Adaptive Boundary Control for Unstable Parabolic PDEs—Part I: Lyapunov Design

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Submitted to *IEEE Transactions on Automatic Control*
June 30, 2005

This paper is Part I of a series of three companion papers which can be all downloaded for the purpose of review from

`http://flyingv.ucsd.edu/part1.pdf` (present paper)
`http://flyingv.ucsd.edu/part2.pdf` (cited as reference [28])
`http://flyingv.ucsd.edu/part3.pdf` (cited as reference [29])

***Abstract*— We develop adaptive controllers for parabolic PDEs controlled from a boundary and containing unknown destabilizing parameters affecting the interior of the domain. These are the first adaptive controllers for unstable PDEs without relative degree limitations, open-loop stability assumptions, or domain-wide actuation. It is the first necessary step towards developing adaptive controllers for physical systems such as fluid, thermal, and chemical dynamics, where actuation can be only applied non-intrusively, the dynamics are unstable, and the parameters, such as the Reynolds, Rayleigh, Prandtl, or Peclet numbers are unknown because they vary with operating conditions. Our method builds upon our explicitly parametrized control formulae in [25] to avoid solving Riccati or Bezout equations at each time step. Most of the designs we present are state feedback but we present two benchmark designs with output feedback which have infinite relative degree.**

I. INTRODUCTION

While for linear finite dimensional systems many adaptive schemes have been proposed [8], adaptive control techniques have been developed for only a few classes of PDEs restricted by relative degree, stability, or domain-wide actuation assumptions. In this paper we develop the first adaptive controllers for parabolic PDEs controlled from a boundary and containing unknown destabilizing parameters affecting the interior of the domain. Our control laws are given by explicit formulae and open the door for the use of a wealth of certainty equivalence and Lyapunov techniques developed for finite dimensional systems. They initiate an effort towards developing adaptive controllers for physical systems such as fluid, thermal, and chemical dynamics, where actuation can be only applied non-intrusively, the dynamics are unstable, and the parameters, such as the Reynolds, Rayleigh, Prandtl, or Peclet numbers are unknown because they vary with operating conditions. Our method builds upon our explicitly parametrized control formulae in [25] to avoid solving Riccati or Bezout equations at each time step.

a) Literature Overview: Early works on adaptive control of infinite-dimensional systems, surveyed by Logemann and Townley [21], were for plants stabilizable by non-identifier based high gain feedback, under a relative degree one assumption. Model reference (MRAC) type schemes were designed by Hong and Bentsman [7], Bohm, Demetriou, Reich, and Rosen [2], Solo and Bamieh [30], Orlov [22], and Bentsman and Orlov [1]. While the strength of these results are the proofs of identifiability of infinite dimensional parameter vectors, their limitation is that they require control action throughout the PDE domain. Other efforts such as Demetriou and Ito [5] and Wen and Balas [32] have employed tools from positive realness; they have also provided some cunning examples that go beyond the relative degree one restriction. Adaptive linear quadratic control with least-squares estimation was pursued by Duncan, Maslowski, and Pasik-Duncan [6] for linear stochastic evolution equations with unbounded input operators and exponentially stable dynamics. Adaptive control of nonlinear PDEs has also received some attention. Liu and Krstic [18] and Kobayashi [11] considered a Burgers equation with various parametric uncertainties; Kobayashi [13] also considered the Kuramoto-Sivashinsky equation. Jovanovic and Bamieh [9] designed adaptive controllers for nonlinear systems on lattices, which include applications like infinite vehicular platoons or infinite arrays of microcantilevers. An experimentally validated adaptive boundary controller for a flexible beam was presented by de Queiroz, Dawson, Agarwal, and Zhang [4].

b) The Results of the Paper: For several unstable parabolic PDE systems controlled from the boundary we assume that physical parameters like reaction, diffusion, or advection coefficients are unknown. No adaptive controllers for such models have been proposed, even though they are frequent in applications that incorporate thermal-fluid or chemically reacting dynamics. An obstacle to the development of adaptive controllers has been the lack of parametrized families of nonadaptive controllers. This obstacle was removed by Smyshlyaev and Krstic [25] who developed explicit formulae for boundary control of a class of parabolic PDEs that includes the problems considered here. Those formulae are not only explicit functions of the spatial coordinates of the PDE, but also depend explicitly on the physical parameters of the plant. This feature is absent from standard methods

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like LQR extensions to PDEs because parametrized solutions to Riccati equations cannot be obtained. While an adaptive version of an LQR approach would require a solution to a high-dimensional Riccati equation at each time step, our approach only requires that new parameter updates be plugged into the control formula.

For clarity, in this paper we present the results for scalar and vector parameter problems. They can be extended to functional parameters as in [1], [2], [7], [22], [30]. With the controllers parametrized in the physical parameters, our schemes are of *indirect* type.

Three basic approaches to the design of parameter identifiers for adaptive control exist [16]: the Lyapunov approach, the passivity-based approach (pursued in [1], [2], [7], [22]), and the swapping approach. The Lyapunov approach, which ensures the best transient performance properties is seldom possible without changing the control law to compensate the potentially destabilizing effect of adaptation, even in the linear case. We exploit the structural opportunities within the class of PDEs we are considering and develop Lyapunov adaptation schemes. In companion papers [28], [29] we develop estimation-based schemes (see Section X for some comments on the contents of those papers).

Our Lyapunov design is inspired by an idea Praly [23] developed for adaptive nonlinear control under growth conditions. Since our PDE problems are linear, we have found a way to significantly simplify this approach, however, we retain its main feature—a logarithm weight on the plant state in the Lyapunov function. This results in a normalization of the update law by a norm on the plant state, which is uncommon for Lyapunov designs. Except for some special examples, projection is needed to keep the parameters within an a priori set. Projection is not used as a robustification tool but to prevent adaptation transients that would require overly conservative restrictions on the size of the adaptation gain. The projection set may be taken conservatively (large), however, in order for stability to be guaranteed, the adaptation gain needs to be taken inversely proportional to the size of the parameter set. The bounds on the gain can be derived explicitly and are a priori verifiable.

Most of the designs presented require full state feedback, however, two examples are given that use only output feedback—scalar sensing at the boundary. These output feedback designs employ adaptive observers which we construct as infinite-dimensional extensions of Kreisselmeier-type filters used in [16].

Only 1D results are presented here. In [28] we show that our tools extend readily to higher dimensions, in appropriate geometries.

No simulation results are given in this paper. In [28] we show simulation results in 2D and inIn [29] we show simulation results for an output feedback design.

Our adaptive boundary control results can be developed both for Dirichlet and Neumann actuation. Most of the controller we present are for the notationally easier Dirichlet actuation but in the introductory example in Section II we use Neumann actuation.

To avoid tedium and keep the concepts clear we present designs for the simplest classes of systems for which the concepts are nontrivial. For example, it is shown separately how to deal with parametric uncertainties in boundary conditions or reaction terms involving boundary values. A skilled designer can combine these tools with the method for reaction-advection-diffusion systems to craft solutions to more general problems.

c) Notation: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The symbols $I_1(\cdot), I_2(\cdot), J_1(\cdot)$, etc., denote the corresponding Bessel functions.

II. Control Design for a System with an Unknown Reaction Coefficient

We start the paper with a design for a benchmark system and present extensions in subsequent sections. The benchmark system has a destabilizing reaction term and employs control only at the boundary. The unknown reaction coefficient is scalar, however, an extension to spatially-varying functional coefficients is discussed in Remark 2. A problem with multiple parameters is discussed in Section IX.

While most of the paper, and this section in particular, assumes availability of full state feedback, Section VIII presents designs that employ only boundary sensing.

Consider the following plant

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,, \qquad (1)$$
$$u(0,t) = 0\,, \qquad (2)$$

where λ is an unknown constant parameter that can have any real value. We use a Neumann boundary controller designed in [25] in the form[1]

$$u_x(1) = -\frac{\hat\lambda}{2}u(1) - \hat\lambda\int_0^1 \xi\frac{I_2\left(\sqrt{\hat\lambda(1-\xi^2)}\right)}{1-\xi^2}u(\xi)d\xi\,, \qquad (3)$$

which employs the measurements of $u(x)$ for $x\in[0,1]$ and an estimate $\hat\lambda$ of λ. Consider an invertible change of variable

$$w(x) = u(x) - \int_0^x k(x,\xi,\hat\lambda)u(\xi)\,d\xi\,, \qquad (4)$$

where

$$k(x,\xi,\hat\lambda) = -\hat\lambda\xi\frac{I_1\left(\sqrt{\hat\lambda(x^2-\xi^2)}\right)}{\sqrt{\hat\lambda(x^2-\xi^2)}}\,. \qquad (5)$$

Lemma A.2 in the Appendix establishes that (4) maps (1)–(3) into

$$w_t = w_{xx} + \dot{\hat\lambda}\int_0^x \frac{\xi}{2}w(\xi)\,d\xi + \tilde\lambda w\,, \qquad (6)$$
$$w(0) = 0\,, \qquad (7)$$
$$w_x(1) = 0\,, \qquad (8)$$

where $\tilde\lambda = \lambda - \hat\lambda$ is the parameter estimation error.

We will show that the update law

$$\dot{\hat\lambda} = \gamma\frac{\|w\|^2}{1+\|w\|^2}\,, \qquad 0<\gamma<1 \qquad (9)$$

[1] In the sequel, to reduce notational overload, the dependence on time will be suppressed whenever possible.

achieves regulation of $u(x,t)$ to zero for all $x \in [0,1]$, for arbitrarily large initial data $u(x,0)$ and for an arbitrarily poor initial estimate $\hat{\lambda}(0)$.

Theorem 1: Suppose that the system (1)–(3), (9) has a well defined classical solution for all $t \geq 0$. Then, for any initial condition $u_0 \in H_1$ and any $\hat{\lambda}(0) \in \mathbb{R}$, the solutions $u(x,t)$ and $\hat{\lambda}(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [0,1]$. Moreover, the following performance bounds hold in the closed-loop nonlinear system:

$$\begin{aligned} u(x,t)^2 \leq\ & 32\left(1 + 3\lambda^2 + \tilde{\lambda}(0)^2 + \gamma \log\left(1 + \|w(0)\|^2\right)\right) \\ & \times \left[\|w_x(0)\|^2 + 3\sqrt{\gamma}\left(1 + \|w(0)\|^2\right) \mathrm{e}^{\frac{1}{\gamma}\tilde{\lambda}(0)^2} \right. \\ & \left. \times \left(\log\left(1 + \|w(0)\|^2\right) + \frac{1}{\gamma}\tilde{\lambda}(0)^2\right)^{3/2}\right] \end{aligned} \tag{10}$$

for all $x \in [0,1]$, $t \geq 0$, and

$$\begin{aligned} & \int_0^\infty u(x,t)^2 dt \leq \\ & 48\left(1 + 3\lambda^2 + \tilde{\lambda}(0)^2 + \gamma \log\left(1 + \|w(0)\|^2\right)\right) \\ & \times \left(1 + \|w(0)\|^2\right) \mathrm{e}^{\frac{1}{\gamma}\tilde{\lambda}(0)^2} \\ & \times \left(\log\left(1 + \|w(0)\|^2\right) + \frac{1}{\gamma}\tilde{\lambda}(0)^2\right) \end{aligned} \tag{11}$$

for all $x \in [0,1]$.

