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Adaptive Control of an Anti-Stable Wave PDE

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Abstract— **Past papers on adaptive control of unstable PDEs with unmatched parametric uncertainties have considered only parabolic PDEs and first-order hyperbolic PDEs. In this note we introduce several tools for approaching adaptive control problems of second-order-in-time PDEs. We present these tools through a benchmark example of an unstable wave equation with an unmatched (non-collocated) anti-damping term, which serves both as a source of instability and of parametric uncertainty. The key effort in the design is to avoid the appearance of the second time derivative of the parameter estimate in the error system.**

I. Introduction

Adaptive control of infinite-dimensional systems is a challenging topic to which several researchers have contributed over the last two decades [1], [2], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [16], [17], [22], [23]. The results have either allowed plant instability but required distributed actuation, or allowed boundary control but required that the plant be at least neutrally stable.

Recently we introduced several designs for boundary control of unstable parabolic PDEs [15], [19], [20]. Subsequently, in [3], [4] we also tackled systems with unknown input delay, i.e., an important class of infinite-dimensional systems with first-order hyperbolic PDE dynamics. The remaining major class of PDEs for which adaptive boundary control results have not been developed yet, at least not in the case where the plant is unstable, are second-order hyperbolic PDEs, namely wave equations. Wave (and beam) equations have been tackled in [2], [5], [6], [11], [13], [14], [17], [22], [23], however, not *unstable* ones.

In this note we present the first adaptive control design for an unstable wave equation controlled from a boundary, and where the source of instability is not collocated (matched) with control, requiring a feedback design outside of the standard catalog of damping-based feedback laws for the wave equation. This note is not meant to present a comprehensive methodology. Quite the opposite, we focus on the (notationally) simplest, most stripped down problem, but a problem that, among all basic wave equation problems with constant coefficients, is the most challenging. The goal of the note is to introduce a few novel ideas of how to deal with boundary control of *second-order-in-time* PDEs, which require parameter estimate-dependent state transformations, and which, if approached in the same way as parabolic or first-order hyperbolic PDEs, would give rise to perturbation terms involving the second derivative (in time) of the parameter estimate. With standard update law choices such perturbations would not be a priori bounded, whereas alternative choices that would make those perturbations bounded would require overparametrization.

The wave equation example that we focus on is a wave equation with an 'anti-damping' type of boundary condition in the boundary opposite from the controlled boundary. This PDE has all of its infinitely many eigenvalues in the right half plane, with arbitrarily large positive real parts. It is exponentially stable in negative time, thus we refer to it as 'anti-stable.' Even in the non-adaptive case, this PDE has been an open problem until a very recent breakthrough by A. Smyshlyaev [21] who constructed a novel 'backstepping transformation' for boundary control of this PDE. This PDE is probably not hugely relevant as a physical model, but it is of considerable importance as a methodological benchmark.

The note is organized as follows. We begin with a problem statement in Section II, followed by the introduction of the adaptive state transformation in Section III. In Sections IV and V we present, respectively, our controller and update law designs. In Section VI we state and prove our global stability and regulation results.

II. Problem Statement

Consider the system

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) && (1)\\ u_x(0,t) &= -qu_t(0,t) && (2)\\ u_x(1,t) &= U(t), && (3) \end{aligned}$$

where $U(t)$ is the input and $(u,u_t) \in H_1(0,1) \times L_2(0,1)$ is the system state. Our goal is to design a feedback law for the input $U(t)$, employing the measurement of the variables $u(0,t), u(1,t), u(x,t), x \in (0,1)$, as well as their time derivatives, if needed, to stabilize the anti-stable wave equation system. The key challenge is the large uncertainty in the antidamping coefficient $q \geq 0$, which will be dealt with by employing an estimate $\hat{q}(t)$ in the adaptive controller, and by designing an update law for $\hat{q}(t)$.

While in this paper we approach the control problem for the system (1), (2) using Neumann actuation (3), the problem can also be solved using Dirchlet actuation, $u(1,t) = U(t)$. However, in that case the system analysis has to take place in a higher space, $H_2(0,1) \times H_1(0,1)$, rather than in $H_1(0,1) \times L_2(0,1)$, which increases the complexity of presentation considerably, though the control design is actually slightly simplified. So, for the sake of clarity, we pursue Neumann actuation.

