation algorithm is illustrated in Fig. 5. The initial conditions are $\Theta(0) = [0, 0, 0, 0]^T$ and $P(0) = 1000I$ where I is $4 \leq 4$ identity matrix. For $0 \leq k < 400$, the closed-loop system was able to slowly reject the disturbance input. The estimated parameters converged to the nominal parameters.

7.2. The RLS Algorithm with a Forgetting Factor

The forgetting factor in this algorithm is a constant $\lambda = .9$. The initial conditions of the algorithm are $\tilde{\Theta}(0) = [0, 0, 0, 0]^T$ and $P(0) = 10I$ where I is 4×4 identity matrix. The performance of the closed loop control system is shown in Fig. 6. It can be seen that the adaptive control system was capable of rejecting the disturbance input even when the

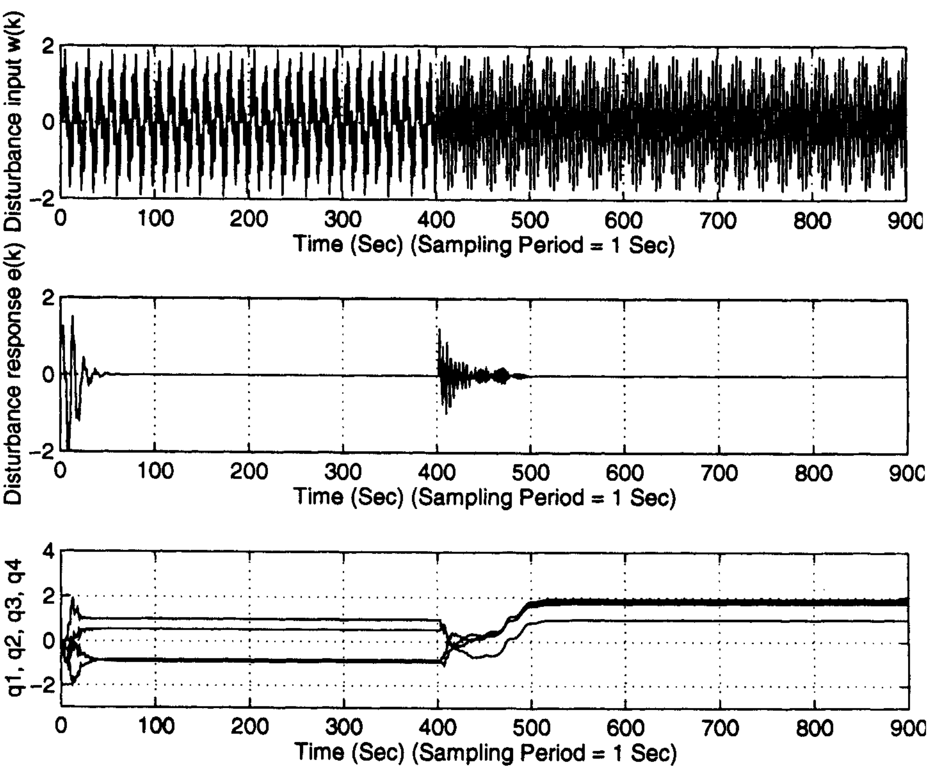

Figure 6 Response of the adaptive control system using the RIS algorithm with forgetting factor. Top: Disturbance input w(k) with time varying frequencies. Middle: Response of the adaptive control system to the disturbance input w(k). Bottom: Parameters of the controller parametrizing mapping Q.

frequency of the disturbance input changes. The estimated parameters converged to the nominal parameters. Hence, for both values of the disturbance frequency, the adaptive control algorithm was able to construct an internal model of the disturbance input in the controller.

8. SUMMARY AND CONCLUSIONS

The problem of adaptively rejecting band-limited disturbance inputs with unknown and/or time varying characteristics was considered. The adaptation approach is based on searching, *on-line*, within a parametrized set of stabilizing controllers, for the controller that achieves asymptotic disturbance rejection. An adaptive disturbance rejection algorithm which is robust to changes in the disturbance input modes is presented and its properties analyzed. Under some mild assumptions, the adaptation results in an on-line implementation of the Internal Model Principle. The off-line designed controller was shown to yield a disturbance rejection performance which is robust in the face of uncertainties in the coprime factor representation of the plant. The case of Adaptive disturbance rejection for plants with uncertainties in the model description is under investigation.

APPENDIX A. PROOF OF LEMMA 2

Necessity ($\Rightarrow$): The Z transform of the disturbance response $\{e(.)\}$ is given by:

$$E(z) = F_{T,Q}(z)W(z) \tag{85}$$

According to (1), the poles of $W(z)$ are all simple and located on the unit circle. Using partial fraction expansion, the disturbance response can be expressed as follows:

$$e(k) = Z^{-1}[\sum_i(\frac{R_{w,i}}{z-p_i})F_{T,Q}(p_i)] + e_0(k) \tag{86}$$

where $R_{\omega,i}$ denotes the residue of $W(z)$ at the pole p_i of $W(z)$ and $e_0(k)$ the sum of responses corresponding to partial fractions with $F_{T,Q}(z)$ poles and the response to non-zero initial conditions. Since the controller K is stabilizing, then the response to nonzero initial conditions is asymptotically zero. Also, since $F_{T,Q}(z) \in RH_\infty$, then the responses corresponding to partial fractions with $F_{T,Q}$ poles asymptotically converge to zero. Therefore, the asymptotic properties of the disturbance response are determined by the first term in the RHS of (86).