Remark 1: While the bound (10) obviously quantifies the "peak transient" performance, the bound (11) quantifies the rate of convergence to zero.

Remark 2: In this paper we consider only parameters without spatial variation. In a future paper [27] we will present an extension to spatially-varying problems [1], [2], [7], [22], [30]. For example, in the case of the benchmark plant (1) but with constant λ replaced by $\lambda(x)$, we will design the adaptive controller

$$u_x(1) = \hat{k}(1,1)u(0) + \int_0^1 \hat{k}_x(1,\xi)u(\xi)d\xi \tag{12}$$

$$\hat{\lambda}_t(t,x) = \gamma \frac{u(t,x)\left(w(t,x) - \int_x^1 \hat{k}(\xi,x)w(t,\xi)d\xi\right)}{1 + \|w(t)\|^2} \tag{13}$$

where $\hat{\lambda}(t,x)$ is the online functional estimate of $\lambda(x)$, $w(x) = u(x) - \int_0^1 \hat{k}(x,\xi)u(\xi)d\xi$, and the kernel $\hat{k}(x,\xi) = \hat{k}_n(x,\xi)$ is obtained recursively from

$$\hat{k}_0(x,\xi) = -\frac{1}{2}\int_{\frac{x-\xi}{2}}^{\frac{x+\xi}{2}} \hat{\lambda}(\zeta)\, d\zeta \tag{14}$$

$$\begin{aligned} \hat{k}_{i+1}(x,\xi) =\ & \hat{k}_i(x,\xi) \\ & + \int_{\frac{x-\xi}{2}}^{\frac{x+\xi}{2}} \int_0^{\frac{x-\xi}{2}} \hat{\lambda}(\zeta - \sigma)\, \hat{k}_i(\zeta + \sigma, \zeta - \sigma) \\ & \times d\sigma d\zeta \end{aligned} \tag{15}$$

for each new update of $\hat{\lambda}(t,x)$. Stability is guaranteed for sufficiently small γ and sufficiently high n. The recursion (15) was proved convergent in [17]. Several methods for its symbolic or numerical computation were proposed and illustrated in [25], noting that the computational effort is at least an order of magnitude lower than solving a Riccati equation.

Remark 3: The non-negative form of the adaptive law (9) is coincidental for this particular benchmark plant and it is further discussed in Section V.

Remark 4: It is also important to note that the update law (9) contains normalization. Normalization is uncommon in Lyapunov designs and is the result of including the logarithm in the Lyapunov function [23]. Normalization is necessary because the control law (3) is of certainty equivalence type—unlike the Lyapunov adaptive controllers in [16] which employ non-normalized adaptation and strengthened nonlinear controllers that compensate for time-varying effects of adaptation. An additional measure of preventing overly fast adaptation in (9) is the restriction on the adaptation gain ($\gamma < 1$).

III. Proof of Theorem 1

Consider a Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1 + \|w\|^2\right) + \frac{1}{2\gamma}\tilde{\lambda}^2\,. \tag{16}$$

The time derivative along the solutions of (6)–(9) can be shown to be

$$\dot{V} = -\frac{\|w_x\|^2}{1 + \|w\|^2} + \frac{\dot{\hat{\lambda}}}{2}\frac{\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx}{1 + \|w\|^2} \tag{17}$$

(the calculation involves one step of integration by parts). Using Lemma A.1 and Poincare's inequality, one gets

$$\left|\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx\right| \leq \frac{2}{\sqrt{3}}\|w_x\|^2\,. \tag{18}$$

Substituting this inequality and (9) into (17), we get

$$\dot{V} \leq -\left(1 - \frac{\gamma}{\sqrt{3}}\right)\frac{\|w_x\|^2}{1 + \|w\|^2}\,. \tag{19}$$

This implies that $V(t)$ remains bounded for all time whenever $0 < \gamma \leq \sqrt{3}$. From the definition of V it follows that $\|w\|$ and $\hat{\lambda}$ remain bounded for all time. However, we need to show that $w(x,t)$ is bounded for all time and for all x. To do this, consider

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_x\|^2 &= \int_0^1 w_x w_{xt}\, dx = -\int_0^1 w_{xx} w_t\, dx \\ &= -\int_0^1 w_{xx}^2\, dx - \tilde{\lambda}\int_0^1 w_{xx} w dx \\ &\quad -\frac{\dot{\hat{\lambda}}}{2}\int_0^1 w_{xx}\int_0^x \xi w(\xi)\, d\xi \\ &= -\|w_{xx}\|^2 + \tilde{\lambda}\int_0^1 w_x^2 dx \\ &\quad +\frac{\dot{\hat{\lambda}}}{2}\int_0^1 x w w_x dx \\ &= -\|w_{xx}\|^2 + \tilde{\lambda}\|w_x\|^2 \\ &\quad +\frac{\dot{\hat{\lambda}}}{4}\left(w(1)^2 - \|w\|^2\right)\,. \end{aligned} \tag{20}$$

Integration by parts was used several times to obtain the above equalities. Using Agmon's inequality (noting that $w(0) = 0$), then Young's inequality, and finally Poincare's inequality (noting that $w_x(1) = 0$), one gets that

$$w(1)^2 - \|w\|^2 \leq \|w_x\|^2 \leq 4\|w_{xx}\|^2 . \quad (21)$$

Substituting (21) into (20), it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_x\|^2 &\leq -(1-\gamma)\|w_{xx}\|^2 + \tilde{\lambda}\|w_x\|^2 \\ &\leq \tilde{\lambda}\|w_x\|^2 . \end{aligned} \quad (22)$$

Integrating the last inequality, we obtain

$$\begin{aligned} \|w_x(t)\|^2 &\leq \|w_x(0)\|^2 \\ &+2 \sup_{0\leq\tau\leq t} |\tilde{\lambda}(\tau)| \int_0^t \|w_x(\tau)\|^2 d\tau . \end{aligned} \quad (23)$$

To obtain this bound, on one hand we have from (16) and (19) that

$$\tilde{\lambda}(t)^2 \leq \tilde{\lambda}(0)^2 + \gamma \log\left(1 + \|w(0)\|^2\right) . \quad (24)$$

On the other hand,

$$\begin{aligned} &\int_0^t \|w_x(\tau)\|^2 d\tau \\ &\leq \sup_{0\leq\tau\leq t} \left(1+\|w(\tau)\|^2\right) \int_0^t \frac{\|w_x(\tau)\|^2}{1+\|w(\tau)\|^2} d\tau . \end{aligned} \quad (25)$$

From (16) and (19) it follows that

$$1 + \|w(\tau)\|^2 \leq \left(1 + \|w(0)\|^2\right) \mathrm{e}^{\frac{1}{\gamma}\tilde{\lambda}(0)^2} . \quad (26)$$

Integrating (19) we get

$$\begin{aligned} &\int_0^t \frac{\|w_x(\tau)\|^2}{1+\|w(\tau)\|^2} d\tau \\ &\leq \frac{1}{2\left(1-\frac{\gamma}{\sqrt{3}}\right)} \left(\log\left(1+\|w(0)\|^2\right) + \frac{1}{\gamma}\tilde{\lambda}(0)^2\right) . \end{aligned} \quad (27)$$

Substituting (26) and (27) into (25), and then, along with (24), into (23), we get

$$\begin{aligned} \|w_x(t)\|^2 &\leq \|w_x(0)\|^2 \\ &+\frac{\sqrt{\gamma}}{1-\frac{\gamma}{\sqrt{3}}} \left(1+\|w(0)\|^2\right) \mathrm{e}^{\frac{1}{\gamma}\tilde{\lambda}(0)^2} \\ &\times \left(\log\left(1+\|w(0)\|^2\right) + \frac{1}{\gamma}\tilde{\lambda}(0)^2\right)^{3/2} . \end{aligned} \quad (28)$$

By combining Agmon's and Poincare's inequalities (and using the fact that $w(0) = 0$), we get $\max_{x\in[0,1]} |w(x)|^2 \leq 4\|w_x\|^2$, thus $w(x,t)$ is uniformly bounded.

Next, we prove regulation of $w(x,t)$ to zero. Using (6)–(8) and Lemma A.1 we obtain

$$\frac{1}{2}\left|\frac{d}{dt}\|w\|^2\right| \leq \|w_x\|^2 + \left(|\tilde{\lambda}| + \frac{\gamma}{4\sqrt{3}}\right)\|w\|^2 . \quad (29)$$

Since $\|w\|$ and $\|w_x\|$ have been proven bounded, it follows that $\frac{d}{dt}\|w\|^2$ is bounded, and thus $\|w(t)\|$ is uniformly continuous. By combining (25)–(27) with Poincare's inequality we also get that $\|w(t)\|^2$ is integrable in time over the infinite time interval. By Barbalat's lemma it follows that $\|w(t)\| \to 0$ as $t \to \infty$.

To show regulation also in the maximum norm, we note that, from Agmon's inequality, $|w(x,t)|^2 \leq 2\|w(t)\|\|w_x(t)\|$. Since $\|w_x\|$ is bounded and $\|w(t)\|$ has been shown convergent to zero, the regulation in maximum norm follows.

Having proved the boundedness and regulation of w, we now set out to establish the same for u. We start by noting that [25]

$$u(x) = w(x) + \int_0^x l(x,\xi,\hat{\lambda}) w(\xi) d\xi , \quad (30)$$

where

$$l(x,\xi,\hat{\lambda}) = -\hat{\lambda}\xi \frac{J_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}} . \quad (31)$$

It is straightforward to show that

$$\|u_x\|^2 \leq 2\left(1+\hat{\lambda}^2+4M\right)\|w_x\|^2 , \quad (32)$$

where

$$M = \int_0^1 \left(\int_0^1 \left|l_x(x,\xi,\hat{\lambda})\right| d\xi\right)^2 dx \quad (33)$$

and

$$l_x(x,\xi,\hat{\lambda}) = \hat{\lambda}x\xi \frac{J_2\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{x^2-\xi^2} . \quad (34)$$

By mimicking the calculation in [25, Equation (101)], we get $\int_0^1 \left|l_x(x,\xi,\hat{\lambda})\right| d\xi \leq |\hat{\lambda}|x + 1$, which implies

$$M \leq \int_0^1 \left(|\hat{\lambda}|x+1\right)^2 dx = \frac{1}{3}\hat{\lambda}^2 + |\hat{\lambda}| + 1 \leq \frac{\hat{\lambda}^2+3}{2} . \quad (35)$$

Thus, it follows that

$$\begin{aligned} \|u_x\|^2 &\leq 2\left(4+3\hat{\lambda}^2\right)\|w_x\|^2 \\ &\leq 8\left(1+3\lambda^2+\tilde{\lambda}^2\right)\|w_x\|^2 . \end{aligned} \quad (36)$$

Noting that $u(x,t)^2 \leq 4\|u_x\|^2$ for all $(x,t) \in [0,1]\times[0,\infty)$, by combining (36), (24), and (28), and using the fact that $\frac{1}{1-\frac{\gamma}{\sqrt{3}}} < 3$ for $\gamma < 1$, we get (10), which proves uniform boundedness of u.

