INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING
Int. J. Adapt. Control Signal Process. 2001; **15**:745–766 (DOI: 10.1002/acs.699)

Adaptive control of Burgers' equation with unknown viscosity

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SUMMARY

In this paper, we propose a fortified boundary control law and an adaptation law for Burgers' equation with unknown viscosity, where no *a priori* knowledge of a lower bound on viscosity is needed. This control law is decentralized, i.e., implementable without the need for central computer and wiring. Using the Lyapunov method, we prove that the closed-loop system, including the parameter estimator as a dynamic component, is globally H^1 stable and well posed. Furthermore, we show that the state of the system is regulated to zero by developing an alternative to Barbalat's Lemma which cannot be used in the present situation. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: Burgers' equation; adaptive control; boundary control; Lyapunov design

1. INTRODUCTION

In this paper, we are concerned with the problem of boundary control of Burgers' equation:

$$\begin{aligned} &u_t - \varepsilon u_{xx} + u u_x = 0, && 0 < x < 1,\ t > 0 \\ &u_x(0,t) = \varphi_0, && t > 0 \\ &u_x(1,t) = \varphi_1, && t > 0 \\ &u(x,0) = u^0(x), && 0 < x < 1 \end{aligned} \tag{1}$$

where the viscosity parameter $\varepsilon > 0$ is unknown. In this problem, φ_0 and φ_1 are control inputs and $u^0(x)$ is an initial state in an appropriate function space.

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Contract/grant sponsor: Office of Naval Research
Contract/grant sponsor: Air Force Office of Scientific Research
Contract/grant sponsor: National Science Foundation
Contract/grant sponsor: Killam Trust

Received 2 November 1999
Revised 28 June 2000

Our objective is to find feedback functions $\varphi_0(u|_{x=0})$ and $\varphi_1(u|_{x=1})$ such that the equilibrium $u(x) \equiv 0$ is globally stable and $u(x,t) \to 0$ as $t \to \infty$ for all $x \in [0,1]$. Since ε is unknown and takes an arbitrary positive value, this objective cannot be achieved by static feedback; hence, we need to design an adaptive controller which incorporates a parameter estimator as a dynamic component of the control law.

The problem of control of Burgers' equation has received extensive attention recently [1–8]. In the present paper, we build upon the design

$$u_x(0,t) = \frac{1}{\varepsilon}[u(0,t)^3 + u(0,t)]$$
$$u_x(1,t) = -\frac{1}{\varepsilon}[u(1,t)^3 + u(1,t)] \tag{2}$$

in Reference [7] where global boundary feedback stabilization was achieved for known ε. Although an adaptive controller was proposed in Reference [7], which achieves L^2 stability, this controller suffered from several deficiencies: (i) it requires the knowledge of a lower bound on ε, a condition under which a static robust controller can be designed, (ii) it did not guarantee H^1-stability, and (iii) it did not guarantee well-posedness. In this paper, we propose a fortified control law and a new adaptation law that remove all of these deficiencies. Our approach follows the basic idea in Reference [9, Section 4.5.1]. However, many new issues arise due to the infinite dimensional character of the present problem. A particularly interesting among them (for adaptive control specialists) is the need to develop an alternative (Lemma 3.1 below) to Barbalat's Lemma for proving regulation of the state of the system to zero.

In this design, we make sure that it is *decentralized*, i.e., the control at each end of the domain would use measurements only from that end. To achieve this, we use two different estimates, denoted η_0 and η_1 at respective ends of the domain. The decentralization is motivated by technological considerations. If Burgers' equation is viewed as a miniature version of a fluid flow control problem with microelectromechanical sensors and actuators, the decentralization would allow control implementation without a central computer and wiring, but using only localized processing which can be embedded in the sensing/actuation micro-hardware.

The rest of the paper is organized as follows. We design an adaptive boundary control and present our main results in Section 2. By establishing a Barbalat-like lemma and using the Lyapunov method, we prove our main results in Section 3.

Notation.