Assume $\lim_{k\to\infty} e(k) = 0$. The response corresponding to$\Sigma_i \frac{R_{w,i}}{z-p_i}$ is not asymptotically zero since the poles p_i, $i = 1,\ldots, n_p$ are all on the unit circle. Therefore, we must have:

$$F_{T,Q}(p_i) = 0, \qquad i = 1,\cdots, n_p \tag{87}$$

which means that the interpolation conditions (14) must be satisfied.

Sufficiency ($\Leftarrow$): Satisfying the interpolation conditions (87) implies that $e(k)$ in (86) asymptotically decays to zero.

APPENDIX B. PROOF OF LEMMA 3

Necessity ($\Rightarrow$): Using the expression for the Youla parameter Q given in (15), the interpolation conditions (14) can be rewritten as follows:

$$[T_{11}(z) + \sum_{i=1}^{n_q} q_i T_{12}(z)\psi_i(z)T_{21}(z)]|_{z=pi} = 0, \qquad i = 1,\cdots, n_p \tag{88}$$

Define the following functions:

$$V_0(z) = T_{11}(z) \tag{89}$$

$$V_i(z) = T_{12}(z)\psi_i(z)T_{21}(z), \qquad i = 1,\cdots,n_q \tag{90}$$

Then (88) can be written as:

$$[V_0(z) + \sum_{i=1}^{n_q} q_i V_i(z)]|_{z=pi} = 0 \tag{91}$$

Without loss of generality, assume the frequency ω_0 in (1) is zero. Then $W(z)$ has one pole at $z = 1$ and k_0 pairs of complex conjugate poles on the unit circle. Since we have that $V_i(\bar{z}) = V_i(\bar{z})$, $i = 0,\ldots,n_q$, where $\bar{z}$ is the complex conjugate of z, then it suffices to consider only $k_0 + 1$ of the $n_p = 2k_0 + 1$ interpolation conditions (91). The $k_0 + 1$ interpolation conditions can be obtained by evaluating $F_{T,Q}(z)$ at the pole at $z = 1$ and at only one pole from each pair of complex conjugate poles. For example, the interpolation conditions can be evaluated at the $k_0 + 1$ poles with positive imaginary parts.

For any complex number z, let $V_{i,re}(z)$ and $V_{i,im}(z)$ denote respectively the real and imaginary parts of the functions $V_i(z)$, $i = 0,\ldots, n_q$. The interpolation condition (91) evaluated at a pole p on the unit circle and different from 1 can be written in the form:

$$[V_{0,re}(p) + \sum_{i=1}^{n_q} q_i V_{i,re}(p)] + j[V_{0,im}(p) + \sum_{i=1}^{n_q} q_i V_{i,im}(p)] = 0 \tag{92}$$

Hence each such pole yields two linear equations of the form:

$$V_{0,re}(p) + \sum_{i=1}^{n_q} q_i V_{i,re}(p) = 0$$

$$V_{0,im}(p) + \sum_{i=1}^{n_q} q_i V_{i,im}(p) = 0$$

Therefore, considering the $k_0 + 1$ poles selected as mentioned above yields $n_p = 2k_0 + 1$ linear equations in the n_q unknowns q_i, $i = 1,\ldots,n_q$. The set of linear equations can be written in the form:

$$\mathbf{A}\Theta + B = 0 \tag{93}$$

where Θ is given by (16), $\mathbf{A}$ is the $n_p \times n_q$ matrix given by:

$$\mathbf{A} = \begin{bmatrix} V_1(1) & \ldots & V_{nq}(1) \\ V_{1,re}(e^{j\omega 1}) & \ldots & V_{n_q,re}(e^{i\omega_1}) \\ V_{1,im}(e^{j\omega 1}) & \ldots & V_{n_q,im}(e^{i\omega_1}) \\ \vdots & \ldots & \vdots \\ V_{1,re}(e^{j\omega_{k_0}}) & \ldots & V_{n_q,re}(e^{i\omega_{k_0}}) \\ V_{1,im}(e^{j\omega_{k_0}}) & \ldots & V_{n_q,im}(e^{i\omega_{k_0}}) \end{bmatrix} \tag{94}$$

and B is the vector given by:

$$\mathbf{B} = \begin{bmatrix} V_{0,re}(1) \\ V_{0,re}(e^{j\omega 1}) \\ V_{1,im}(e^{j\omega 1}) \\ \vdots \\ V_{0,re}(e^{j\omega_{k_0}}) \\ V_{0,im}(e^{j\omega_{k_0}}) \end{bmatrix} \tag{95}$$

Sufficiency ($\Leftarrow$): For a given $F_{T,Q}$, and according to the expressions given above for **A** and B, satisfying (17) implies that (14) is satisfied.