To prove regulation of $u(x,t)$ to zero for all $x \in [0,1]$, we start by noting that

$$\|u\|^2 \leq 2(1+L)\|w\|^2 \quad (37)$$

where

$$L = \max_{0\leq\xi\leq x\leq 1} l(x,\xi,\hat{\lambda})^2 \quad (38)$$

is finite whenever $\hat{\lambda}$ is finite (which we have proved using Lyapunov analysis). Since $\|w\|$ is regulated to zero, so is $\|u\|$. By Agmon's inequality $u(x,t)^2 \leq 2\|u\|\|u_x\|$, where $\|u_x\|$ is bounded by (36), (24), and (28). This completes the proof of regulation of u.

The bound (11) is obtained in a similar manner to (10), by combining (36) with (24)–(27).

IV. Well Posedness

Since the purpose of our paper is stabilization, we focus our effort on proving boundedness and regulation. As evident from Section III, this is not a routine task due to the nonlinear character of the closed-loop system

$$w_t = w_{xx} + \frac{\gamma}{2}\frac{\|w\|^2}{1+\|w\|^2}\int_0^x \xi w(\xi)\, d\xi + \tilde{\lambda} w \quad (39)$$

$$w(0) = w_x(1) = 0\,, \quad (40)$$

$$\dot{\hat{\lambda}} = -\gamma\frac{\|w\|^2}{1+\|w\|^2}\,. \quad (41)$$

The analysis of existence and uniqueness of solutions is even more involved. One of the steps in proving *global* existence and uniqueness of *classical* solutions is to prove boundedness of $w_t(t,x)$ and $w_{xx}(t,x)$, which proceeds as follows. It is first observed from the first line of (22) that $\|w_{xx}\|$ is square integrable over infinite time. The same property holds for $\|w_t\|$. It is then shown that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_t\|^2 + \|w_{tx}\|^2 = \\ &\tilde{\lambda}\|w_t\|^2 + \frac{\ddot{\hat{\lambda}}}{2}\int_0^1 w_t(x)\int_0^x \xi w(\xi) d\xi dx \\ &+\dot{\hat{\lambda}}\int_0^1 w_t(x)\left(\int_0^x \frac{\xi}{2} w_t(\xi)d\xi - w(x)\right)dx \end{aligned} \quad (42)$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_{tx}\|^2 + \|w_{txx}\|^2 = \\ &\tilde{\lambda}\|w_{tx}\|^2 + \frac{\ddot{\hat{\lambda}}}{2}\int_0^1 x w_{tx}(x)w(x)dx \\ &+\dot{\hat{\lambda}}\int_0^1 w_{tx}(x)\left(\frac{x}{2}w_t(x) - w_x(x)\right)dx\,, \end{aligned} \quad (43)$$

where

$$\ddot{\hat{\lambda}} = \frac{\gamma}{(1+\|w\|^2)^2}\frac{d}{dt}\|w\|^2 \quad (44)$$

is bounded because of (29). From the boundedness of $\|w\|, \|w_x\|, \dot{\hat{\lambda}}, \ddot{\hat{\lambda}}$ and the square integrability in time of $\|w\|, \|w_t\|$, by integrating (42) it follows that $\|w_t\|$ is bounded and $\|w_{tx}\|$ is square integrable. Then, by integrating (43) and using the square integrability of $\|w_x\|$ and the other functions mentioned above, it follows that $\|w_{tx}\|$ is bounded and $\|w_{txx}\|$ is square integrable. By Agmon's inequality, we get that $w_t(t,x)$ is uniformly bounded for all values of its arguments, and the same holds for $w_{xx}(t,x)$. Those properties are also valid in the original variable $u(t,x)$ using the smoothly invertible variable change (4)–(5).

Existence and uniqueness of appropriately defined *weak* solutions can be studied in the same way as in [18, Section 4]. One writes the system in the form of two integral equations, using the "heat equation" Green function for the PDE for w, and then applies the Banach fixed point theorem. The main difference in using that idea here would be that the Green function used in [18] was for Neumann boundary conditions at both ends, whereas in our case one boundary condition is Dirichlet and the other is Neumann, which would necessitate a slightly different Green function.

We shall not belabor on well posedness issues in the rest of the paper both in the interest of space and due to the parabolic character of the system which ensures it. As in Theorem 1, in the rest of the paper we shall simply assume well posedness.

V. Parametric Robustness

Let us suppose that the adaptation is turned off, i.e., $\gamma = 0$, i.e., $\dot{\hat{\lambda}} \equiv 0$. Then the closed loop system is

$$w_t = w_{xx} + \left(\lambda - \hat{\lambda}\right)w\,, \quad (45)$$

$$w(0) = 0\,, \quad (46)$$

$$w_x(1) = 0\,, \quad (47)$$

where $\hat{\lambda}$ is a constant parameter estimate. By studying the eigenvalue problem of this system, it can be shown that parameter estimates $\hat{\lambda}$ which are greater than $\lambda - \frac{\pi^2}{4}$ are exponentially stabilizing, whereas those smaller than $\lambda - \frac{\pi^2}{4}$ are destabilizing. This means that, if an upper bound on λ is known—let us denote this bound by $\bar{\lambda}$—then (3) is a stabilizing linear controller whenever $\hat{\lambda}$ is replaced by $\bar{\lambda}$ (or any constant value higher than $\bar{\lambda}$).

This robustness property explains why $\dot{\hat{\lambda}}$ in the adaptation law (9) is nonnegative: overestimating $\hat{\lambda}$ cannot be harmful within the controller structure (3).[2] A caveat however is that, in the presence of measurement noise, the parameter estimate will drift. In the update law (9) the estimate has nowhere to drift but up[3] (which is consistent with the structure of the control law but still undesirable). In practical implementation one would add leakage, deadzone, or projection [8] to reduce or completely stop the drift.

The linear/frozen-parameter robustness is an unusual feature of the control formula (3). It is different than the "infinite gain margin" property of inverse optimal controllers, which allow an arbitrary increase of a scalar gain in front of the optimal control law. Infinite gain margin allows only an unplanned increase in the "control authority" but does not guarantee robustness to changes in the physical parameters of the system. The robustness exhibited with (3) is with respect to the physical parameter λ.

Due to the ability of the controller (3) to remain stabilizing when λ is overestimated, it might be tempting to view the backstepping design as being "high-gain." This would not be appropriate because (3) resorts to high gain only when λ generates a high number of unstable eigenvalues in the plant.

The form of high gain that controller (3) is capable of employing should not be confused with adaptive high gain controllers surveyed in [21] where a multiplicative gain is tuned for a controller of the form

$$u_x(1) = G\{Cu\} \quad (48)$$

[2] While the update law (9) can take the estimate $\hat{\lambda}$ only "up," the growth of the estimate stops as $\|w(t)\|$ goes to zero. Since $V(t)$ is nonincreasing and bounded from below (by zero), it has a limit. Hence $\tilde{\lambda}(t)^2$ has a limit. So does $\hat{\lambda}(t)$ and it is higher than $\lambda - \frac{\pi^2}{4}$. The size of $\hat{\lambda}(\infty)$ depends on the size of the initial condition u_0.

[3] This issue is no less critical with update laws that are sign-indefinite, however, with (9) it is obvious.

where G is the gain and C is an output operator such that $u_x(1) \mapsto Cu$ is relative degree one. For the present system an operator C independent of the unknown λ cannot be found, therefore, tuning of a multiplicative gain G could not be successful.

VI. An Alternative Approach

The use of a logarithm in the Lyapunov function (16) was inspired by Praly's Lyapunov adaptation designs in [23]. We do not exactly follow that method in this paper because our PDE plants are linear. It is however of interest to see what an exact application of that method results in, as it has potential beyond our class of problems.

Let us start by denoting

$$A = \frac{\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx}{1+\|w\|^2} \tag{49}$$

$$B = 2\frac{A}{1+\|w\|^2} \tag{50}$$

$$\begin{aligned} H &= -A^2 + \frac{1}{1+\|w\|^2}\int_0^1 \left(\left(\int_0^x \xi w(\xi)d\xi\right)^2 \right.\\ &\quad \left. + w(x)\left(\int_0^x \xi\left(\int_0^\xi w(\eta)d\eta\right)d\xi\right)\right)dx\,. \end{aligned} \tag{51}$$

This method employs two estimates working in tandem, $\hat{\lambda}$ and $\hat{\theta}$. A long Lyapunov based derivation, briefly justified after the statement of the theorem below, yields

$$\begin{aligned} \dot{\hat{\lambda}} &= \gamma\frac{\beta\gamma}{\beta\gamma(1-\gamma H)-1} \\ &\quad \times\left(\frac{\frac{3}{2}\|w\|^2+2A\|w_x\|^2}{1+\|w\|^2}\right. \\ &\quad -\left(\left(1+\frac{1}{\gamma^2}\right)-\frac{1}{\beta\gamma^2}\right)\beta\gamma B\left(\hat{\lambda}-\hat{\theta}-\gamma A\right) \\ &\quad \left.-\sigma\left(\hat{\lambda}-\hat{\theta}-\gamma A\right)\right) \end{aligned} \tag{52}$$

$$\dot{\hat{\theta}} = \gamma\left(\frac{2\|w\|^2}{1+\|w\|^2}-\beta\gamma B\left(\hat{\lambda}-\hat{\theta}-\gamma A\right)\right). \tag{53}$$

We have written the two update laws in a way to highlight as much as possible the parts that are similar about them. Three gains are employed, which need to satisfy the following conditions:

$$\gamma < 3 \tag{54}$$

$$\beta > \frac{1}{\gamma(1-\frac{\gamma}{3})} \tag{55}$$

$$\sigma > 0\,. \tag{56}$$

The conditions (54) and (55) are related to the fact that $|H| \le \frac{1}{3}$.[4] These conditions ensure that the denominator in the first line of (52) remains positive.

Besides its complexity, a disadvantage of the update law (52) is that it employs $\|w_x\|$, i.e., it requires the measurement of the spatial derivative $u_x(x,t)$.