We now introduce notation used throughout the paper. $H^s(0,1)$ denotes the usual Sobolev space (see References [10,11]) for any $s \in \mathbb{R}$. For $s \geqslant 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$. We use the following H^1 norm of $H^1(0,1)$:

$$\|u\|_{H^1} = \left(u(0)^2 + \int_0^1 u_x^2\,\mathrm{d}x\right)^{1/2}, \quad u \in H^1(0,1)$$

which is equivalent to the usual one. The norm on $L^2(0, 1)$ is denoted by $\|\cdot\|$. It is easy to see that

$$\|u\|^2 \leqslant 2\|u\|_{H^1}^2. \tag{3}$$

The H^2 norm is defined in the usual way, $\|u\|_{H^2}^2 = \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2$. Let X be a Banach space and $T > 0$. We denote by $C^n([0, T]; X)$ the space of n times continuously differentiable functions defined on $[0, T]$ with values in X, and write $C([0, T]; X)$ for $C^0([0, T]; X)$. In what follows, for simplicity, we omit the indication of the varying range of x and t in equations and we understand that x varies from 0 to 1 and t from 0 to ∞.

2. MAIN RESULT

For notational convenience, in what follows, we denote

$$w_0 = u|_{x=0}, \quad w_1 = u|_{x=1} \tag{4}$$

and

$$\tilde{\eta}_0 = \eta_0 - \frac{1}{6\varepsilon}, \quad \tilde{\eta}_1 = \eta_1 - \frac{1}{6\varepsilon} \tag{5}$$

where η_0 and η_1 shall be used as estimates of $1/6\varepsilon$. Estimating $1/\varepsilon$ rather than ε is the key for eliminating the need for the knowledge of a lower bound on ε.

We follow the Lyapunov approach. To this end, we introduce the energy function

$$E = \int_0^1 u^2 \, \mathrm{d}x \tag{6}$$

and the Lyapunov function

$$V = E + \frac{\varepsilon}{\gamma}(\tilde{\eta}_0^2 + \tilde{\eta}_1^2) \tag{7}$$

where γ is a positive constant. Note that the Lyapunov function V is the energy function E augmented by the estimation error $\tilde{\eta}_0^2 + \tilde{\eta}_1^2$. Let us calculate the time derivative of V. Using Equation (1) and integrating by parts, we obtain

$$\begin{aligned}\dot{V} &= 2\int_0^1 u(\varepsilon u_{xx} - u u_x)\,\mathrm{d}x + \frac{2\varepsilon}{\gamma}\left(\eta_0 - \frac{1}{6\varepsilon}\right)\dot{\eta}_0 + \frac{2\varepsilon}{\gamma}\left(\eta_1 - \frac{1}{6\varepsilon}\right)\dot{\eta}_1 \\ &= 2\varepsilon w_1 \varphi_1 - 2\varepsilon w_0 \varphi_0 - 2\varepsilon \int_0^1 u_x^2\,\mathrm{d}x - \frac{2}{3}(w_1^3 - w_0^3) \\ &\quad + \frac{2\varepsilon}{\gamma}\left(\eta_0 - \frac{1}{6\varepsilon}\right)\dot{\eta}_0 + \frac{2\varepsilon}{\gamma}\left(\eta_1 - \frac{1}{6\varepsilon}\right)\dot{\eta}_1\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\varepsilon w_1\varphi_1 - 2\varepsilon w_0\varphi_0 - 2\varepsilon\int_0^1 u_x^2\,\mathrm{d}x \\
&\quad + \frac{1}{3}(w_0^4 + w_0^2 + w_1^4 + w_1^2) + \frac{2\varepsilon}{\gamma}\left(\eta_0 - \frac{1}{6\varepsilon}\right)\dot{\eta}_0 + \frac{2\varepsilon}{\gamma}\left(\eta_1 - \frac{1}{6\varepsilon}\right)\dot{\eta}_1 \\
&= 2\varepsilon w_1\varphi_1 - 2\varepsilon w_0\varphi_0 - 2\varepsilon\int_0^1 u_x^2\,\mathrm{d}x \\
&\quad + 2\varepsilon\left(\frac{1}{6\varepsilon} - \eta_0 + \eta_0\right)(w_0^4 + w_0^2) + 2\varepsilon\left(\frac{1}{6\varepsilon} - \eta_1 + \eta_1\right)(w_1^4 + w_1^2) \\
&\quad + \frac{2\varepsilon}{\gamma}\left(\eta_0 - \frac{1}{6\varepsilon}\right)\dot{\eta}_0 + \frac{2\varepsilon}{\gamma}\left(\eta_1 - \frac{1}{6\varepsilon}\right)\dot{\eta}_1 \\
&= -2\varepsilon\int_0^1 u_x^2\,\mathrm{d}x + 2\varepsilon w_1[\varphi_1 + \eta_1(w_1^3 + w_1)] - 2\varepsilon w_0[\varphi_0 - \eta_0(w_0^3 + w_0)] \\
&\quad + 2\varepsilon\left(\frac{1}{6\varepsilon} - \eta_0\right)\left(w_0^4 + w_0^2 - \frac{\dot{\eta}_0}{\gamma}\right) + 2\varepsilon\left(\frac{1}{6\varepsilon} - \eta_1\right)\left(w_1^4 + w_1^2 - \frac{\dot{\eta}_1}{\gamma}\right) \qquad (8)
\end{aligned}
$$