APPENDIX C. PROOF OF LEMMA 4

Necessity ($\Rightarrow$): Since $\{e(., \Theta)\}$ is the response of an asymptotically stable system with transfer function $F_{T,Q}(z)$ to a quasi-stationary input $\{w(.)\}$, then it is quasi-stationary [34]. Let $\Phi_w(\cdot)$ and $\Phi_e(., \Theta)$ denote the power spectra of the disturbance signal $\{w(.)\}$ and the disturbance response $\{e(., \Theta)\}$ respectively. Then we have:

$$\Phi_e(\omega,\Theta) = |F_{T,Q}(j\omega)|^2\Phi_w(\omega) \tag{96}$$

Since the power spectrum of $\{e(., \Theta)\}$ represents the Fourier Transform of the auto-correlation function of $\{e(., \Theta)\}$, then by definition of the Inverse Fourier Transform we have [34]:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} e^2(k,\Theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_e(\omega,\Theta)\, d\omega \tag{97}$$

The above equality implies:

$$\begin{aligned} arg\min_{\Theta} \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} e^2(k, \Theta) &= arg\min_{\Theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_e(\omega,\Theta)d\omega \\ &= arg\min_{\Theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} |F_{T,Q}(e^{j\omega})|^2\Phi_w(\omega)d\omega \end{aligned} \tag{98}$$

The power spectrum of the disturbance signal (1) computed as mentioned above is given by:

$$\Phi_w(\omega) = \sum_{n=-k_l}^{k_0} |a_n|^2 \sum_{l=-\infty}^{l=\infty} 2\pi\delta(\omega - \omega_n - 2\pi l) \tag{99}$$

where $a_n \neq 0, -k_0 \leq n \leq k_0$, are complex coefficients in the expression of $w(k)$ when the latter is written as a sum of complex exponentials. The integral on the RHS of (98) can be evaluated as follows:

$$\int_{-\pi}^{\pi} | F_{T,Q}(e^{j\omega})|^2\Phi_w(\omega)d\omega = \int_{-\pi}^{\pi} | F_{T,Q}(e^{j\omega})|^2 \sum_{n=-k_0}^{k_o} | a_n|^2 \sum_{l=-\infty}^{l=\infty} 2\pi\, \delta(\omega - \omega_n - 2\pi l)d\omega$$

$$= \int_{-\pi}^{\pi} |F_{T,Q}(e^{j\omega})|^2 \sum_{n=-k_0}^{k_o} 2\pi |a_n|^2 \delta(\omega - \omega_n) d\omega$$
$$= \sum_{n=-k_0}^{k_o} 2\pi |a_n|^2 |F_{T,Q}(e^{j\omega_n})|^2 \tag{100}$$

The RHS of the above equation can be written in the following quadratic form:

$$\sum_{n=-k_0}^{k_0} 2\pi |a_n|^2 |F_{T,Q}(e^{j\omega} n)|^2 = (\mathbf{A}\Theta + B)^T \mathbf{M} (\mathbf{A}\Theta + B) \tag{101}$$

where Θ is as in (16), the matrices $\mathbf{A}$ and B are as in (17) and $\mathbf{M}$ is a positive definite diagonal matrix given by:

$$\mathbf{M} = \begin{bmatrix} \alpha_0^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{k_0}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{k_0}^2 \end{bmatrix} \tag{102}$$

and where

$$\alpha^2{}_n = \begin{cases} 2\pi |a_0|^2 & \text{if} \quad n = 0 \\ 4\pi |a_n|^2 & \text{if} \quad n = 1, \cdots, k_0 \end{cases} \tag{103}$$

Taking the derivative of the RHS of (101) with respect to Θ and setting it equal to zero, we get:

$$\mathbf{A}^T \mathbf{M} (\mathbf{A}\Theta + B) = 0 \tag{104}$$

Since $\mathbf{A}$ has rank n_p and $\mathbf{M}$ is positive definite, we recover the interpolation condition (17):

$$\mathbf{A}\Theta + B = 0 \tag{105}$$

Moreover, under assumption **A1**, the above system of equations admits at least one solution. Hence, minimizing the LHS of (97) implies solving the interpolation conditions (17).

Sufficiency ($\Leftarrow$): If the parameter vector Θ satisfies the interpolation conditions (17), the RHS of (101) is zero which means that the LHS of (97) is at its global minimum.

APPENDIX D. PROOF OF LEMMA 5

Since $\{v_1(.)\}$ is persistently exciting of order n_q, then the matrix $[\lim_{N\to\infty} \frac{1}{N}\Sigma_{k=1}^{N} ø(k)ø(k)^T]$ is nonsingular. Therefore, there exists a unique vector Θ_{min} such that:

$$\Theta_{min} = arg \min_{\Theta} \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} e^2(k,\Theta)$$

$$= (\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \phi(k)\phi(k)^T)^{-1} (\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \phi(k)^T v_0(k)) \tag{106}$$