[4] A fairly obvious bound is $|H| \le 3$ but a careful calculation in the vein of Lemma A.1 can establish a tighter bound $|H| \le \frac{1}{3}$.

Theorem 2: Suppose that the system (1)–(3), (52), (53) has a well defined classical solution for all $t \ge 0$. Then, for any initial condition $u_0 \in L_2$ and any $\hat{\lambda}(0), \hat{\theta}(0) \in \mathbb{R}$, the spatial L_2 norm $\|u(t)\|$ remains bounded and the spatial H_1 norm $\|u_x(t)\|$ is square integrable over an infinite time interval. Moreover, the estimates $\hat{\lambda}(t), \hat{\theta}(t)$ are uniformly bounded.

The proof of this result employs a Lyapunov function

$$\begin{aligned} V &= \frac{\beta\gamma^2}{2}\frac{\beta\gamma+1}{\beta\gamma-1} + \log\left(1+\|w\|^2\right) \\ &\quad -\frac{1}{2\gamma}\left(\hat{\lambda}-\hat{\theta}\right)^2 + \frac{1}{2\gamma}\left(\lambda-\hat{\theta}\right)^2 \\ &\quad +\frac{\beta}{2}\left(\hat{\lambda}-\hat{\theta}-\gamma A\right)^2. \end{aligned} \tag{57}$$

It is possible to prove that

$$\begin{aligned} V &\ge \log\left(1+\|w\|^2\right) \\ &\quad +\frac{1}{2\gamma}\left(\left(\hat{\lambda}-\hat{\theta}\right)^2+\frac{\beta\gamma-1}{2}\left(\lambda-\hat{\theta}\right)^2\right), \end{aligned} \tag{58}$$

i.e., V is positive definite around the equilibrium $w(x) \equiv 0, \hat{\lambda} = \hat{\theta} = \lambda$. Then, a *very* long calculation yields

$$\dot{V} = -2\frac{\|w_x\|^2}{1+\|w\|^2}\,. \tag{59}$$

The properties stated in Theorem 2 readily follow from this equation.

VII. Other Benchmark Problems

In this section we will show that our method extends beyond the basic reaction-diffusion class of parabolic PDEs. We will consider two benchmark problems–one with a boundary value appearing on the right-hand-side of the PDE model and another with a parametric uncertainty in an uncontrolled boundary condition. Both benchmark problems are unstable in the absence of feedback.

These benchmarks will expose one limitation of the 'log-Lyapunov paradigm:' in general it requires not only a restriction on the value of the adaptation gain γ but also the use of parameter projection. A small γ is the main tool for preventing destabilizing transients. Projection is only used to make the restriction on γ a priori verifiable.

The projection operator that would be used in implementation is defined as

$$\mathrm{Proj}_{[\underline{\theta},\bar{\theta}]}\{\tau\} = \tau\begin{cases} 0, & \hat{\theta}=\underline{\theta} \text{ and } \tau<0 \\ 0, & \hat{\theta}=\bar{\theta} \text{ and } \tau>0 \\ 1, & \text{else} \end{cases} \tag{60}$$

where $\hat{\theta}$ is the parameter estimate (θ is used as a generic symbol for an unknown parameter, which will in subsequent presentation be replaced by specific parameters labeled by $g, q, \varepsilon, b, \lambda$), the interval $\left[\underline{\theta}, \bar{\theta}\right]$ is the interval within which $\hat{\theta}$ is being kept by projection, and τ denotes the nominal update law.

Unfortunately, the projection operator (60) is discontinuous. This presents a problem at two levels: (1) in the analysis it is not possible to obtain classical solutions but only Filippov solutions; (2) in implementation the presence of noise may induce frequent switching of the update law. This issue is not as serious as controller switching in sliding mode control because the projection operator does not drive an actuator. Since the projection drives only the update law $\dot{\hat{\theta}}$ there would be no discontinuities in $\hat{\theta}(t)$ and therefore no jumps in the control action. However, obtaining classical solutions and not having to deal with Filippov solutions is a good enough reason to consider a continuous version of the projection operator where, instead of a hard switch, a boundary layer of width $\delta > 0$ is introduced:

$$\text{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\} = \tau \begin{cases} \frac{\hat{\theta}-\underline{\theta}+\delta}{\delta}, & \underline{\theta} - \delta \leq \hat{\theta} < \underline{\theta} \text{ and } \tau < 0 \\ \frac{\bar{\theta}+\delta-\hat{\theta}}{\delta}, & \bar{\theta} < \hat{\theta} \leq \bar{\theta} + \delta \text{ and } \tau > 0 \\ 1, & \text{else} \end{cases} \tag{61}$$

where the update law τ is scaled linearly with θ in the boundary layer. With the help of [16, Lemma E.1] we get:

Lemma 3: The following properties of the projection operator (61) are guaranteed

1) The operator is a locally Lipschitz function of $\hat{\theta}, \tau$ on $\left[\underline{\theta} - \delta, \bar{\theta} + \delta\right] \times \mathbb{R}$.
2) $\left(\text{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\}\right)^2 \leq \tau^2$.
3) For $\hat{\theta}(0) \in \left[\underline{\theta} - \delta, \bar{\theta} + \delta\right]$, the solution of $\dot{\hat{\theta}} = \text{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\}$ remains in $\left[\underline{\theta} - \delta, \bar{\theta} + \delta\right]$.
4) $-\tilde{\theta}\text{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\} \leq -\tilde{\theta}\tau$ for all $\hat{\theta} \in \left[\underline{\theta} - \delta, \bar{\theta} + \delta\right], \theta \in \left[\underline{\theta}, \bar{\theta}\right]$.

All of the properties in Lemma 3 except Lipschitzness also hold for the discontinuous projection (60), with $\delta = 0$. The discontinuous projection would be preferable in applications for its simplicity which does not come at the expense of control switching, and because it is a standard feature in the integrator block in Simulink. For these reasons and to avoid clutter in our further presentation, we employ (60) where projection is needed.

Now we return to our presentation of the benchmark problems.

A. Example 1

Consider the plant

$$\begin{aligned} u_t &= u_{xx} + g u(0,t) && (62) \\ u_x(0) &= 0\,, && (63) \end{aligned}$$

where g is a constant, unknown parameter and $u(0,t)$ is the boundary value of $u(x,t)$ at $x = 0$. This system is inspired by a model of a thermal instability in solid propellant rockets [3]. We will control this system via Dirichlet actuation, $u(1,t)$. In the absence of control, $u(1,t) \equiv 0$, the system is unstable if and only if $g > 2$. We assume that this is indeed the case, $g > 2$. Let us further assume that an upper bound $\bar{g}$ on g is known to us. It is important to note that such an assumption was not made on λ in Section II. We will design an adaptive controller in this section whose update law incorporates the standard projection operator [16] to keep the parameter estimate $\hat{g}$ in the interval $[2, \bar{g}]$, while driving $u(x,t)$ to zero.

A stabilizing control formula was designed in [25] as

$$u(1) = -\int_0^1 \sqrt{\hat{g}} \sinh\left(\sqrt{\hat{g}}(1-\xi)\right) u(\xi) d\xi\,. \tag{64}$$

Consider the variable change

$$w(x) = u(x) + \int_0^x \sqrt{\hat{g}} \sinh\left(\sqrt{\hat{g}}(x-\xi)\right) u(\xi) d\xi\,. \tag{65}$$

It can be shown that

$$\begin{aligned} w_t &= w_{xx} + \dot{\hat{g}} \int_0^x w(\xi) \frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}} d\xi \\ &\quad + \tilde{g} w(0) \cosh\left(\sqrt{\hat{g}}x\right) && (66) \\ w_x(0) &= 0 && (67) \\ w(1) &= 0\,, && (68) \end{aligned}$$

where $\tilde{g} = g - \hat{g}$. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1 + \|w\|^2\right) + \frac{1}{2\gamma}\tilde{g}^2\,. \tag{69}$$

Taking its time derivative we arrive at the update law

$$\dot{\hat{g}} = \frac{\gamma}{1 + \|w\|^2} \text{Proj}_{[2,\bar{g}]}\left\{ w(0) \int_0^1 w(x) \cosh\left(\sqrt{\hat{g}}x\right) dx \right\}. \tag{70}$$

The derivative of the Lyapunov function is

$$\dot{V} = -\frac{\|w_x\|^2}{1 + \|w\|^2} + \dot{\hat{g}} \frac{\int_0^1 w(x) \int_0^x w(\xi) \frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}} d\xi dx}{1 + \|w\|^2}. \tag{71}$$

It can be shown that

$$\dot{V} \leq -\left(1 - 2\gamma \mathrm{e}^{2\sqrt{\bar{g}}}\right) \frac{\|w_x\|^2}{1 + \|w\|^2}\,. \tag{72}$$

Stability is thus achieved whenever

$$\gamma < \frac{1}{2}\mathrm{e}^{-2\sqrt{\bar{g}}}\,. \tag{73}$$

This condition highlights the key differences between the design for the PDE in Section II and for the PDE (62):

1) The adaptation gain, which was limited by 1 in Section II, needs to decrease as g increases in (62).
2) The knowledge of the parameter's upper bound is needed for the plant (62). Projection is used to keep the parameter within the a priori bound, such that the condition (73) is sufficient to achieve stability. It should also be noted that stability can be achieved without projection, by selecting γ to satisfy
$$\gamma < \frac{1}{2}\mathrm{e}^{-2\left(\sqrt{2\bar{g}+\hat{g}(0)} + \left(\gamma \log\left(1+\|w_0\|^2\right)\right)^{1/4}\right)}\,, \tag{74}$$
where $w_0(x)$ is determined using the initial state $u_0(x)$ and the initial parameter estimate $\hat{g}(0)$. While it may be unusual to choose the adaptation gain based on the initial state u_0, it is acceptable as a theoretical result and consistent with the Lyapunov function (69), yielding estimates on $\|u(t)\|$ and $\tilde{g}(t)$ that depend on $\|u_0\|$ and $\tilde{g}(0)$. However, in application one would prefer projection due to its added assurance against parameter drift.