What is the significance of the adaptive boundary control problem for the system (1), (2)? First, the significance

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of the *non*-adaptive boundary control problem is that the uncontrolled system (1), (2) has infinitely many unstable eigenvalues, with arbitrarily large positive real parts. Under a Dirichlet boundary condition $u(1,t)=0$, they are

$$\lambda_n = \frac{1}{2}\ln\left|\frac{1+q}{1-q}\right| + j\pi \left\{ \begin{array}{ll} n+1/2, & 0\le q\le 1 \\ n, & q>1 \end{array}\right. \tag{4}$$

for $n\in\mathbb{Z}$. Additionally, the source of instability, the anti-damping term in (2), is on the opposite boundary from the boundary that is controlled. To deal with this challenge, we employ the following transformation invented by A. Smyshlyaev [21] (for known q):

$$w(x,t) = u(x,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\left(-\hat{q}(t)u(0,t) + \int_0^x u_t(y,t)dy\right). \tag{5}$$

The second significance of the adaptive boundary control problem for (1), (2) is that, as the reader shall see, the feedback law depends in a non-trivial (non-linear) manner on the estimate of the uncertain parameter q (this would not be the case if we dealt with a simpler problem where the unknown parameter is the propagation speed coefficient and the wave equation is of the standard kind, with eigenvalues on the real axis). So, this problem is a good non-routine example of adaptive control for second-order hyperbolic PDEs.

The third significance of the problem, and the most important adaptive control design issue specific to second order hyperbolic PDEs such as the wave equation, is that, if we proceed to differentiate (5) twice with respect to time, to get $w_{tt}(x,t)$, a second derivative of the parameter estimate $\ddot{\hat{q}}(t)$ would arise, as opposed to only the first derivative as in the parabolic PDE case [15].

III. System Transformation

To deal with the difficulty with $\ddot{\hat{q}}(t)$, we approach the problem differently, by redefining the wave equation model (1)–(3) as a first-order-in-time evolution equation,

$$\begin{aligned} u_t(x,t) &= v(x,t) && (6)\\ v_t(x,t) &= u_{xx}(x,t) && (7)\\ u_x(0,t) &= -qv(0,t) && (8)\\ u_x(1,t) &= U(t), && (9) \end{aligned}$$

where the variable $v(x,t)$ is the velocity. The state of this model is (u,v).

Starting from (5), we derive the transformation $(u,v)\mapsto(w,\omega)$ as

$$\omega(x,t) = v(x,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u_x(x,t) \tag{10}$$

$$w_x(x,t) = u_x(x,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}v(x,t) \tag{11}$$

$$\begin{aligned} w(1,t) =& \frac{1-\hat{q}^2(t)}{1+\hat{q}(t)c}u(1,t) \\ &+ \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\int_0^1 \left(v(x,t)+\hat{q}(t)u_x(x,t)\right)dx. \end{aligned} \tag{12}$$

The reason for the choice of variables (v,u_x,u_1) and (ω,w_x,w_1), where $u_1(t)=u(1,t), w_1(t)=w(1,t)$, is that the system norm we will employe for the system (6)–(9) is

$$\Omega(t) = \int_0^1 v^2(x,t)dx + \int_0^1 u_x^2(x,t)dx + u^2(1,t). \tag{13}$$

The inverse of the transformation (10)–(12), namely $(w,\omega)\mapsto(u,v)$, is given by

$$v(x,t) = \frac{(1+\hat{q}(t)c)^2}{(1-\hat{q}^2(t))(1-c^2)}\left(\omega(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}w_x(x,t)\right) \tag{14}$$

$$u_x(x,t) = \frac{(1+\hat{q}(t)c)^2}{(1-\hat{q}^2(t))(1-c^2)}\left(w_x(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\omega(x,t)\right) \tag{15}$$

$$\begin{aligned} u(1,t) =& \frac{1+\hat{q}(t)c}{1-\hat{q}^2(t)}w(1,t) \\ &+ \frac{(\hat{q}(t)+c)(1+\hat{q}(t)c)}{(1-\hat{q}^2(t))(1-c^2)} \\ &\times \int_0^1 \left(-\omega(x,t)+cw_x(x,t)\right)dx. \end{aligned} \tag{16}$$

In the non-adaptive case, namely, when $\hat{q}(t)\equiv q$, the transformation (5) would convert the plant (1), (2) into the 'target system' $w_{tt}=w_{xx}, w_x(0,t)=cw_t(0,t)$, which has a damping boundary condition at $x=0$.