Since **A** is nonsingular, there exists a unique vector Θ^0 such that the interpolation conditions are satisfied. Therefore, we have:

$$\lim_{k\to\infty} e(k,\Theta^0) = 0 \tag{107}$$

Assuming zero initial conditions, then $\{e(., \Theta^0)\}$ represents the impulse response of an exponentially stable linear system. Therefore, there exists $\alpha > 0$ and $\beta > 0$ such that:

$$|e(k,\Theta^0)| \leq \alpha e^{-\beta k} \tag{108}$$

Therefore, the infinite sum $\lim_{N\to\infty}\Sigma_{k=1}^{N} e^2(k,\Theta^0)$ exists. In fact, let $r = e^{-2\beta} < 1$. We have:

$$\lim_{N\to\infty}\sum_{k=1}^{N} e(k,\Theta^0)^2 \leq \lim_{N\to\infty}\sum_{k=1}^{N} \alpha^2 e^{-2\beta k}$$

$$\leq \alpha^2 \lim_{N\to\infty}\sum_{k=1}^{N} r^k$$

$$\leq \alpha^2 \lim_{N\to\infty}\frac{r(1-r^N)}{1-r}$$

$$\leq \alpha^2 \frac{r}{1-r} \tag{109}$$

Therefore,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} e(k,\Theta^0)^2 = 0 \tag{110}$$

Since Θ_{min} is the unique minimizer of $\lim_{N\to\infty}\frac{1}{N}\Sigma_{i=1}^{N} e_2(k,\Theta)$, then we have:

$$0 \leq \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} e^2(k,\Theta_{min}) \leq \lim_{N\to\infty}\frac{1}{N}\Sigma_{i=1}^{N} e^2(k,\Theta^0) = 0 \tag{111}$$

which implies:

$$\lim_{N\to\infty} \frac{1}{\mathrm{N}} \sum_{i=1}^{N} e^2 (k, \Theta_{min}) = 0 \tag{112}$$

However, since Θ_{min} is the unique minimizer and since the infinite sum corresponding to Θ_{min} has the same value as that corresponding to Θ^0, then we must have $\Theta_{min} = \Theta^0$ that is, Θ_{min} satisfies the interpolation conditions.

APPENDIX E. PROOF OF LEMMA 6

Assume the plant is such that:

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} P_I & P_2 \\ P_I & P_2 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{113}$$

that is, $e = y$. Then we have:

$$\begin{aligned} T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix} \\ &= \begin{bmatrix} P_1 + P_2 UM_0 P_1 & P_2 M_0 \\ M_0 P_1 & 0 \end{bmatrix} \end{aligned} \tag{114}$$

Using the Bezout identity (3), we get:

$$\begin{aligned} P_1 + P_2UM_0P_1 &= VM_0 P_1 \\ &= VT_{21} \end{aligned} \tag{115}$$

The transfer function $F_{T,Q}$ is then given as follows:

$$\begin{aligned} F_{T,Q} &= T_{11} + T_{12}QT_{21} \\ &= VT_{21} + N_0QT_{21} \\ &= (V + N_0Q)T_{21} \end{aligned} \tag{116}$$

Using the factorization in (25), we get:

$$\begin{aligned} E(z) &= (T_{11} + T_{12}QT_{21})W(z) \\ &= (V + N_0Q)T_{21}W(z) \\ &= (V + N_0Q)M_d^{-1}N_d \end{aligned}$$

Under the assumption of Lemma 5, the controller obtained by using Θ_{min} in (15) achieves regulation. Therefore, we have:

$$E(z) = (V + N_0 Q)M_d^{-1}N_d \in RH_\infty \Rightarrow (V + N_0Q)M_d^{-1}[N_d \;\; M_d] \in RH_\infty \tag{117}$$

$$\Rightarrow (V + N_0Q)M_d^{-1} \in RH_\infty \tag{118}$$

where the last implication results from the fact that (25) is a coprime factorization (*i.e.*, $\exists\,[X\ Y] \in RH_\infty$ such that $[N_d\ M_d][X\ Y]^T = 1$). Since the poles of $M_d^{-1}(z)$ are the poles of $W(z)$ (all simple and on the unit circle), then we must have:

$$(V + N_0Q)|_{p_i} = 0 \tag{119}$$

where p_i is a pole of $W(z)$. The last equality can be shown using arguments similar to those used in the proof of Lemma 2. Since we have $K = (U + M_0Q)(V + N_0Q)^{-1}$, then the controller K must contain all the poles of $W(z)$ (a model of the disturbance input).

APPENDIX F. PROOF OF THEOREM 1

(a) At any given time k, we have:

$$e(k) = v_0(k) - \phi(k)^T\hat{\Theta}\,(k-1) \tag{120}$$

$$e_{min}(k) = v_0(k) - \phi(k)^T\Theta_{min} \tag{121}$$

Adding and subtracting $e_{min}(k)$ in (120), we get:

$$e(k) = \phi(k)^T\tilde{\Theta}\,(k-1) + e_{min}(k) \tag{122}$$

where

$$\tilde{\Theta}(k) = \Theta_{min} - \hat{\Theta}\,(k) \tag{123}$$

We also have:

$$\tilde{\Theta}(k+1) = \tilde{\Theta}(k) - \lambda(k+1)P(k+1)\phi(k+1)e(k+1) \tag{124}$$

$$P(k+1)^{-1} = P(k)^{-1} + \lambda(k+1)\phi(k+1)\phi(k+1)^T \tag{125}$$

Consider the following Lyapunov function candidate:

$$V(k) = \tilde{\Theta}(\mathrm{k})^T P(k)^{-1}\tilde{\Theta}k) \tag{126}$$

Substitute (124) for $\tilde{\Theta}(k+1)$ to get:

$$\begin{aligned} V(k+1) = {} & \tilde{\Theta}(k)^T P(k+1)^{-1}\tilde{\Theta}(k) - 2\lambda(k+1)\,\phi\,(k+1)^T\tilde{\Theta}(k)e(k+1) \\ & + \lambda^2(k+1)\phi(k+1)^T P(k+1)\phi(k+1)e^2(k+1) \end{aligned} \tag{127}$$