Other than the use of projection, the rest of the results of this section are qualitatively the same as those in Section II. One can prove boundedness in the maximum norm in a similar manner as in Section III. A lengthy calculation yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_x\|^2 = \\
&-\|w_{xx}\|^2 - w_x(1)\Big[\tilde{g}\cosh\left(\sqrt{\hat{g}}\right)w(0) \\
&+\dot{\hat{g}}\int_0^1 w(x)\frac{\sinh\left(\sqrt{\hat{g}}(1-x)\right)}{\sqrt{\hat{g}}}dx\Big] \\
&-\dot{\hat{g}}\sqrt{\hat{g}}\int_0^1 w(x)\int_0^x \sinh\left(\sqrt{\hat{g}}(x-\xi)\right)w(\xi)d\xi dx \\
&-\tilde{g}\hat{g}w(0)\int_0^1 w(x)\cosh\left(\sqrt{\hat{g}}x\right)dx\,,
\end{aligned} \tag{75}$$

which can be majorized by

$$\frac{1}{2}\frac{d}{dt}\|w_x\|^2 \le 8\left(\gamma^2+\tilde{g}^2\right)\mathrm{e}^{4\sqrt{\bar{g}}}\|w_x\|^2\,. \tag{76}$$

Integrating (72) and (76) one gets boundedness of $\|w_x\|$. Regulation is shown similar as in Section III. The results in the $u(x,t)$ variable follow from the inverse transformation

$$u(x) = w(x) + \hat{g}\int_0^x (x-\xi)w(\xi)d\xi\,. \tag{77}$$

Theorem 4: Suppose that the system (62)–(64), (70) has a well defined classical solution for all $t \ge 0$. Then, for any initial condition $u_0 \in H_1$ and any $\hat{g}(0) \in [2,\bar{g}]$, the solutions $u(x,t)$ and $\hat{g}(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [0,1]$.

B. Example 2

Consider the plant

$$\begin{aligned}
u_t &= u_{xx} && (78)\\
u_x(0) &= -qu(0,t)\,, && (79)
\end{aligned}$$

where q is a constant, unknown parameter. This system is also inspired by the solid propellant rocket instability [3]. We will control this system via Dirichlet actuation, $u(1,t)$. In the absence of control, $u(1,t) \equiv 0$, the system is unstable if and only if $q > 1$. We assume that $q > 1$ and also that an upper bound $\bar{q}$ on q is known to us. We will design an adaptive controller with projection [16] to keep the parameter estimate $\hat{q}$ in the interval $[1,\bar{q}]$, while achieving stability.

A stabilizing control formula for this system is

$$u(1) = -\int_0^1 \hat{q}\mathrm{e}^{\hat{q}(1-\xi)}u(\xi)d\xi\,. \tag{80}$$

The idea for this choice is due to Andrey Smyshlyaev [29]. Consider the variable change

$$w(x) = u(x) + \int_0^x \hat{q}\mathrm{e}^{\hat{q}(x-\xi)}u(\xi)d\xi\,. \tag{81}$$

It can be shown that

$$\begin{aligned}
w_t &= w_{xx} + \dot{\hat{q}}\int_0^x w(\xi)\mathrm{e}^{\hat{q}(x-\xi)}d\xi && (82)\\
w_x(0) &= -\tilde{q}w(0) && (83)\\
w(1) &= 0\,, && (84)
\end{aligned}$$

where $\tilde{q} = q - \hat{q}$. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1+\|w\|^2\right) + \frac{1}{2\gamma}\tilde{q}^2\,. \tag{85}$$

Taking its time derivative we arrive at the update law

$$\dot{\hat{q}} = \frac{\gamma}{1+\|w\|^2}\mathrm{Proj}_{[1,\bar{q}]}\left\{w(0)^2\right\}\,. \tag{86}$$

The derivative of the Lyapunov function is

$$\dot{V} = -\frac{\|w_x\|^2}{1+\|w\|^2} + \dot{\hat{q}}\frac{\int_0^1 w(x)\int_0^x w(\xi)\mathrm{e}^{\hat{q}(x-\xi)}d\xi dx}{1+\|w\|^2}\,. \tag{87}$$

With a lengthy, careful calculation, applying twice the Cauchy-Schwartz inequality, one can show that

$$\begin{aligned}
&\left|\int_0^1 w(x)\int_0^x w(\xi)\mathrm{e}^{\hat{q}(x-\xi)}d\xi dx\right| \\
&\le \frac{\sqrt{\mathrm{e}^{2\hat{q}}-1-2\hat{q}}}{2\hat{q}}\|w\|^2 \\
&\le \frac{\mathrm{e}^{\hat{q}}}{\sqrt{2}}\|w\|^2\,.
\end{aligned} \tag{88}$$

Using projection and Agmon's inequality, it then follows that

$$\dot{V} \le -\left(1-\sqrt{2}\gamma\mathrm{e}^{\bar{q}}\right)\frac{\|w_x\|^2}{1+\|w\|^2}\,. \tag{89}$$

Stability is thus achieved whenever

$$\gamma < \frac{\sqrt{2}}{2}\mathrm{e}^{-\bar{q}}\,. \tag{90}$$

Again, projection and slow adaptation are needed to mitigate the effect of $\dot{\hat{q}}$ in the Lyapunov analysis.

We have thus proved L_2 stability in the w variable. Square integrability of $\|w_x(t)\|$ in time also readily follows from the Lyapunov analysis. From (89) it follows that $\dot{V}$ is bounded *from above*. This property is not sufficient to conclude uniform continuity of $\|w(t)\|$ and ensure the applicability of the classical Barbalat lemma, however, it is sufficient to meet the conditions of the less restrictive Lemma 3.1 in [18], which implies that $\|w(t)\| \to 0$ as $t \to \infty$. All of the above boundedness and regulation properties for the w variable are also valid in the original u variable due to the inverse transformation

$$u(x) = w(x) + \hat{q}\int_0^x w(\xi)d\xi\,. \tag{91}$$

Unfortunately, boundedness of $u(x)$ and its convergence to zero with time (uniformly in x) are difficult to prove because of the presence of the time-varying parameter error $\tilde{q}$ in (83). This difficulty is consistent with similar observations made in [1]. Boundedness and regulation despite uncertainty in the boundary condition was achieved in [18] but this was done using a particular "nonlinear damping" feedback, which is not possible here because we do not allow actuation at $x = 0$.

Theorem 5: Suppose that the system (78)–(80), (86) has a well defined classical solution for all $t \ge 0$. Then, for any initial condition $u_0 \in L_2$ and any $\hat{q}(0) \in [1,\bar{q}]$, the spatial L_2 norm $\|u(t)\|$ and the estimate $\hat{q}(t)$ remain uniformly bounded, $\|u(t)\|$ converges to zero as $t \to \infty$, and $u(t,x)$ is square integrable in t for all $x \in [0,1]$.

Let us now consider the "frozen adaptation" version of (82)–(84), with $\dot{\hat{q}} = 0$ and with a constant parameter error $\tilde{q}$. This system is exponentially stable if and only if the estimate is $\hat{q} > q - 1$. The same parametric robustness observations as those made in Section V hold for the plant-controller pair (78)–(80). Likewise, those observations justify the use of the estimator of the type (86) where the product of the "estimation error and regressor" is always nonnegative.

VIII. Output-Feedback Designs

A. Example 1

As in Section VII-A, we consider the plant

$$u_t = u_{xx} + gu(0,t) \tag{92}$$

$$u_x(0) = 0\,. \tag{93}$$

Suppose that only $u(0,t)$, the boundary value of $u(x,t)$ at $x = 0$, is available for measurement, whereas $u(1,t)$ is available for actuation. The transfer function from the input $u(1,t)$ to the output $u(0,t)$ has infinitely many poles and no zeros (the relative degree is infinite).

Instead of the unmeasurable state $u(x)$, we will employ an adaptive observer which consists of the input filter

$$\eta_t = \eta_{xx} \tag{94}$$

$$\eta_x(0) = 0 \tag{95}$$

$$\eta(1) = u(1)\,, \tag{96}$$

the output filter

$$v_t = v_{xx} + u(0) \tag{97}$$

$$v_x(0) = 0 \tag{98}$$

$$v(1) = 0\,, \tag{99}$$

and an estimate of $u(x)$ given by

$$\hat{g}v(x) + \eta(x)\,. \tag{100}$$

Our adaptive controller employs the control law

$$u(1) = -\int_0^1 \sqrt{\hat{g}}\sinh\left(\sqrt{\hat{g}}(1-\xi)\right)(\hat{g}v(\xi) + \eta(\xi))\,d\xi\,, \tag{101}$$

and the update law

$$\dot{\hat{g}} = \frac{\gamma}{1 + \|w\|^2 + a\|v\|^2}\text{Proj}_{[2,\bar{g}]}\left\{v(0) \times \int_0^1 \left(av(x) + \hat{g}\cosh\left(\sqrt{\hat{g}}x\right)w(x)\right)dx\right\} \tag{102}$$

where a and γ are positive, sufficiently small normalization and adaptation gains. The variable change $(\eta, v) \mapsto w(x)$ is defined as

$$w(x) = \hat{g}v(x) + \eta(x) + \int_0^x \sqrt{\hat{g}}\sinh\left(\sqrt{\hat{g}}(x-\xi)\right) \times (\hat{g}v(\xi) + \eta(\xi))\,d\xi\,. \tag{103}$$

Theorem 6: Suppose that the system (92)–(101), (102), (94)–(96), (97)–(99) has a well defined classical solution for all $t \geq 0$. Then, there exists $a^* > 0$, such that for all $a \in (0, a^*)$ there exists $\gamma^*(a) > 0$ [where both a^* and $\gamma^*(a)$ can be a priori estimated by the designer], such that for all $\gamma \in (0, \gamma^*)$ the following holds: For any initial condition $u_0, \eta_0, v_0 \in H_1$ and any $\hat{g}(0) \in [2, \bar{g}]$, the solutions $u(x,t), \eta(x,t), v(x,t)$ and $\hat{g}(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \eta(x,t) = \lim_{t\to\infty} v(x,t) = 0$ for all $x \in [0,1]$.