This leads us to select the adaptive feedback control

$$\dot{\eta}_0 = \gamma(w_0^2 + w_0^4) \qquad (9)$$

$$\dot{\eta}_1 = \gamma(w_1^2 + w_1^4) \qquad (10)$$

$$\varphi_0 = k(w_0 + w_0^7) + \eta_0(w_0 + w_0^3) \qquad (11)$$

$$\varphi_1 = -k(w_1 + w_1^7) - \eta_1(w_1 + w_1^3) \qquad (12)$$

where k is any positive constant, and the reasons for including w_0^7 and w_1^7 will become apparent in the H^1 analysis. With this control, we obtain

$$\dot{V} \leqslant -2\varepsilon k(w_0^2 + w_0^8 + w_1^2 + w_1^8) - 2\varepsilon\int_0^1 u_x^2\,\mathrm{d}x \qquad (13)$$

which implies the L^2 stability. The L^2 stability is not sufficient because it does not guarantee boundedness of solutions (the L^2 energy can be bounded even though the solution $u(x)$ can take infinite values on a measure zero subset of $[0, 1]$). For this reason we also pursue H^1 stability, which implies boundedness. The closed-loop system

$$
\begin{aligned}
& u_t - \varepsilon u_{xx} + u u_x = 0 \\
& u_x|_{x=0} = k(w_0 + w_0^7) + \eta_0(w_0 + w_0^3) \\
& u_x|_{x=1} = -k(w_1 + w_1^7) - \eta_1(w_1 + w_1^3) \\
& \dot{\eta}_0 = \gamma(w_0^2 + w_0^4) \\
& \dot{\eta}_1 = \gamma(w_1^2 + w_1^4) \\
& u(x,0) = u^0(x), \quad \eta_0(0) = \eta_0^0, \quad \eta_1(0) = \eta_1^0
\end{aligned}
\tag{14}
$$

satisfies the following theorem.

Theorem 2.1

Suppose that $k > 0$ and $\gamma > 0$ and the initial conditions $u^0 \in H^2(0, 1)$ and $\eta_0^0 \geqslant 0$, $\eta_1^0 \geqslant 0$. If the problem (14) has a global classical solution (u, η_0, η_1), then we have

(1) the equilibrium $u(x) \equiv 0$, $\tilde{\eta}_0 = \tilde{\eta}_1 = 0$ is globally L^2-stable, i.e.

$$
\|u(t)\|^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_0(t)^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_1(t)^2 \leqslant \|u^0\|^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_0(0)^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_1(0)^2 \tag{15}
$$

for all $t \geqslant 0$, and u is regulated to zero in L^2 sense:

$$
\lim_{t\to\infty} \|u(t)\| = 0 \tag{16}
$$

(2) the equilibrium $u(x) \equiv 0$, $\tilde{\eta}_0 = \tilde{\eta}_1 = 0$ is globally H^1-stable, i.e.