Using the expression (125) for $P(k+1)^{-1}$ we get:

$$\begin{aligned} V(k+1) &= \tilde{\Theta}(k)[P(k)^{-1} + \lambda(k+1)\phi(k+1)\phi(k+1)^T]\tilde{\Theta}(k)^T \\ &\quad - 2\lambda(k+1)\phi(k+1)^T\tilde{\Theta}(k)e(k+1) \\ &\quad + \lambda^2(k+1)\phi(k+1)^T P(k+1)\phi(k+1)e^2(k+1) \\ &= \tilde{\Theta}(k)P(k)^{-1}\tilde{\Theta}(k)^T + \lambda(k+1)\tilde{\Theta}(k)^T\phi(k+1)\phi(k+1)^T\tilde{\Theta}(k) - \\ &\quad 2\lambda(k+1)\phi(k+1)^T\tilde{\Theta}(k)e(k+1) \\ &\quad + \lambda^2(k+1)\phi(k+1)^T P(k+1)\phi(k+1)e^2(k+1) \end{aligned} \tag{128}$$

Using the identity:

$$\lambda(k+1)\phi(k+1)^T P(k+1)\phi(k+1) = \frac{\lambda(k+1)\phi(k+1)^T P(k)\phi(k+1)}{1 + \lambda(k+1)\phi(k+1)^T P(k)\phi(k+1)} \tag{129}$$

and the expression for $e(k+1)$ in (122), we get:

$$\begin{aligned} V(k+1) &= V(k) + \lambda(k+1)\left[e_{min}^2(k+1) - \frac{e^2(k+1)}{1 + \lambda(k+1)\phi(k+1)^T P(k)\phi(k+1)} \right] \\ &\le V(k) + \lambda(k+1)\left[\alpha_{max}^2 e^{-2\beta_{min}(k+1)} - \frac{e^2(k+1)}{1 + \lambda(k+1)\phi(k+1)^T P(k)\phi(k+1)} \right] \end{aligned} \tag{130}$$

If $\lambda(k)$ is chosen as indicated in (39), then the function $V(\cdot)$ is positive nonincreasing. Let S_N denote the set of integers such that $\lambda(k) = 1$ for $k \in S_N$ and $k \le N$. We have:

$$\sum_{k=1}^{N} \lambda(k)\left[\frac{e^2(k)}{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)} - \alpha_{max}^2 e^{-2\beta_{min}k} \right] \le V(0) - V(N)$$

$$\Rightarrow \quad \sum_{k \in S_N}\left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} - \alpha_{max}^2 e^{-2\beta_{min}k} \right] \le V(0) - V(N) \tag{131}$$

Two cases can arise:

CASE 1: $\lambda(k) = 1$ infinitely many times. Then:

$$\sum_{k \in S_\infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} - \alpha_{max}^2 e^{-2\beta_{min}k} \right] \leq V(0) - V(\infty) \leq \infty \tag{132}$$

which implies the following:

$$\lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} - \alpha_{max}^2 e^{-2\beta_{min}k} \right] = 0$$

$$\Rightarrow \lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} \right] = \lim_{K \to \infty} \alpha_{max}^2 e^{-2\beta_{min}k}$$

$$\Rightarrow \lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} \right] = 0 \tag{133}$$

CASE 2 $\lambda(k) = 1$ only a finite number of times. Then:

$$\lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} - \alpha_{max}^2 e^{-2\beta_{min}k} \right] \leq 0$$

$$\Rightarrow \lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} \right] \leq \lim_{k \to \infty} - \alpha_{max}^2 e^{-2\beta_{min}k}$$

$$\Rightarrow \lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} \right] = 0 \tag{134}$$

Therefore, in either case we have:

$$\lim_{k \to \infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} \right] = 0 \tag{135}$$

(b) We have:

$$\lim_{N \to \infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{e^2(k)}{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)} - \alpha_{max}^2 e^{-2\beta_{min}k} \right] \leq \infty$$

$$\Rightarrow \lim_{N \to \infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{e^2(k)}{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)} \right] - \lim_{N \to \infty} \sum_{k=1}^{N} \lambda(k) \left[\alpha_{max}^2 e^{-2\beta_{min}k} \right] \leq \infty$$

$$\Rightarrow \lim_{N\to\infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{e^2(k)}{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)} \right] \leq \infty$$

$$\Rightarrow \lim_{N\to\infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{(1 + \lambda(k)\phi(k)^T P(k-1)\phi(k))e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right] \leq \infty$$

$$\Rightarrow \lim_{N\to\infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right] +$$

$$\lim_{N\to\infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{\lambda(k)\phi(k)^T P(k-1)\phi(k)e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right] \leq \infty$$

Using the fact that $\lim_{N\to\infty} \sum_{k=1}^{N} \lambda(k) \left[\frac{e^2(k)}{\{1+\lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2}\right] < \infty$ from part (a) of the proof, we get:

$$\lim_{N\to\infty} \sum_{k=1}^{N} \lambda^2(k) \left[\frac{\phi(k)^T P(k-1)\phi(k)e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right] \leq \infty \tag{136}$$