The same approach applied to (5) would result in $\ddot{\hat{q}}(t)$ appearing on the right-hand side of the PDE. This problem is avoided with the introduction of the transformed velocity state (10). After a lengthy calculation, the 'target system' is shown to be

$$w_t(x,t) = \omega(x,t) + \tilde{q}(t)\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u_t(0,t) + \theta(x,t)\dot{\hat{q}}(t) \tag{17}$$

$$\omega_t(x,t) = w_{xx}(x,t) + \beta(x,t)\dot{\hat{q}}(t) \tag{18}$$

$$w_x(0,t) = c\omega(0,t) - \tilde{q}(t)\frac{1-c^2}{1+\hat{q}(t)c}u_t(0,t), \tag{19}$$

where

$$\tilde{q}(t) = q - \hat{q}(t) \tag{20}$$

is the parameter estimation error and

$$\begin{aligned} \theta(x,t) =& \frac{1}{1-\hat{q}^2(t)}\left(\int_0^1 \left(\omega(y,t)+\hat{q}(t)w_x(y,t)\right)dy \right.\\ &\left. - \frac{2\hat{q}(t)+c+c\hat{q}^2(t)}{1+\hat{q}(t)c}\right) - \int_x^1 \alpha(y,t)dy \end{aligned} \tag{21}$$

$$\alpha(x,t) = \frac{1}{1-\hat{q}^2(t)}\left(\omega(x,t) - -\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}w_x(x,t)\right) \tag{22}$$

$$\beta(x,t) = \frac{1}{1-\hat{q}^2(t)}\left(w_x(x,t) - \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\omega(x,t)\right). \tag{23}$$

In the non-adaptive case, the 'target system' $w_{tt}=w_{xx}, w_x(0,t)=cw_t(0,t)$ combined with an appropriate boundary condition at $x=1$, is exponentially stable. Our task is to achieve stability of the target system (17)–(19) in the adaptive case, namely, when $\tilde{q}(t)\neq 0$ and $\dot{\hat{q}}(t)\neq 0$ are acting as perturbations to the system, by choosing a boundary controller at $x=1$ and by designing an update law for $\hat{q}(t)$.

Before we proceed with the feedback design, in the remainder of this section we give several more relations that are satisfied by the transformed state variables. They will come in handy in the design and in the stability proof. The first of these relations is

$$w_{xt}(x,t) = \omega_x(x,t) + \alpha(x,t)\dot{\hat{q}}(t) \tag{24}$$

and the second one is

$$u_t(0,t) = \frac{1+\hat{q}(t)c}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\omega(0,t)\,, \tag{25}$$

which, along with (19), gives a damper-like boundary condition at $x=0$, but with time-varying damping:

$$w_x(0,t) = \frac{c - \tilde{q}(t)\frac{1+\hat{q}(t)c}{1-\hat{q}^2(t)}}{1-\tilde{q}(t)\frac{\hat{q}(t)+c}{1-\hat{q}^2(t)}}\omega(0,t) \tag{26}$$

$$= \frac{c + \frac{\tilde{q}(t)}{q\hat{q}(t)-1}}{1 + c\frac{\tilde{q}(t)}{q\hat{q}(t)-1}}\omega(0,t) \tag{27}$$

$$= \frac{\hat{q}(t)+c-q(1+\hat{q}(t)c)}{1+\hat{q}(t)c-q(\hat{q}(t)+c)}\omega(0,t)\,. \tag{28}$$

Each of the three forms of the damping coefficient (multiplying $\omega(0,t)$ on the right-hand side) will be useful in the subsequent analysis. We will have to restrict the size of the gain c to prevent the denominator of this time-varying damping coefficient from going through zero as our estimate $\hat{q}(t)$ undergoes possibly broad transients. Note however that none of the three forms (26)–(28) of the damping boundary condition at $x=0$, which are not linear in $\tilde{q}(t)$, will be used for update law design. The form (19), which is linear in $\tilde{q}(t)$, will be used for update law design.