$$
\begin{aligned}
\|u(t)\|_{H^1}^2 + \tilde{\eta}_0(t)^2 + \tilde{\eta}_1(t)^2 \leqslant C[\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^8 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2] \\
\times \exp(C(\|u^0\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2))
\end{aligned}
\tag{17}
$$

for all $t \geqslant 0$, where $C = C(k, \varepsilon, \gamma)$ is a positive constant, and u is regulated to zero for all $x \in [0, 1]$:

$$
\lim_{t\to\infty} \max_{x\in[0,1]} |u(x,t)| = \lim_{t\to\infty} \|u(t)\|_{H^s} = 0 \tag{18}
$$

for any $s < 2$.

Remark 2.1

Both Parts 1 and 2 of the theorem claim global stability for entire closed-loop system $(u, \tilde{\eta}_0, \tilde{\eta}_1)$ but claim regulation only for u. This is standard in adaptive control where the parameter estimate convergence to the true value is not a prerequisite for regulation (additional 'persistence of excitation' conditions are needed to make $\tilde{\eta}_0$ and $\tilde{\eta}_1$ go to zero).

Remark 2.2

(1) If $\eta_0^0 < 0$ and $\eta_1^0 < 0$, Part 1 of Theorem 2.1 remains valid but the problem of existence of solutions of (14) is open. We stress here that (16) holds even though for $\eta_0^0, \eta_1^0 < 0$ we cannot establish the stronger result (18). To prove (16), using the arguments standard in finite dimensional adaptive controls (based on Barbalat's Lemma), we would need to show that $\|u\|$ is uniformly continuous, which is not possible for η_0^0, $\eta_1^0 < 0$. For this reason, we develop Lemma 3.1, an alternative to Barbalat's Lemma, which requires that $\|u\|$ has an upper-bounded time derivative. Neither of the lemmas is implied by the other one.
(2) If $\gamma \geqslant 0$, $\eta_0^0 > 1/6\varepsilon$, $\eta_1^0 > 1/6\varepsilon$ and $k \geqslant 0$, the whole Theorem 2.1 is still valid. Moreover, for $\gamma = 0$, the system (14) is globally exponentially stable. For details, we refer to References [1,7].
(3) For $\gamma = 0$ and for any η_0^0, $\eta_1^0 \in \mathbb{R}$, we have global ultimate L^2-boundedness

$$\|u(t)\|^2 \leqslant \|u^0\|^2 e^{-\sigma t} + \frac{2}{9\varepsilon k \sigma}, \quad \forall t \geqslant 0 \tag{19}$$

where

$$\sigma = \min\{\varepsilon, \varepsilon k/4\} \tag{20}$$

To prove this, we start with

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u^2\,dx &= 2\int_0^1 u(\varepsilon u_{xx} - u u_x)\,dx \\
&= -2\varepsilon k(w_0^2 + w_0^8 + w_1^2 + w_1^8) \\
&\quad - 2\varepsilon[\eta_0(w_0^2 + w_0^4) + \eta_1(w_1^2 + w_1^4)] \\
&\quad - 2\varepsilon\int_0^1 u_x^2\,dx - \frac{2}{3}(w_1^3 - w_0^3) \\
&\text{(use Young's inequality and note that } a^6 \leqslant a^8 + a^2\text{)} \\
&\leqslant -\varepsilon k(w_0^2 + w_0^8 + w_1^2 + w_1^8) \\
&\quad - 2\varepsilon[\eta_0(w_0^2 + w_0^4) + \eta_1(w_1^2 + w_1^4)] \\
&\quad - 2\varepsilon\int_0^1 u_x^2\,dx + \frac{2}{9\varepsilon k} \\
&\text{(use (3))} \\
&\leqslant -\frac{\varepsilon k}{2}(w_0^2 + w_0^8 + w_1^2 + w_1^8) \\
&\quad - \sigma\int_0^1 u^2\,dx + \frac{2}{9\varepsilon k}
\end{aligned} \tag{21}$$

It therefore follows that

$$\|u(t)\|^2 \leqslant \|u^0\|^2 \mathrm{e}^{-\sigma t} + \frac{2}{9\varepsilon k\sigma}(1 - \mathrm{e}^{-\sigma t}), \quad \forall t \geqslant 0 \tag{22}$$

which proves (19).