From (37), we have:

$$\| \hat{\Theta}(k) - \hat{\Theta}(k-1) \|^2 = \lambda^2(k) \left[\frac{\phi(k)^T P(k-1)^2 \phi(k)e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right]$$

$$\leq \lambda^2(k) \left[\frac{\phi(k)^T P(k-1)\phi(k)e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right] \lambda_{max}(P(0)) \tag{137}$$

which yields:

$$\sum_{k=1}^{\infty} \| \hat{\Theta}(k) - \hat{\Theta}(k-1) \|^2 = \sum_{k=1}^{\infty} \lambda^2(k) \left[\frac{\phi(k)^T P(k-1)\phi(k)e^2(k)}{\{1 + \lambda(k)\phi(k)^T P(k-1)\phi(k)\}^2} \right] \lambda_{max}(P(0)) \leq \infty \tag{138}$$

Therefore:

$$\lim_{k\to\infty} \| \hat{\Theta}(k) - \hat{\Theta}(k-1) \| = 0 \tag{139}$$

Using the Schwarz inequality, we get:

$$\lim_{k\to\infty} \| \hat{\Theta}(k) - \hat{\Theta}(k-l) \| = 0 \qquad \forall\, l < 0 \tag{140}$$

(c) Assume $\{\phi(.)\}$ is bounded, that is, there exists $M > 0$ such that $\| \phi(.) \|^2 \le M$. *Hence* We have:

$$\lim_{k\to\infty} \left[\frac{e^2(k)}{1 + \phi(k)^T P(k-1)\phi(k)} \right] = 0 \Rightarrow \lim_{k\to\infty} \left[\frac{e^2(k)}{1 + \lambda_{max}(P(k-1)) \| \phi(k) \|^2} \right] = 0$$

$$\Rightarrow \lim_{k\to\infty} \left[\frac{e^2(k)}{1 + \lambda_{max}(P(0))\, M} \right] = 0$$

$$\Rightarrow \lim_{k\to\infty} e(k) = 0 \tag{141}$$

APPENDIX G. PROOF OF THEOREM 2

(a) Convergence analysis of $\tilde{\Theta}_1(k)$:
Let $V_1(k) = \tilde{\Theta}_1^T(k)P^{-1}(k)\tilde{\Theta}_1(k)$. From (44) we have:

$$P^{-1}(k) = \frac{1}{\lambda(k+1)} P^{-1}(k+1) - \phi(k+1)\phi(k+1)^T \tag{142}$$

Using (55) with (142) we get:

$$P^{-1}(k+1)\tilde{\Theta}_1(k+1) = \lambda(k+1)P^{-1}(k)\tilde{\Theta}_1(k)$$

$$= P^{-1}(k+1)\tilde{\Theta}_1(k) - \lambda(k+1)\phi(k+1)\phi(k+1)^T\tilde{\Theta}_1(k) \tag{143}$$

Hence:

$$V_1(k+1) = \tilde{\Theta}_1^T(k+1)\, P^{-1}(k+1)\tilde{\Theta}_1^T(k+1)$$

$$= [\lambda(k+1)\tilde{\Theta}_1^T(k)P^{-1}(k)P(k+1)]$$

$$[P^{-1}(k+1)\tilde{\Theta}_1(k) - \lambda(k+1)\phi(k+1)\phi(k+1)^T\tilde{\Theta}_1(k)]$$

$$= [\lambda(k+1)\tilde{\Theta}_1(k)^T P^{-1}(k)\tilde{\Theta}_1(k)] -$$

$$[\lambda^2(k+1)\tilde{\Theta}_1(k)^T P^{-1}(k)P(k+1)\phi(k+1)\phi(k+1)^T\tilde{\Theta}_1(k)] \tag{144}$$

From (46) we have:

$$P(k+1)\phi(k+1) = \frac{1}{\lambda\,(k+1)}\left[\frac{P(k)\phi(k+1)}{1+\phi(k+1)^T P(k)\phi(k+1)}\right] \tag{145}$$

Substituting in (144):

$$\begin{aligned}
V_1(k+1) &= \lambda(k+1)V_1(k) - \lambda(k+1)\left[\frac{\phi^T(k+1)\tilde{\Theta}_1(k)\tilde{\Theta}_1^T(k)\phi(k+1)}{1+\phi(k+1)^T P(k)\phi(k+1)}\right] \\
&= \lambda(k+1)V_1(k) - \lambda(k+1)\left[\frac{(\phi^T(k+1)\tilde{\Theta}_1(k))^2}{1+\phi(k+)^T P(k)\phi(k+1)}\right] \\
&\le \lambda(k+1)V_1(k) \\
&\le [\prod_{i=0}^{k}\lambda(i+1)]V_1(0) \\
&\le [\lambda_{max}^{k+1}]V_1(0) \\
&\le [\lambda_{max}^{k+1}]\tilde{\Theta}_1^T(0)\,P^{-1}(0)\tilde{\Theta}_1(0)
\end{aligned} \tag{146}$$

which yields:

$$[\lambda_{min}(P^{-1}(k))]\,\|\,\tilde{\Theta}_1(k+1)\,\|^2 \le [\,\lambda_{max}^{k+1}][\lambda_{max}(P(0))]\,\|\,\tilde{\Theta}_1(0)\,\|^2 \tag{147}$$