IV. Controller Selection

Our design will employ several Lyapunov functions of increasing complexity. To select the control law, with start with a simple Lyapunov function which represents the total (kinetic and potential) energy of the wave equation:

$$\mathscr{E}(t) = \frac{1}{2}\left(\int_0^1 \omega^2(x,t)dx + \int_0^1 w_x^2(x,t)dx + c_1 w^2(1,t)\right). \tag{29}$$

The derivative of this energy function, after an integration by parts and substitution of (18) and (24), is obtained in the form

$$\begin{aligned}\dot{\mathscr{E}}(t) =& \int_0^1 \left(\omega(x,t)w_{xx}(x,t) - w_x(x,t)\omega_x(x,t)\right)dx \\ &+ \int_0^1 \left(w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t)\right)dx\dot{\hat{q}}(t) \\ &+ c_1 w(1,t)w_t(1,t) \\ =& \int_0^1 \left(w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t)\right)dx\dot{\hat{q}}(t) \\ &+ \omega(1,t)w_x(1,t) - \omega(0,t)w_x(0,t) + c_1 w(1,t)w_t(1,t)\,,\end{aligned} \tag{30}$$

and, after the substitution of (19) and of (17) for $x=1$, it becomes

$$\begin{aligned}\dot{\mathscr{E}}(t) =& \int_0^1 \left(w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t)\right)dx\dot{\hat{q}}(t) \\ &+ c_1 w(1,t)\theta(1,t)\dot{\hat{q}}(t) - c\omega^2(0,t) \\ &+ \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t) \\ &+ \omega(1,t)w_x(1,t) + c_1 w(1,t)\omega(1,t)\,.\end{aligned} \tag{31}$$

The basis of our controller selection is dealing with the cross-terms in the last line of (31). We make a simple choice of Robin boundary condition

$$w_x(1,t) = -c_1 w(1,t)\,. \tag{32}$$

This boundary condition yields the control law

$$\begin{aligned}U(t) =& -\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u_t(1,t) - c_1 u(1,t) \\ &+ c_1\hat{q}(t)\frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u(0,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\int_0^1 u_t(y,t)dy\,.\end{aligned} \tag{33}$$

With this control law, (31) becomes

$$\begin{aligned}\dot{\mathscr{E}}(t) =& \int_0^1 \left(w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t)\right)dx\dot{\hat{q}}(t) \\ &+ c_1 w(1,t)\theta(1,t)\dot{\hat{q}}(t) - c\omega^2(0,t) \\ &+ \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t)\,.\end{aligned} \tag{34}$$

Even in the nonadaptive case, this result is insufficient for proving stability because, when $\tilde{q}(t) \equiv \dot{\hat{q}}(t) \equiv 0$, we only have $\dot{\mathscr{E}}(t) = --c\omega^2(0,t)$, which is only negative semidefinite relative to the Lyapunov function (29). Before we proceed to the design of the update law, we augment the Lyapunov function as follows:

$$E(t) = \mathscr{E}(t) + \delta\int_0^1 (-2+x)\omega(x,t)w_x(x,t)dx\,, \tag{35}$$

where δ is positive. It is easy to see, with the help of Young's inequality, that

$$-2\delta\mathscr{E}(t) \le \delta\int_0^1 (-2+x)\omega(x,t)w_x(x,t)dx \le 2\delta\mathscr{E}(t)\,, \tag{36}$$

which implies that

$$(1-2\delta)\mathscr{E}(t) \le E(t) \le (1+2\delta)\mathscr{E}(t)\,. \tag{37}$$

Hence, E is positive definite for $\delta \in (0,1/2)$.