Remark 2.3

Noting that for each non-negative value of η_0 and η_1 the right-hand sides of (11) and (12) are invertible functions, the feedback law can be written as

$$w_0 = h(u_x|_{x=0}, \eta_0) \tag{23}$$

$$w_1 = h(-u_x|_{x=1}, \eta_1) \tag{24}$$

where h is smooth in its first argument for each non-negative value of the second argument. Noting that $\eta_0^0, \eta_1^0 \geqslant 0$ ensures that η_0 and η_1 remain non-negative, the feedback law can be implemented by measuring u_x and actuating u at the boundary. By substituting (23) and (24) into (9) and (10), one can also view the update laws as dependent upon $u_x|_{x=0,1}$.

Remark 2.4

Since there is only one parameter ε in the Burgers equation, it is plausible to introduce only one parameter update law η. Indeed, we can design such an update law as it was already done in Krstic [7], where a lower bound on ε was also required. To this end, we introduce the Lyapunov function

$$V = E + \frac{\varepsilon}{\gamma}\left(\eta - \frac{1}{6\varepsilon}\right)^2 \tag{25}$$

where γ is a positive constant. By a straightforward calculation, we obtain

$$\begin{aligned}
\dot{V} &= 2\int_0^1 u(\varepsilon u_{xx} - u u_x)\,\mathrm{d}x + \frac{2\varepsilon}{\gamma}\left(\eta - \frac{1}{6\varepsilon}\right)\dot{\eta} \\
&= 2\varepsilon w_1 \varphi_1 - 2\varepsilon w_0 \varphi_0 - 2\varepsilon \int_0^1 u_x^2\,\mathrm{d}x - \frac{2}{3}(w_1^3 - w_0^3) + \frac{2\varepsilon}{\gamma}\left(\eta - \frac{1}{6\varepsilon}\right)\dot{\eta} \\
&\leqslant 2\varepsilon w_1 \varphi_1 - 2\varepsilon w_0 \varphi_0 - 2\varepsilon \int_0^1 u_x^2\,\mathrm{d}x \\
&\quad + \frac{1}{3}(w_0^4 + w_0^2 + w_1^4 + w_1^2) + \frac{2\varepsilon}{\gamma}\left(\eta - \frac{1}{6\varepsilon}\right)\dot{\eta}
\end{aligned}$$

$$= 2\varepsilon w_1 \varphi_1 - 2\varepsilon w_0 \varphi_0 - 2\varepsilon \int_0^1 u_x^2 \, \mathrm{d}x$$

$$+ 2\varepsilon \left(\frac{1}{6\varepsilon} - \eta + \eta \right) (w_0^4 + w_0^2 + w_1^4 + w_1^2) + \frac{2\varepsilon}{\gamma} \left(\eta - \frac{1}{6\varepsilon} \right) \dot{\eta}$$

$$= -2\varepsilon \int_0^1 u_x^2 \, \mathrm{d}x + 2\varepsilon w_1 [\varphi_1 + \eta(w_1^3 + w_1)] - 2\varepsilon w_0 [\varphi_0 - \eta(w_0^3 + w_0)]$$

$$+ 2\varepsilon \left(\frac{1}{6\varepsilon} - \eta \right) \left(w_0^4 + w_0^2 + w_1^4 + w_1^2 - \frac{\dot{\eta}}{\gamma} \right) \tag{26}$$