Taking the limit of both sides as $k \to \infty$:

$$\begin{aligned}
&\lim_{k\to\infty}[\lambda_{min}(P^{-1}(k))]\,\|\,\tilde{\Theta}_1(k+1)\,\|^2 \le 0 \\
&\lim_{k\to\infty}\lambda_{min}(P^{-1}(k))\lim_{k\to\infty}\|\,\tilde{\Theta}_1(k+1)\,\|^2 \le 0
\end{aligned} \tag{148}$$

To show convergence of $\tilde{\Theta}_1(k+1)$, it is enough to have $\lim_{k\to\infty}\lambda_{min}(P^{-1}(k))$ bounded away from zero. Such condition is satisfied under assumption **A6**. Therefore, we have $\lim_{k\to\infty}\tilde{\Theta}_1(k) = 0$.

(b) Convergence analysis of $\tilde{\Theta}_2(k)$:

We have:

$$\tilde{\Theta}_2(k+1) = [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\Theta}_2(k) + \Delta\Theta_{min}(k)$$

Assume the time interval between any two successive changes in $\Theta_{min}(k)$ is long enough to allow the parameter estimates to converge. If at time k_0, $\Delta\Theta_{min}(k_0)$ takes a value different from zero, then $\Theta_2(k_0+1)$ is taken as an initial condition and the analysis of the convergence of $\tilde{\Theta}_2(k)$ would be the same as that for $\Theta_1(k)$ except that the initial time is k_0+1 instead of 0.

(c) Convergence analysis of $\tilde{\Theta}_3(k)$:

We have:

$$\tilde{\Theta}_3(k+1) = [\lambda(k+1)P(k+1)P^{-1}(k)]\tilde{\Theta}_3(k) - \lambda(k+1)P(k+1)\phi(k+1)e_{min}(k+1), \qquad \tilde{\Theta}_3(0) = 0 \quad (149)$$

Define the following Lyapunov function candidate $V_3(k) = \tilde{\Theta}_3^T(k)P^{-1}(k)\tilde{\Theta}_3(k)$. Equation (57) yields:

$$P^{-1}(k+1)\tilde{\Theta}_3(k+1) = [\lambda(k+1)P^{-1}(k)]\tilde{\Theta}_3(k) - \lambda(k+1)\phi(k+1)e_{min}(k+1) \quad (150)$$

Using the expression of $P^{-1}(k)$ from (142) in the above equation, we get:

$$\tilde{\Theta}_3(k+1) = \tilde{\Theta}_3(k) - \lambda(k+1)P(k+1)\phi(k+1)\phi(k+1)^T\tilde{\Theta}_3(k) - \lambda(k+1)P(k+1)\phi(k+1)e_{min}(k+1) \quad (151)$$

Hence we have:

$$\begin{aligned} V_3(k+1) &= \tilde{\Theta}_3^T(k+1)P^{-1}(k+1)\tilde{\Theta}_3(k+1) \\ &= \lambda(k+1)V_3(k) + \lambda(k+1)\{-\tilde{\Theta}_3^T(k)\phi(k+1) + \\ &\quad \lambda(k+1)\tilde{\Theta}_3^T(k)\phi(k+1)\phi(k+1)^TP(k+1)\phi(k+1) - \\ &\quad \lambda(k+1)\phi(k+1)^TP(k+1)P^{-1}(k)\tilde{\Theta}_3(k)\}e_{min}(k+1) - \\ &\quad \lambda^2(k+1)\tilde{\Theta}_3^T(k)\phi(k+1)\phi(k+1)^TP(k+1)P^{-1}(k)\tilde{\Theta}_3(k) + \\ &\quad \lambda^2(k+1)\phi(k+1)^TP(k+1)\phi(k+1)\, e_{min}^2(k+1) \end{aligned} \quad (152)$$

The following identities can be easily derived:

$$\tilde{\Theta}_3^T(k)\phi(k+1)[-1 + \lambda(k+1)\phi(k+1)^TP(k+1)\phi(k+1)] = -\frac{\tilde{\Theta}_3^T(k)\phi(k+1)}{1 + \phi(k+1)^TP(k)\phi(k+1)}$$

$$-\lambda(k+1)\phi(k+1)^TP(k+1)P^{-1}(k)\tilde{\Theta}_3(k) = -\frac{\tilde{\Theta}_3^T(k)\phi(k+1)}{1 + \phi(k+1)^TP(k)\phi(k+1)}$$

$$\tilde{\Theta}^T{}_3(k)\phi(k+1)[\lambda(k+1)\phi(k+1)^TP(k+1)P^{-1}(k)]\tilde{\Theta}_3(k) = \frac{(\tilde{\Theta}_3^T(k)\phi(k+1))^2}{1 + \phi(k+1)^TP(k)\phi(k+1)}$$

$$\lambda(k+1)\phi(k+1)^TP(k+1)\phi(k+1) = \frac{\phi^T(k+1)P(k)\phi(k+1)}{1 + \phi(k+1)^TP(k)\phi(k+1)}$$