With another lengthy calculation we obtain

$$\begin{aligned}\dot{E}(t) =& -\delta\int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right)dx \\ &- \frac{\delta}{2}\left(\omega^2(1,t) + w_x^2(1,t)\right) + \delta\left(\omega^2(0,t) + w_x^2(0,t)\right) \\ &- c\omega^2(0,t) + \eta(t)\dot{\hat{q}}(t) \\ &+ \tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t)\,.\end{aligned} \tag{38}$$

where

$$\eta(t) = \int_0^1 (w_x(x,t)\alpha(x,t) + \omega(x,t)\beta(x,t))\,dx + \delta\int_0^1 (-2+x)(w_x(x,t)\beta(x,t) + \omega(x,t)\alpha(x,t))\,dx + c_1 w(1,t)\theta(1,t). \tag{39}$$

Substituting (27) and (32), we obtain

$$\begin{aligned}\dot{E}(t) = &-\delta\int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right)dx \\ &-\frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ &-(c-\delta(1+n(t)))\,\omega^2(0,t) \\ &+\eta(t)\dot{\hat{q}}(t) \\ &+\tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c}u_t(0,t),\end{aligned} \tag{40}$$

where

$$n(t) = \left(\frac{c + \frac{\tilde{q}(t)}{q\hat{q}(t)-1}}{1 + c\frac{\tilde{q}(t)}{q\hat{q}(t)-1}}\right)^2 = \left(\frac{\hat{q}(t) + c - q(1+\hat{q}(t)c)}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\right)^2. \tag{41}$$

Having selected the controller, we are now ready to proceed with the update law design.

V. Update Law Selection

To design the update law, we further augment the Lyapunov function, bringing it into its final, complete form:

$$V(t) = \ln(1+E(t)) + \frac{1}{2\gamma}\tilde{q}^2(t), \tag{42}$$

where $\gamma > 0$. The derivative of the Lyapunov function is

$$\begin{aligned}\dot{V}(t) = &\frac{1}{1+E(t)}\Big\{-\delta\int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right)dx \\ &-\frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ &-(c-\delta(1+n(t)))\,\omega^2(0,t) \\ &+\eta(t)\dot{\hat{q}}(t)\Big\} \\ &+\tilde{q}(t)\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{(1+\hat{q}(t)c)(1+E(t))}u_t(0,t) \\ &-\frac{1}{\gamma}\tilde{q}(t)\dot{\hat{q}}(t),\end{aligned} \tag{43}$$

The update law is chosen to achieve cancellation of the last two terms:

$$\dot{\hat{q}}(t) = \gamma\mathrm{Proj}\left\{\frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{(1+\hat{q}(t)c)(1+E(t))}u_t(0,t)\right\}, \tag{44}$$

where Proj is the standard projection operator

$$\mathrm{Proj}_{[\underline{q},\bar{q}]}\{\tau\} = \tau\begin{cases} 0, & \hat{q} = \underline{q} \text{ and } \tau < 0 \\ 0, & \hat{q} = \bar{q} \text{ and } \tau > 0 \\ 1, & \text{else}\end{cases} \tag{45}$$

and $\hat{q}(0) \in [\underline{q}, \bar{q}]$. For implementation of the update law, we represent $\omega(0,t)$ and $w(1,t)$ as

$$\omega(0,t) = u_t(0,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}u_x(0,t) \tag{46}$$

$$w(1,t) = u(1,t) + \frac{\hat{q}(t)+c}{1+\hat{q}(t)c}\left(-\hat{q}(t)u(0,t) + \int_0^1 u_t(y,t)dy\right). \tag{47}$$

Now, substituting (25) into (44), we get

$$\dot{\hat{q}}(t) = \gamma\frac{\mathrm{Proj}\{\xi(t)\}}{1+E(t)} \tag{48}$$

$$\xi(t) = \frac{(1-c^2)\omega(0,t) + c_1(\hat{q}(t)+c)w(1,t)}{1+\hat{q}(t)c - q(\hat{q}(t)+c)}\omega(0,t) \tag{49}$$

and thus

$$\begin{aligned}\dot{V}(t) \le &\frac{1}{1+E(t)}\Big\{-\delta\int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right)dx \\ &-\frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ &-(c-\delta(1+n(t)))\,\omega^2(0,t) \\ &+\gamma\frac{\eta(t)\mathrm{Proj}\{\xi(t)\}}{1+E(t)}\Big\}.\end{aligned} \tag{50}$$