Taking

$$\dot{\eta} = \gamma (w_0^4 + w_0^2 + w_1^4 + w_1^2) \tag{27}$$

$$\varphi_0 = \eta(w_0^3 + w_0) + (1 + \dot{\eta})(2w_0^3 + w_0) \tag{28}$$

$$\varphi_1 = -\eta(w_1^3 + w_1) - (1 + \dot{\eta})(2w_1^3 + w_1) \tag{29}$$

we obtain

$$\dot{V} \leqslant -2\varepsilon(1 + \dot{\eta})(2w_0^4 + w_0^2 + 2w_1^4 + w_1^2) - 2\varepsilon \int_0^1 u_x^2 \, \mathrm{d}x \tag{30}$$

which implies the L^2 stability. The term $\dot{\eta}$ in (28) and (29) plays a role in establishing H^1 stability and the full result as in Theorem 2.1. However, the control law (27)–(29) is less desirable than (9)–(12) since it requires exchange of information between the two ends, i.e. it is not *decentralized*.

Remark 2.5

The problem of existence and uniqueness of a *classical* solution of problem (14) remains open. However, in Section 4, we shall show that it has a unique *weak* solution in the sense defined there.

3. PROOF OF STABILITY

In this section, we prove our main result by using the Lyapunov method. To prove the regulation result (16) independent of (18) as we explain in Remark 2.2 (Part 1), we establish the following alternative to Barbalat's lemma (see, e.g. Reference [9, Lemma A.6, p. 491]).

Lemma 3.1

Suppose that the function $f(t)$ defined on $[0, \infty)$ satisfies the following conditions:

(i) $f(t) \geqslant 0$ for all $t \in [0, \infty)$,
(ii) $f(t)$ is differentiable on $[0, \infty)$ and there exists a constant M such that

$$f'(t) \leqslant M, \quad \forall t \geqslant 0 \tag{31}$$

(iii) $\int_0^\infty f(t)\,\mathrm{d}t < \infty$.

Then we have

$$\lim_{t \to \infty} f(t) = 0 \tag{32}$$

Remark 3.1

It is important to see that this lemma and the standard Barbalat's lemma do not imply each other. While Barbalat's lemma assumes that $f(t)$ is uniformly continuous, Lemma 3.1 assumes that $f'(t)$ is bounded, but only from above.

We are now in the position to prove our main result.

Proof of Theorem 2.1. *Step 1:* Stability Estimate (15). By (13), we obtain

$$\|u(t)\|^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_0(t)^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_1(t)^2 \leqslant \|u^0\|^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_0(0)^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_1(0)^2, \quad t \geqslant 0 \tag{33}$$

and

$$\begin{aligned} 2\varepsilon \int_0^\infty \int_0^1 u_x^2\,\mathrm{d}x\,\mathrm{d}t + 2\varepsilon k \int_0^\infty [w_0(t)^2 + w_0(t)^8 + w_1(t)^2 + w_1(t)^8]\,\mathrm{d}t \\ \leqslant \|u^0\|^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_0(0)^2 + \frac{\varepsilon}{\gamma}\tilde{\eta}_1(0)^2 \end{aligned} \tag{34}$$

Hence (15) is established.

Step 2: Regulation (16). To prove (16), it suffices to verify conditions (ii) and (iii) of Lemma 3.1. By (3) and (34), we obtain

$$\int_0^\infty \|u(t)\|^2\,\mathrm{d}t \leqslant C(\varepsilon, \gamma)(\|u^0\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2) \tag{35}$$

Here and in the sequel, $C = C(\varepsilon, \gamma, k)$ denotes a generic positive constant depending on ε, γ, k, which may vary from line to line. Thus condition (iii) of Lemma 3.1 is fulfilled. On the other hand, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2) &= 2\int_0^1 u(\varepsilon u_{xx} - uu_x)\,\mathrm{d}x \\ &= 2\varepsilon w_1 u_x(1) - 2\varepsilon w_0 u_x(0) - 2\varepsilon \int_0^1 u_x^2\,\mathrm{d}x - \frac{2}{3}(w_1^3 - w_0^3) \\ &\leqslant -2\varepsilon k(w_0^8 + w_0^2 + w_1^8 + w_1^2) - 2\varepsilon[\eta_0(w_0^4 + w_0^2) + \eta_1(w_1^4 + w_1^2)] \\ &\quad + \tfrac{1}{3}(w_0^4 + w_0^2 + w_1^4 + w_1^2) \\ &\text{(use Young's inequality and note that } a^4 \leqslant a^8 + a^2) \end{aligned}$$