Proof: We start with a target system derived in [29],

$$w_t = w_{xx} + \dot{\hat{g}}Q + \tilde{g}\cosh\left(\sqrt{\hat{g}}x\right)(e(0) + w(0)) \tag{104}$$

$$w_x(0) = 0 \tag{105}$$

$$w(1) = 0\,, \tag{106}$$

where $\tilde{g} = g - \hat{g}$ is the parameter estimation error, signal Q is defined by

$$Q(x) = v(x) - \int_0^x (\hat{g}v(\xi) + w(\xi))\frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}}d\xi\,, \tag{107}$$

and $e(x,t)$ is an observer error defined as

$$e = u - gv - \eta\,, \tag{108}$$

and governed by

$$e_t = e_{xx} \tag{109}$$

$$e_x(0) = 0 \tag{110}$$

$$e(1) = 0\,. \tag{111}$$

Consider the Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1 + \|w\|^2 + a\|v\|^2\right) + \frac{b}{2}\|e\|^2 + \frac{1}{2\gamma}\tilde{g}^2\,, \tag{112}$$

where $a \in (0,1)$ and b are positive constants yet to be defined. We note that

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 = -\|e_x\|^2 \tag{113}$$

and, with (108), (103), and (97)–(99), that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -\|v_x\|^2 + (w(0) + e(0) + \tilde{g}v(0))\int_0^1 v(\xi)d\xi\,. \tag{114}$$

With (112), (113), (114), and (104)–(106), we get

$$\dot{V} = \frac{1}{1 + \|w\|^2 + a\|v\|^2}\left\{-\|w_x\|^2 - a\|v_x\|^2 + aw(0)\int_0^1 v(x)dx + e(0)\int_0^1 \left(av(x) + \hat{g}\cosh\left(\sqrt{\hat{g}}x\right)w(x)\right)dx + \dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right\} - b\|e_x\|^2\,, \tag{115}$$

which can be majorized by

$$\dot{V} \leq \frac{1}{1 + \|w\|^2 + a\|v\|^2}\left\{-(1-8a)\|w_x\|^2 - \frac{a}{2}\|v_x\|^2 - b\|e_x\|^2 + ae(0)\int_0^1 v(x)dx + e(0)\int_0^1 \hat{g}\cosh\left(\sqrt{\hat{g}}x\right)w(x)dx + \dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right\}\,. \tag{116}$$

By applying Young's inequality to the two cross-terms with $e(0)$, we get

$$\begin{aligned}\dot{V} \quad \leq \quad & \frac{1}{1+\|w\|^2+a\|v\|^2}\left\{-\left(1-8a-\frac{2}{\mu_1}\right)\|w_x\|^2\right.\\ & -\left(\frac{a}{2}-\frac{2}{\mu_2}\right)\|v_x\|^2\\ & -\left(b-2\mu_2 a^2-2\mu_1\bar{g}^2\mathrm{e}^{2\sqrt{\bar{g}}}\right)\|e_x\|^2\\ & \left.+\dot{g}\int_0^1 w(x)Q(x)dx\right\}, \end{aligned} \tag{117}$$

where μ_1 and μ_2 are positive constants that we can arbitrarily choose in our analysis. It can be shown that

$$\left|\int_0^1 w(x)Q(x)dx\right| \leq 2\mathrm{e}^{2\sqrt{\bar{g}}}\left(\|w\|^2+\|v\|^2\right), \tag{118}$$

which can then be used to prove that

$$\begin{aligned}&\left|\dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right|\\ &\leq 2\frac{\gamma}{a}\mathrm{e}^{2\sqrt{\bar{g}}}|v(0)|\left(a\|v\|+\mathrm{e}^{2\sqrt{\bar{g}}}\|w\|\right). \end{aligned} \tag{119}$$

With further calculations involving Young's, Poincare's, and Agmon's inequalities, and using that fact that $a, \gamma \in (0,1)$, one arrives at a conservative bound

$$\begin{aligned}&\left|\dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right|\\ &\leq 80\frac{\gamma}{a^2}\mathrm{e}^{8\sqrt{\bar{g}}}\|v_x\|^2+\frac{1}{4}\|w_x\|^2. \end{aligned} \tag{120}$$

Substituting this bound into (117), we get

$$\begin{aligned}\dot{V} \quad \leq \quad & \frac{1}{1+\|w\|^2+a\|v\|^2}\left\{-\left(\frac{3}{4}-8a-\frac{2}{\mu_1}\right)\|w_x\|^2\right.\\ & -\left(\frac{a}{2}-\frac{2}{\mu_2}-80\frac{\gamma}{a^2}\mathrm{e}^{8\sqrt{\bar{g}}}\right)\|v_x\|^2\\ & \left.-\left(b-2\mu_2 a^2-2\mu_1\bar{g}^2\mathrm{e}^{2\sqrt{\bar{g}}}\right)\|e_x\|^2\right\}. \end{aligned} \tag{121}$$

Selecting now

$$a^* = \frac{1}{16} \tag{122}$$

$$\gamma^* = \frac{a^3}{320}\mathrm{e}^{-8\sqrt{\bar{g}}} \tag{123}$$

$$\mu_1 = 16 \tag{124}$$

$$\mu_2 = \frac{16}{a} \tag{125}$$

$$b = 64\left(a+\bar{g}^2\mathrm{e}^{2\sqrt{\bar{g}}}\right), \tag{126}$$

for $a \in (0, a^*]$ and $\gamma \in (0, \gamma^*]$ we obtain

$$\dot{V} \quad \leq \quad -\frac{1}{8}\frac{\|w_x\|^2+a\|v_x\|^2+4b\|e_x\|^2}{1+\|w\|^2+a\|v\|^2}. \tag{127}$$

From (127) one can conclude the boundedness of $\|w\|, \|v\|$ and the integrability in time of $\|w_x\|^2, \|v_x\|^2$. From this, one can conclude that $\|Q\|$ is bounded and, with Agmon's inequality, that $\dot{\hat{g}}$ is square integrable over infinite time, which implies that $\dot{\hat{g}}\|Q\|$ is square integrable. Agmon's inequality also guarantees that $\tilde{g}\cosh\left(\sqrt{\hat{g}}x\right)(e(0)+w(0))$, which appears in (104), is square integrable. These properties can be used to show that $\|w_x\|$ is bounded. A similar argument, showing that $u(0) = w(0)+e(0)+\tilde{g}v(0)$ is square integrable over infinite time, can be used to conclude that $\|v_x\|$ is bounded. One can show next that $\dot{\hat{g}}Q + \tilde{g}\cosh\left(\sqrt{\hat{g}}x\right)(e(0)+w(0))$ and $u(0)$ are bounded and use that to prove that the time derivatives of $\|w\|^2, \|v\|^2$ are bounded. By Barbalat's lemma this implies the regulation of $\|w\|, \|v\|$, and, by Agmon's inequality, the regulation of $w(x), v(x)$ for all $x \in [0,1]$. To obtain the corresponding boundedness and regulation results for u, we first use the inverse transformation

$$\eta(x) = w(x) - \hat{g}v(x) + \hat{g}\int_0^x (x-\xi)w(\xi)d\xi, \tag{128}$$

which establishes the boundedness and regulation of η, and then invoke (108). ■

It is clear that the conservative values of a^* and γ^* are for the purposes of the proof only. In an implementation one would be safe to choose higher values of a and γ.

B. Example 2

As in Section VII-B, we consider the plant

$$u_t = u_{xx} \tag{129}$$

$$u_x(0) = -qu(0,t), \tag{130}$$

where only $u(0,t)$, the boundary value of $u(x,t)$ at $x = 0$, is available for measurement. The transfer function from the input $u(1,t)$ to the output $u(0,t)$ has infinitely many poles and no zeros (the relative degree is infinite).

Our output feedback adaptive controller uses the same input filter (94)–(96), but with an output filter

$$v_t = v_{xx} \tag{131}$$

$$v_x(0) = -u(0) \tag{132}$$

$$v(1) = 0, \tag{133}$$

a control law

$$u(1) = -\int_0^1 \hat{q}\mathrm{e}^{\hat{q}(1-\xi)}\left(\hat{q}v(\xi)+\eta(\xi)\right)d\xi, \tag{134}$$

and an update law

$$\begin{aligned}\dot{\hat{q}} \quad = \quad & \frac{\gamma}{1+\|w\|^2+a\|v\|^2}\mathrm{Proj}_{[1,\bar{q}]}\left\{v(0)\right.\\ & \left.\times\left(av(0)+\hat{q}\left(w(0)+\hat{q}\int_0^1 \mathrm{e}^{\hat{q}x}w(x)dx\right)\right)\right\}. \end{aligned} \tag{135}$$

The variable change $(\eta, v) \mapsto w(x)$ is defined as

$$w(x) = \hat{q}v(x) + \eta(x) + \int_0^x \hat{q}\mathrm{e}^{\hat{q}(x-\xi)}\left(\hat{q}v(\xi)+\eta(\xi)\right)d\xi. \tag{136}$$

Theorem 7: Suppose that the system (129)–(134), (135), (94)–(96), has a well defined classical solution for all $t \geq 0$. Then, there exists $a^* > 0$, such that for all $a \in (0, a^*)$ there exists $\gamma^*(a) > 0$ [where both a^* and $\gamma^*(a)$ can be a priori estimated by the designer], such that for all $\gamma \in (0, \gamma^*)$ the following holds: For any initial condition $u_0, \eta_0, v_0 \in L_2$ and any $\hat{q}(0) \in [1, \bar{q}]$, the spatial L_2 norms $\|u(t)\|, \|\eta\|, \|v\|$ and the estimate $\hat{q}(t)$ remain uniformly bounded, $\|u(t)\|, \|\eta\|, \|v\|$

converge to zero as $t \to \infty$, and $u(t,x), \eta(t,x), v(t,x)$ are square integrable in t for all $x \in [0,1]$.

To prove this result we start with a target system derived in [29],

$$\begin{aligned} w_t &= w_{xx} \\ &\quad +\dot{\hat{q}}\left\{v + \int_0^x \mathrm{e}^{\hat{q}(x-\xi)}\left(\hat{q}v(\xi) + w(\xi)\right)d\xi\right\} \\ &\quad +\hat{q}^2\mathrm{e}^{\hat{q}x}\left(e(0) + \tilde{q}v(0)\right) && (137) \\ w_x(0) &= -\hat{q}\left(e(0) + \tilde{q}v(0)\right) && (138) \\ w(1) &= 0\,, && (139) \end{aligned}$$

and then proceed with the Lyapunov function (112), with $\tilde{q}$ instead of $\tilde{g}$, going through inequalities as in Section VIII-A. The regulation is deduced as in Section VII-B, using the *upper* boundedness of $\dot{V}(t)$, the square integrability in time of $\|w_x\|, \|v_x\|$, and $\|e_x\|$ [all those properties are obtained from an inequality similar to (127)], and Lemma 3.1 in [18]. The inverse transformation needed for deducing the properties of η and u from the properties of w, v, e is

$$\eta(x) = w(x) - \hat{q}v(x) + \hat{q}\int_0^x w(\xi)d\xi\,. \qquad (140)$$

IX. Design for Systems with Unknown Diffusion and Advection Coefficients

For the sake of clarity we started in Section II with a reaction-diffusion system with only an unknown reaction coefficient. In this section we show how one can also incorporate adaptation for unknown diffusion and advection coefficients. Consider the system

$$\begin{aligned} u_t &= \varepsilon u_{xx} + bu_x + \lambda u && (141) \\ u(0) &= 0\,, && (142) \end{aligned}$$

where ε, b, λ are unknown constants.