Due to standard properties of the projection operator, we obtain that

$$\hat{q}(t) \in [\underline{q}, \bar{q}], \quad \forall t \ge 0 \tag{51}$$

and

$$\begin{aligned}\dot{V}(t) \le &\frac{1}{1+E(t)}\Big\{-\delta\int_0^1 \left(\omega^2(x,t) + w_x^2(x,t)\right)dx \\ &-\frac{\delta}{2}\left(\omega^2(1,t) + c_1^2 w^2(1,t)\right) \\ &-(c-\delta(1+n(t)))\,\omega^2(0,t) \\ &+\gamma\frac{|\eta(t)\xi(t)|}{1+E(t)}\Big\}.\end{aligned} \tag{52}$$

VI. Stability Analysis

To complete the Lyapunov calculation, it remains to deal with the last two terms in (52), namely, with the positive effects of $n(t)$ and $|\eta(t)\xi(t)|$, with the aid of the parameters δ and γ, which will be chosen sufficiently small for this purpose. In addition, the following reasonable and non-restrictive assumption is made.

Assumption 1: Non-negative constants $\underline{q}$ and $\bar{q}$ are known such that $q \in [\underline{q}, \bar{q}]$ and either $\bar{q} < 1$ or $\underline{q} > 1$.

This is simply a stabilizability assumption. When $q = 1$, the real part of all the plant eigenvalues is $+\infty$, requiring infinite control gains for stabilization.

Let us denote a constant c^* as

$$c^* = \frac{1}{\bar{q} - \underline{q}}\begin{cases} \underline{q}^2 - 1, & \underline{q} > 1 \\ 1 - \bar{q}^2, & \bar{q} < 1\end{cases} \tag{53}$$

Then, due to (51), choosing c such that $0 < c < c^*$, guarantees that

$$1 + \hat{q}(t)c - q(\hat{q}(t)+c) \neq 0 \tag{54}$$

for all time.

For a given $c \in (0, c^*)$, let us denote

$$n^* = \max_{\hat{q} \in [\underline{q}, \bar{q}]} \left(\frac{\hat{q} + c - q(1 + \hat{q}c)}{1 + \hat{q}c - q(\hat{q} + c)} \right)^2 < \infty . \tag{55}$$

Hence, we get

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & - \frac{\delta}{2} \left(\omega^2(1,t) + c_1^2 w^2(1,t) \right) \\ & - \left(c - \delta \left(1 + n^* \right) \right) \omega^2(0,t) \\ & \left. + \gamma \frac{|\eta(t) \xi(t)|}{1+E(t)} \right\} . \end{aligned} \tag{56}$$

By carefully examining the expressions (39) and (21)–(23), with the help of (51) and the Cauchy-Schwartz inequality, it can be shown that there exists a positive constant m_1 such that $|\eta(t)| \leq m_1 \mathscr{E}(t)$. Hence, with the help of (37), we get

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & - \frac{\delta}{2} \left(\omega^2(1,t) + c_1^2 w^2(1,t) \right) \\ & - \left(c - \delta \left(1 + n^* \right) \right) \omega^2(0,t) \\ & \left. + \gamma \frac{m_1}{1 - 2\delta} |\xi(t)| \right\} . \end{aligned} \tag{57}$$

Finally, with $c \in (0, c^*)$, (51), and Young's inequality, it follows that there exist positive constants m_2 and m_3 such that

$$\frac{m_1}{1 - 2\delta} |\xi(t)| \leq m_2 \omega(0,t)^2 + m_3 w(1,t)^2 . \tag{58}$$