Using the above identities in the expression for $V(k+1)$, we get:

$$V_3(k+1) = \lambda(k+1)V_3(k) - \frac{\lambda(k+1)}{1+\phi(k+1)^T P(k)\phi(k+1)}[(\tilde{\Theta}_3^T(k)\phi(k+1))^2 + 2\phi(k+1)^T\tilde{\Theta}_3(k)e_{min}(k+1)] + \lambda(k+1)\frac{\phi(k+1)^T P(k)\phi(k+1)}{1+\phi(k+1)^T P(k)\phi(k+1)} e_{min}^2(k+1) \quad (153)$$

Completing the square in the above equation we get:

$$V_3(k+1) = \lambda(k+1)V_3(k) - \frac{\lambda(k+1)}{\phi(k+1)^T P(k+1)\phi(k+1)}[(\tilde{\Theta}_3^T(k)\phi(k+1) + e_{min}(k+1))^2] + \frac{\lambda(k+1)}{\phi(k+1)^T P(k+1)\phi(k+1)} e_{min}^2(k+1) + \lambda(k+1)\frac{\phi(k+1)^T P(k+1)\phi(k+1)}{1+\phi(k+1)^T P(k+1)\phi(k+1)} e_{min}^2(k+1) \quad (154)$$

The square term involves the parameter error $\tilde{\Theta}_3(k)$. To facilitate the study of the convergence properties of the algorithm, it is convenient to eliminate the terms with $\tilde{\Theta}_3(k)$ since the latter is an unknown. This can be done by eliminating the negative term from the above equation. We get:

$$V_3(k+1) \le \lambda(k+1)V_3(k) + \lambda(k+1)e_{min}^2(k+1) \quad (155)$$

Using the above equation iteratively and knowing that $\tilde{\Theta}_3(0) = 0$ yields:

$$\begin{aligned} V_3(k+1) &\le [\prod_{i=0}^{k} \lambda(i+1)]V_3(0) + \sum_{i=1}^{k+1} \{[\prod_{j=i}^{k+1} \lambda(j)]e_{min}^2(i)\} \\ &\le \sum_{i=1}^{k} \lambda_{max}^{k+1-i}\, e_{min}^2(i) \\ &\le \sum_{i=1}^{k} e_{min}^2(i) \\ &\le \alpha^2 \frac{r(1-r^k)}{1-r} \end{aligned} \quad (156)$$

where the RHS of the last inequality is taken from the results in the proof of Lemma 5. A new inequality can be written as follows:

$$\lambda_{min}(P^{-1}(k)) \| \tilde{\Theta}_3(k+1) \|^2 \leq \alpha^2 \frac{r(1-r^k)}{1-r} \tag{157}$$

Taking the limit on both sides:

$$\lim_{k\to\infty} [\lambda_{min}(P^{-1}(k)) \| \tilde{\Theta}_3(k+1) \|^2 \leq \alpha^2 \frac{r}{1-r}$$

$$\lim_{k\to\infty} [\lambda_{min}(P^{-1}(k))] \lim_{k\to\infty} \| \tilde{\Theta}_3(k+1) \|^2 \leq \alpha^2 \frac{r}{1-r} \tag{158}$$

Using assumption **A5**, we get:

$$\lim_{k\to\infty} \tilde{\Theta}_3(k) = 0 \tag{159}$$

APPENDIX H. PROOF OF LEMMA 9

Assume the plant is such that:

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{160}$$

that is: $e = y$. Then we have:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$= \begin{bmatrix} P_1 + P_2UMP_1 & P_2M \\ MP_1 & R \end{bmatrix} \tag{161}$$

Using the double Bezout identity (3), we get:

$$\begin{aligned} P_1 + P_2UMP_1 &= VMP_1 \\ &= VT_{21} \end{aligned} \tag{162}$$

The transfer function $F_{T,Q}$ is then given as follows:

$$\begin{aligned} F_{T,Q} &= T_{11} + T_{12} \frac{Q}{1-RQ} T_{21} \\ &= VT_{21} + N \frac{Q}{1-RQ} T_{21} \\ &= (V + (N_0 + RV) \frac{Q}{1-RQ}) T_{21} \\ &= (V + N_0Q) \frac{T_{21}}{1-RQ} \end{aligned} \tag{163}$$

Using the coprime factorization of $T_{21}W$, we get:

$$E(z) = (T_{11} + T_{12} \frac{Q}{1 - RQ} T_{21})W(z)$$
$$= (V + N_o Q) \frac{1}{1 - RQ} T_{21}W(z)$$
$$= (V + N_o Q) \frac{1}{1 - RQ} M_d^{-1} N_d$$

Under the assumptions of Lemma 6, $(V + N_0Q)$ is such that:

$$(V + N_0Q)|_{p_i} = 0 \tag{164}$$

at any pole p_i of $W(z)$. Hence, since the poles of M_d^{-1} are by assumption poles of $W(z)$, then $(V + N_0Q)M_d^{-1} \in RH_\infty$ which yields:

$$E(z) = (V + N_0Q) \frac{1}{1 - RQ} M_d^{-1}N_d \in RH_\infty \tag{165}$$

The above result is due to the fact that $(1 - RQ)^{-1} \in RH_\infty$. Therefore, the disturbance rejection performance of the controller designed off-line based on the nominal plant model is robust to plant model uncertainties as long as the closed loop system is stable.

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