The control law for this system is [25]

$$\begin{aligned} u(1) &= -\int_0^1 \frac{\hat{\lambda}+c}{\hat{\varepsilon}}\xi e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(1-\xi)} \\ &\quad \times \frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}}u(\xi)d\xi\,, && (143) \end{aligned}$$

where $\hat{\varepsilon}, \hat{b}, \hat{\lambda}$ are the estimates of ε, b, λ and $c \geq 0$ is a design gain. Using the transformation

$$\begin{aligned} w(x) &= u(x) - \int_0^x k(x,\xi)u(\xi)\,d\xi && (144) \\ k(x,\xi) &= -\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\xi e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(x-\xi)}\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}}, && (145) \end{aligned}$$

and its inverse

$$\begin{aligned} u(x) &= w(x) + \int_0^x l(x,\xi)w(\xi)\,d\xi && (146) \\ l(x,\xi) &= -\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\xi e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(x-\xi)}\frac{J_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}}, && (147) \end{aligned}$$

we get

$$\begin{aligned} w_t &= \varepsilon w_{xx} + bw_x - cw \\ &\quad +\dot{\hat{\varepsilon}}\int_0^x \varphi_0(x,\xi)w(\xi)\,d\xi \\ &\quad +\dot{\hat{b}}\int_0^x \varphi_1(x,\xi)w(\xi)\,d\xi \\ &\quad +\dot{\hat{\lambda}}\int_0^x \varphi_2(x,\xi)w(\xi)\,d\xi \\ &\quad -\tilde{\varepsilon}\left(\frac{\hat{\lambda}+c}{\hat{\varepsilon}}w + \frac{\hat{b}}{\hat{\varepsilon}}\int_0^x \varphi_3(x,\xi)w(\xi)\,d\xi\right) \\ &\quad +\tilde{b}\int_0^x \varphi_3(x,\xi)w(\xi)\,d\xi + \tilde{\lambda}w && (148) \\ w(0) &= 0\,, && (149) \\ w(1) &= 0\,, && (150) \end{aligned}$$

where

$$\begin{aligned} \varphi_0(x,\xi) &= -\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\varphi_2(x,\xi) - \frac{\hat{b}}{\hat{\varepsilon}}\varphi_1(x,\xi) && (151) \\ \varphi_1(x,\xi) &= \frac{x-\xi}{2\hat{\varepsilon}}k(x,\xi) \\ &\quad +\frac{1}{2\hat{\varepsilon}}\int_\xi^x (x-\sigma)k(x,\sigma)l(\sigma,\xi)\,d\sigma && (152) \\ \varphi_2(x,\xi) &= \frac{\xi}{2\hat{\varepsilon}}e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(x-\xi)} && (153) \\ \varphi_3(x,\xi) &= \mathrm{div}k(x,\xi) + \int_\xi^x (\mathrm{div}k(x,\sigma))l(\sigma,\xi)\,d\sigma && (154) \end{aligned}$$

and

$$\begin{aligned} \mathrm{div}k(x,\xi) &= \frac{1}{\xi}k(x,\xi) + \frac{\hat{\lambda}+c}{\hat{\varepsilon}}e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(x-\xi)}\frac{\xi}{x+\xi} \\ &\quad \times I_2\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}\right) && (155) \end{aligned}$$

Based on (148) and the Lyapunov function

$$V = \frac{1}{2}\left(\log(1+\|w\|^2) + \frac{\tilde{\varepsilon}^2 + \tilde{b}^2 + \tilde{\lambda}^2}{\gamma}\right) \qquad (156)$$

we choose the update laws

$$\begin{aligned} \dot{\hat{\lambda}} &= \gamma\frac{\|w\|^2}{1+\|w\|^2} && (157) \\ \dot{\hat{b}} &= \gamma\frac{\int_0^1 w(x)\int_0^x \varphi_3(x,\xi)w(\xi)\,d\xi dx}{1+\|w\|^2} && (158) \\ \dot{\hat{\varepsilon}} &= -\gamma\frac{\left(\hat{\lambda}+c\right)\|w\|^2 + \hat{b}\int_0^1 w(x)\int_0^x \varphi_3(x,\xi)w(\xi)\,d\xi dx}{\hat{\varepsilon}\left(1+\|w\|^2\right)} \\ & && (159) \end{aligned}$$

where projection is used (though we don't explicitly include it in the definition of the update laws) to keep the parameter estimates within a priori bounds $[\underline{\lambda}, \bar{\lambda}]$, $[\underline{b}, \bar{b}]$, and $[\underline{\varepsilon}, \bar{\varepsilon}]$, where $\underline{\varepsilon} > 0$. As in the previous problems, γ is limited by an upper bound which can be a priori computed.

Theorem 8: Suppose that the system (141)–(143), (157)–(159) has a well defined classical solution for all $t \geq 0$. Then, there exists $\gamma^* > 0$ such that, for all $\gamma \in (0, \gamma^*)$, for any initial condition $u_0 \in H_1$ and any $\hat{\lambda}(0) \in [\underline{\lambda}, \bar{\lambda}]$, $\hat{b}(0) \in [\underline{b}, \bar{b}]$, and $\hat{\varepsilon}(0) \in [\underline{\varepsilon}, \bar{\varepsilon}]$, the solutions $u(x,t)$ and $\hat{\lambda}(t), \hat{b}(t), \hat{\varepsilon}(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [0,1]$.

Proof: It can be shown that

$$\dot{V} = \frac{1}{1+\|w\|^2}\left(-\varepsilon\|w_x\|^2 - c\|w\|^2 + \dot{\hat{\varepsilon}}F_0 + \dot{\hat{b}}F_1 + \dot{\hat{\lambda}}F_2\right), \tag{160}$$

where

$$F_i(x) = \int_0^1 w(x)\int_0^x \varphi_i(x,\xi)w(\xi)\,d\xi dx \tag{161}$$

for $i = 0,1,2,3$. By applying the Cauchy-Schwartz inequality twice to (161), we get

$$|F_i| \le \|w\|^2\left(\int_0^1\int_0^x \varphi_i(x,\xi)^2 d\xi dx\right)^{1/2}. \tag{162}$$

Because the functions $\varphi_i(x,\xi)$ are continuous in $x,\xi,\hat{\varepsilon},\hat{b},\hat{\lambda}$ over the domain of their definition given by $\mathcal{T}\times[\underline{\varepsilon},\bar{\varepsilon}]\times[\underline{b},\bar{b}]\times[\underline{\lambda},\bar{\lambda}]$, where $\underline{\varepsilon} > 0$ and $\mathcal{T} = \{x,\xi \in \mathbb{R} | 0 \le \xi \le x \le 1\}$, it can be shown that there exist continuous, nonnegative-valued, nondecreasing functions $M_i : \mathbb{R}^5_+ \to \mathbb{R}_+$ such that

$$\int_0^1 w(x)\int_0^x \varphi_i(x,\xi)w(\xi)\,d\xi dx \le M_i\left(\frac{1}{\underline{\varepsilon}}, |\underline{b}|, |\bar{b}|, |\underline{\lambda}|, |\bar{\lambda}|\right). \tag{163}$$

The simplest one among these functions is

$$M_2 = \frac{1}{4\sqrt{3}\underline{\varepsilon}}\mathrm{e}^{\frac{1}{\underline{\varepsilon}}\max\{|\underline{b}|,|\bar{b}|\}}. \tag{164}$$

From (160)–(163), it follows that

$$\dot{V} \le \frac{1}{1+\|w\|^2}\left[-\varepsilon\|w_x\|^2 - c\|w\|^2 + \frac{\gamma\|w\|^4}{1+\|w\|^2} \times\left(\frac{|\bar{\lambda}|+c}{\underline{\varepsilon}}M_0 + \frac{|\bar{b}|}{\underline{\varepsilon}}M_3M_0 + M_3M_1 + M_2\right)\right] \tag{165}$$

where we emphasize the emergence of the fourth power of $\|w\|$ in the last term of the first line of (165). By applying Poincare's inequality we obtain

$$\dot{V} \le -\frac{\varepsilon(1-\gamma/\gamma^*)\|w_x\|^2 + c\|w\|^2}{1+\|w\|^2}, \tag{166}$$

where

$$\gamma^* = \frac{\varepsilon}{4}\left(\frac{|\bar{\lambda}|+c}{\underline{\varepsilon}}M_0 + \frac{|\bar{b}|}{\underline{\varepsilon}}M_3M_0 + M_3M_1 + M_2\right)^{-1}. \tag{167}$$

This establishes the boundedness of $\|w\|$ for $\gamma < \gamma^*$.

To prove the boundedness of $\|w_x\|^2$, we show that

$$\frac{1}{2}\frac{d}{dt}\|w_x\|^2 = -\varepsilon\|w_{xx}\|^2 - \frac{\varepsilon c + \varepsilon\tilde{\lambda} + \lambda\tilde{\varepsilon}}{\hat{\varepsilon}}\|w_x\|^2 - \int_0^1 w_{xx}(x)G(x)dx, \tag{168}$$

where

$$G(x) = bw_x + \dot{\hat{\varepsilon}}\int_0^x \varphi_0(x,\xi)w(\xi)\,d\xi + \dot{\hat{b}}\int_0^x \varphi_1(x,\xi)w(\xi)\,d\xi + \dot{\hat{\lambda}}\int_0^x \varphi_2(x,\xi)w(\xi)\,d\xi + \frac{\varepsilon\hat{b} - \hat{\varepsilon}b}{\hat{\varepsilon}}\int_0^x \varphi_3(x,\xi)w(\xi)\,d\xi. \tag{169}$$

Next we note that

$$\left|\dot{\hat{\lambda}}\right| \le \gamma \tag{170}$$

$$\left|\dot{\hat{b}}\right| \le \gamma M_3 \tag{171}$$

$$\left|\dot{\hat{\varepsilon}}\right| \le \gamma M_4 \tag{172}$$

$$\left|\frac{\varepsilon\hat{b} - \hat{\varepsilon}b}{\hat{\varepsilon}}\right| \le M_5, \tag{173}$$

where

$$M_4 = \frac{\max\{|\underline{\lambda}|, |\bar{\lambda}|\} + c + M_3\max\{|\underline{b}|, |\bar{b}|\}}{\underline{\varepsilon}} \tag{174}$$

$$M_5 = 2\frac{\bar{\varepsilon}}{\underline{\varepsilon}}\max\{|\underline{b}|, |\bar{b}|\}. \tag{175}$$

With Young's inequality we get

$$-\int_0^1 w_{xx}(x)G(x)dx \le \varepsilon\|w_{xx}\|^2 + \frac{1}{4\varepsilon}\|G\|^2. \tag{176}$$

Let us denote

$$H_i(x) = \int_0^x \varphi_i(x,\xi)w(\xi)\,d\xi \tag{177}$$

for $i = 0,1,2,3$, for which, with the Cauchy-Schwartz inequality, we get

$$\|H_i\| \le M_i\|w\|. \tag{178}$$

Then, from (169)–(178), with the triangle inequality and Poincare's inequality we obtain

$$\|G\|^2 \le 8\left[b + \gamma\left(M_4M_0^2 + M_3M_1^2 + M_2^2\right) + M_5M_3^2\right]\|w_x\|^2. \tag{179}$$

Substituting (179) into (176) and then into (168), we get

$$\frac{1}{2}\frac{d}{dt}\|w_x\|^2 \le N\|w_x\|^2, \tag{180}$$

where

$$N(t) = \frac{2}{\varepsilon}\left[b + \gamma\left(M_4M_0^2 + M_3M_1^2 + M_2^2\right) + M_5M_3^2\right] - \frac{\varepsilon c + \varepsilon\tilde{\lambda} + \lambda\tilde{\varepsilon}}{\hat{\varepsilon}} \tag{181}$$

is bounded. With $\|w\|$ bounded, from (166) we get that $\|w_x\|^2$ is integrable over infinite time. By integrating (180), it follows that $\|w_x\|$ is bounded. By Agmon's inequality, $w(x,t)$ is also bounded for all $t \ge 0$ and for all $x \in [0,1]$.