Consequently,

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{1+E(t)} \left\{ -\delta \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & - \frac{\delta}{2} \omega^2(1,t) - \left(\frac{\delta c_1^2}{2} - \gamma m_3 \right) w^2(1,t) \\ & \left. - \left(c - \gamma m_2 - \delta \left(1 + n^* \right) \right) \omega^2(0,t) \right\} . \end{aligned} \tag{59}$$

To finish the Lyapunov analysis, we first chose γ sufficiently small (as a function of δ) to make $\frac{\delta c_1^2}{2} - \gamma m_3 > 0$ and then δ sufficiently small to make $c - \gamma m_2 - \delta \left(1 + n^* \right)$. Indeed, there exist constants

$$\begin{aligned} \delta^*(\underline{q}, \bar{q}, c, c_1) &= \frac{2c}{c_1^2 m_2 + 4(1 + n^*)} \qquad (60) \\ \gamma^*(\underline{q}, \bar{q}, c, c_1) &= \frac{c_1^2 \delta^*}{4} , \qquad (61) \end{aligned}$$

which can be found (conservatively) as explicit functions of their arguments, so that, for any $\delta \in (0, \delta^*)$ and any $\gamma \in (0, \gamma^*)$ there exists a positive constant μ such that

$$\begin{aligned} \dot{V}(t) \leq & - \frac{\mu}{1+E(t)} \left\{ \int_0^1 \left(\omega^2(x,t) + w_x^2(x,t) \right) dx \right. \\ & \left. + \omega^2(1,t) + w^2(1,t) + \omega^2(0,t) \right\} . \end{aligned} \tag{62}$$

From the negative semi-definiteness of (62) it is clear that global stability follows since

$$V(t) \leq V(0) , \quad \forall t \geq 0 , \tag{63}$$

however, we want to derive a specific stability estimate. Towards that end, we introduce the norm

$$\Sigma(t) = \int_0^1 \omega^2(x,t) dx + \int_0^1 w_x^2(x,t) dx + w^2(1,t) \tag{64}$$

and note that

$$\frac{\min\{1, c_1\}}{2} \Sigma(t) \leq \mathscr{E}(t) \leq \frac{\max\{1, c_1\}}{2} \Sigma(t) . \tag{65}$$

From (10)–(16), with the help of (51), it follows that there exist positive constants s_1, s_2 such that

$$s_1 \Sigma(t) \leq \Omega(t) \leq s_2 \Sigma(t) . \tag{66}$$

Hence, from (37), (65), (66) it follows that

$$\frac{2 s_1}{(1 + 2\delta) \max\{1, c_1\}} E(t) \leq \Omega(t) \leq \frac{2 s_2}{(1 - 2\delta) \min\{1, c_1\}} E(t) . \tag{67}$$

From (42) it follows that

$$\begin{aligned} \tilde{q}^2(t) &\leq 2\gamma V(t) \leq 2\gamma \left(e^{V(t)} - 1 \right) \qquad (68) \\ E(t) &\leq e^{V(t)} - 1 . \qquad (69) \end{aligned}$$

The last norm we introduce is

$$\Upsilon(t) = \Omega(t) + \tilde{q}^2(t) . \tag{70}$$

Then, with (67)–(69) and (63) we obtain

$$\Upsilon(t) \leq 2 \left(\frac{s_2}{(1 - 2\delta) \min\{1, c_1\}} + \gamma \right) \left(e^{V(0)} - 1 \right) . \tag{71}$$

Finally, with (42) and (67) we get

$$\begin{aligned} V(0) &\leq E(0) + \frac{1}{2\gamma} \tilde{q}^2(0) \\ &\leq \frac{1}{2} \left(\frac{(1 + 2\delta) \max\{1, c_1\}}{s_1} + \frac{1}{\gamma} \right) \Upsilon(0) . \qquad (72) \end{aligned}$$

Denoting

$$\begin{aligned} R &= 2 \left(\frac{s_2}{(1 - 2\delta) \min\{1, c_1\}} + \gamma \right) \qquad (73) \\ \rho &= \frac{1}{2} \left(\frac{(1 + 2\delta) \max\{1, c_1\}}{s_1} + \frac{1}{\gamma} \right) , \qquad (74) \end{aligned}$$

we obtain a global stability estimate $\Upsilon(t) \leq R \left(e^{\rho \Upsilon(0)} - 1 \right)$.

It still remains to prove a regulation result. Due to the boundedness of $E(t)$, by integrating (62) in time, we get that $\Sigma(t)$ is integrable. Thus, by (66), $\int_0^\infty \Omega(t) dt < \infty$, which means that $\operatorname{ess\,lim}_{t \to \infty} \Omega(t) = 0$. It would be desirable to also have that $\lim_{t \to \infty} \Omega(t) = 0$, however, for this result we need the boundedness of $\omega(0,t)$, which requires that stability analysis also be conducted in the sense of a higher norm, $\int_0^1 \left(\omega_x^2(x,t) + w_{xx}(x,t) \right) dx$. We forego this extra analysis here, recognizing that the issue is associated with the fact that the parametric uncertainty in (2) multiplies a trace term $u_t(0,t)$ which is not a part of the natural norm of the system given by $\Omega(t)$.

In summary, we have proved the following result.

Theorem 1: Consider the closed-loop system consisting of the plant (1)–(3), the control law (33), and the parameter update law (44). Let Assumption 1 hold and pick $c \in (0, c^*)$, where c^* is given by (53). There exist $\gamma^*, \delta^* > 0$ such that for any $\gamma \in (0, \gamma^*)$ and $\delta \in (0, \delta^*)$, the zero solution of the system $(u, v, \hat{q} - q)$ is globally stable in the sense that there exist positive constants R and ρ (independent of the initial conditions) such that for all initial conditions satisfying $(u_0, v_0, \hat{q}_0) \in H_1(0,1) \times L_2(0,1) \times [\underline{q}, \bar{q}]$, the following holds:

$$\Upsilon(t) \leq R\left(e^{\rho\Upsilon(0)} - 1\right), \quad \forall t \geq 0. \tag{75}$$

Furthermore, regulation is achieved in the sense that $\mathrm{ess}\lim_{t\to\infty} \Omega(t) = 0$.

VII. Conclusions

We presented an adaptive feedback law for boundary control of an unstable wave equation with an unmatched parametric uncertainty. The basic idea introduced here for how to approach second-order-in-time PDE problems is potentially usable in other similar PDEs, from variations on the wave equation to beam equations. For example, if we consider the wave equation, but with the anti-damping boundary condition (2) replaced by an anti-stiffness boundary condition given as $u_x(0,t) = -qu(0,t)$, we would introduce the transformation

$$w(x,t) = u(x,t) + (c + \hat{q}(t)) \int_0^x e^{\hat{q}(t)(x-y)} u(y,t) dy \tag{76}$$

$$\omega(x,t) = u_t(x,t) + (c + \hat{q}(t)) \int_0^x e^{\hat{q}(t)(x-y)} u_t(y,t) dy, \tag{77}$$

and then proceed with adaptive control design as in this paper. The update law would be obtained as

$$\dot{\hat{q}}(t) = \gamma \mathrm{Proj}\left\{\frac{\zeta(t) u(0,t)}{1 + E(t)}\right\} \tag{78}$$

$$\zeta(t) = \omega(0,t) + (c + \hat{q}(t)) \int_0^1 e^{\hat{q}(t)x} \omega(x,t) dx \tag{79}$$

$$\begin{aligned} E(t) = &\frac{1}{2}\left(\int_0^1 \omega^2(x,t)dx + \int_0^1 w_x^2(x,t)dx + cw^2(0,t)\right) \\ &+ \delta \int_0^1 (1+x)\omega(x,t) w_x(x,t) dx. \end{aligned} \tag{80}$$

It is important to point out that, at the moment, the area of boundary control of second order hyperbolic PDEs is still quite underdeveloped. Designs for a very general class of partial integro-differential equations (which includes first derivatives in time and space, and a second mixed derivative in space and time, with non-constant coefficients) within that class are yet to be developed, as was done for parabolic PIDEs in [18]. It is only after the non-adaptive problem is properly solved (with explicit gains) that general efforts on adaptive control of second order hyperbolic PDEs can be undertaken.

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