To show regulation, we calculate

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 = -\varepsilon\|w_x\|^2 - c\|w\|^2 + \dot{\hat{\varepsilon}}F_0 + \dot{\hat{b}}F_1 + \dot{\hat{\lambda}}F_2 - \tilde{\varepsilon}\left(\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\|w\|^2 + \frac{\hat{b}}{\hat{\varepsilon}}F_3\right) + \tilde{b}F_3 + \tilde{\lambda}\|w\|^2 . \quad (182)$$

All of the terms on the right hand side of this inequality have been proved to be bounded. Therefore $\frac{d}{dt}\|w\|^2$ is bounded. Since $\|w\|^2$ is also integrable over infinite time, by Barbalat's lemma $\|w(t)\| \to 0$ as $t \to \infty$. Regulation in maximum norm follows from Agmon's inequality and the boundedness of $\|w_x\|$.

To infer the results for the original variable $u(x,t)$ from those for $w(x,t)$, we recall the inverse transformation (146)–(147), which is a bounded operator in both L_2 and H_1. ■

While the Lyapunov design requires the use of projection and a low adaptation gain, one of its remarkable properties is that, even though the plant has parametric uncertainties multiplying u_x and u_{xx}, the adaptive scheme does not require the measurement of neither u_x nor u_{xx}. The update laws (157)–(159) employ only the measurement of u. This is in contrast with adaptive controllers in [1], [2], [7], [30] for reaction-advection-diffusion systems which require the measurement of u_{xx} to estimate the unknown diffusion coefficient ε.

The update laws employ $\varphi_3(x,\xi)$ which is given in quadratures. The integral in (154) would be calculated numerically, just like the other integrals appearing in the update laws and depending on the measured state $u(x,t)$.

Remark 5: It should be pointed out that in the Lyapunov approach the diffusion coefficient ε need not be estimated directly. This is analogous to the finite dimensional adaptive control [16] where the "high frequency gain" need not be estimated directly in the Lyapunov approach, whereas in the passive or swapping approaches it needs to be estimated. The estimation of ε is avoided by denoting the unknown parameters $\alpha = (\lambda + c)/\varepsilon$ and $\beta = b/\varepsilon$ and by replacing the adaptive controller (143) by

$$u(1) = -\int_0^1 \hat{\alpha}\xi e^{-\frac{\hat{\beta}}{2}(1-\xi)} \frac{I_1\left(\sqrt{\hat{\alpha}(1-\xi^2)}\right)}{\sqrt{\hat{\alpha}(1-\xi^2)}} u(\xi)d\xi\,, \quad (183)$$

by replacing the update laws (157)–(159) by

$$\dot{\hat{\alpha}} = \gamma\frac{\|w\|^2}{1+\|w\|^2} \quad (184)$$

$$\dot{\hat{\beta}} = \gamma\frac{\int_0^1 w(x)\int_0^x \varphi_3(x,\xi)w(\xi)\,d\xi dx}{1+\|w\|^2} \quad (185)$$

(equipped with appropriate projection), and by using w and φ_3 as defined in (144) and (154), respectively, with $k(x,\xi)$, $l(x,\xi)$, and div$k(x,\xi)$ redefined as

$$k(x,\xi) = -\hat{\alpha}\xi e^{-\frac{\hat{\beta}}{2}(x-\xi)} \frac{I_1\left(\sqrt{\hat{\alpha}(x^2-\xi^2)}\right)}{\sqrt{\hat{\alpha}(x^2-\xi^2)}}, \quad (186)$$

$$l(x,\xi) = -\hat{\alpha}\xi e^{-\frac{\hat{\beta}}{2}(x-\xi)} \frac{J_1\left(\sqrt{\hat{\alpha}(x^2-\xi^2)}\right)}{\sqrt{\hat{\alpha}(x^2-\xi^2)}} \quad (187)$$

$$\mathrm{div}k(x,\xi) = \frac{1}{\xi}k(x,\xi) + \hat{\alpha}e^{-\frac{\hat{\beta}}{2}(x-\xi)}\frac{\xi}{x+\xi}I_2\left(\sqrt{\hat{\alpha}(x^2-\xi^2)}\right) . \quad (188)$$

X. Conclusions and Future Work

The need for projection and a bound on the adaptation gain are the key limitations of the Lyapunov approach. In a companion paper on "estimation-based" approaches to adaptive control of PDEs [28] we present methods which do not require projection and which work without limits on the adaptation gain. These methods employ 'passivity/observer-based' and 'swapping-based' identifiers presented for finite-dimensional systems in [16]. However, in the case of uncertain diffusion and advection coefficients, these schemes require the measurement of $u_x(x,t)$ (and in some cases of $u_{xx}(x,t)$), like the schemes in [1], [2], [7], [30]. The Lyapunov schemes in Section IX require only the measurement of $u(x,t)$.

While, for the sake of clarity, we chose to present our design tools through benchmark problems, it is possible to develop an adaptive controller for the class of systems

$$u_t = \varepsilon u_{xx} + bu_x + \lambda u + gu(0) \quad (189)$$

$$u_x(0) = -qu(0)\,, \quad (190)$$

where $\varepsilon, b, \lambda, g, q$ are unknown. It is also possible to do so when these coefficients are spatially varying, as explained in Remark 2.

At present we have not worked out how to extend the result of Section IX to the output-feedback case. Even though boundary observers for this class of systems were developed in [26] for the case where ε, b, λ are known, the design of adaptive observers will be more complex than for the systems in Section VIII. In [29] we present the estimation-based versions of the Lyapunov output-feedback designs presented here.

It is possible to extend the results of this paper to special geometries in arbitrary dimension. For example, in 3D it is possible to extend them to domains in the shape of a rectangular parallelepiped with u_{xx} in (189) replaced by Δu and bu_x replaced by $b_1u_x + b_2u_y + b_3u_z$. It is shown in [28] how to deal with higher dimensions, thus we do not pursue them here.

Appendix

Lemma A.1:

$$\left|\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx\right| \le \frac{1}{2\sqrt{3}}\|w\|^2 . \quad (A.1)$$

Proof: Using the Cauchy-Schwartz inequality twice we obtain the following sequence of inequalities:

$$\begin{aligned}
&\left|\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx\right| \\
&\le \int_0^1 |w(x)|\left(\int_0^x \xi|w(\xi)|d\xi\right)dx \\
&\le \int_0^1 |w(x)|\left(\int_0^x \xi^2 d\xi\right)^{1/2}\left(\int_0^x w(\xi)^2 d\xi\right)^{1/2}dx \\
&\le \|w\|\int_0^1 |w(x)|\frac{1}{\sqrt{3}}x^{3/2}dx \\
&\le \frac{\|w\|}{\sqrt{3}}\|w\|\left(\int_0^1 x^3 dx\right)^{1/2} \\
&\le \frac{1}{2\sqrt{3}}\|w\|^2 . \qquad (A.2)
\end{aligned}$$

■

Lemma A.2: The transformation (4)–(5) maps the system (1)–(3) into (6)–(8).

Proof: Boundary conditions (7) and (8) are obviously satisfied. Substituting (4) into (1) we get

$$\begin{aligned}
w_t(x,t) &= w_{xx}(x,t) - \dot{\hat{\lambda}}\int_0^x k_{\hat{\lambda}}(x,y,\hat{\lambda})u(y,t)\,dy \\
&\quad + \tilde{\lambda}w\,, \qquad (A.3)
\end{aligned}$$

To replace u with w we use an inverse transformation (30) with a kernel (31). We have

$$\begin{aligned}
&\int_0^x k_{\hat{\lambda}}(x,\xi,\hat{\lambda})u(\xi,t)\,d\xi \\
&= \int_0^x k_{\hat{\lambda}}(x,\xi,\hat{\lambda}) \\
&\quad \times\left(w(\xi,t) + \int_0^\xi l(\xi,\eta,\hat{\lambda})w(\eta,t)\,d\eta\right)d\xi \\
&= \int_0^x \left(k_{\hat{\lambda}}(x,\xi,\hat{\lambda}) + \int_\xi^x k_{\hat{\lambda}}(x,\eta,\hat{\lambda})l(\eta,\xi,\hat{\lambda})\,d\eta\right) \\
&\quad \times w(\xi,t)\,d\xi\,. \qquad (A.4)
\end{aligned}$$

The inner integral is computed as follows

$$\begin{aligned}
&\int_\xi^x k_{\hat{\lambda}}(x,\eta,\hat{\lambda})l(\eta,\xi,\hat{\lambda})\,d\eta \\
&= \int_\xi^x \frac{\hat{\lambda}\eta\xi}{2}I_0\left(\sqrt{\hat{\lambda}(x^2-\eta^2)}\right)\frac{J_1\left(\sqrt{\hat{\lambda}(\eta^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(\eta^2-\xi^2)}}\,d\eta \\
&= \frac{\xi}{2}\int_0^{\sqrt{\hat{\lambda}(x^2-\xi^2)}} I_0\left(\sqrt{\hat{\lambda}(x^2-\xi^2)-s^2}\right)J_1(s)\,ds \\
&= \frac{\xi}{2}\left(I_0\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)-1\right) . \qquad (A.5)
\end{aligned}$$

Here the last integral was computed with a help of [24]. Finally we get

$$\begin{aligned}
&\int_0^x k_{\hat{\lambda}}(x,\xi,\hat{\lambda})u(\xi,t)\,d\xi \\
&= \int_0^x \left\{-\frac{\xi}{2}I_0\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)\right. \\
&\quad \left.+\frac{\xi}{2}\left(I_0\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)-1\right)\right\}w(\xi,t)\,d\xi \\
&= -\int_0^x \frac{\xi}{2}w(\xi,t)\,d\xi\,. \qquad (A.6)
\end{aligned}$$

■

ACKNOWLEDGEMENTS

I would like to thank Andrey Smyshlyaev for much help that his parallel effort in [28], [29] has provided to this paper through the derivation of the target/error-systems (6), (66), (82), (104), (137), and (148). I would also like to thank Laurent Praly, Petar Kokotovic, and Yuri Orlov for helpful suggestions. This effort has also benefited from stimulating discussions with Brian Anderson and Steve Morse. The work was supported by National Science Foundation under a grant number CMS-0329662.

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