$$\begin{aligned}
&\leqslant -2\varepsilon k(w_0^8+w_0^2+w_1^8+w_1^2)\\
&\quad +\varepsilon k(w_0^8+w_0^2+w_1^8+w_1^2)+C(\varepsilon,k)(\eta_0^2+\eta_1^2)\\
&\quad +\varepsilon k(w_0^8+w_0^2+w_1^8+w_1^2)+C(\varepsilon,k)\\
&\leqslant C(\varepsilon,k)(1+\eta_0^2+\eta_1^2)
\end{aligned}\tag{36}$$

which, combining with (33), implies condition (ii) of Lemma 3.1.

Step 3: Stability Estimate (17). Using Equation (14) and integrating by parts, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_x^2\,\mathrm{d}x &= 2\int_0^1 u_x u_{xt}\,\mathrm{d}x\\
&= -2w_{1t}[k(w_1+w_1^7)+\eta_1(w_1^3+w_1)]\\
&\quad -2w_{0t}[k(w_0+w_0^7)+\eta_0(w_0^3+w_0)]\\
&\quad -2\int_0^1 u_{xx}(\varepsilon u_{xx}-uu_x)\,\mathrm{d}x\\
&= -k\frac{\mathrm{d}}{\mathrm{d}t}\left(w_0^2+\frac{1}{4}w_0^8+w_1^2+\frac{1}{4}w_1^8\right)\\
&\quad -\eta_0\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}w_0^4+w_0^2\right)-\eta_1\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}w_1^4+w_1^2\right)\\
&\quad -2\int_0^1 u_{xx}(\varepsilon u_{xx}-uu_x)\,\mathrm{d}x\\
&= -k\frac{\mathrm{d}}{\mathrm{d}t}\left(w_0^2+\frac{1}{4}w_0^8+w_1^2+\frac{1}{4}w_1^8\right)\\
&\quad -\frac{\mathrm{d}}{\mathrm{d}t}\left[\eta_0\left(\frac{1}{2}w_0^4+w_0^2\right)+\eta_1\left(\frac{1}{2}w_1^4+w_1^2\right)\right]\\
&\quad +\dot\eta_0\left(\frac{1}{2}w_0^4+w_0^2\right)+\dot\eta_1\left(\frac{1}{2}w_1^4+w_1^2\right)\\
&\quad -2\int_0^1 u_{xx}(\varepsilon u_{xx}-uu_x)\,\mathrm{d}x
\end{aligned}\tag{37}$$

Since

$$|u(x,t)|=\left|w_0(t)+\int_0^x u_\xi(\xi,t)\,\mathrm{d}\xi\right|\leqslant|w_0(t)|+\|u_x(t)\|\tag{38}$$

it follows from Young's inequality that

$$-2\int_0^1 u_{xx}(\varepsilon u_{xx} - uu_x)\,\mathrm{d}x \leqslant \frac{1}{\varepsilon}\int_0^1 u^2 u_x^2\,\mathrm{d}x$$
$$\leqslant \frac{2}{\varepsilon}\left(w_0^2 + \int_0^1 u_x^2\,\mathrm{d}x\right)\int_0^1 u_x^2\,\mathrm{d}x \tag{39}$$

Moreover, we have

$$\begin{aligned}
&\dot{\eta}_0\left(\frac{1}{2}w_0^4 + w_0^2\right) + \dot{\eta}_1\left(\frac{1}{2}w_1^4 + w_1^2\right)\\
&= \gamma(w_0^4 + w_0^2)\left(\frac{1}{2}w_0^4 + w_0^2\right) + \gamma(w_1^4 + w_1^2)\left(\frac{1}{2}w_1^4 + w_1^2\right)\\
&\leqslant \gamma(w_0^4 + w_0^2)^2 + \gamma(w_1^4 + w_1^2)^2\\
&\leqslant 2\gamma(w_0^4 + w_0^8 + w_1^4 + w_1^8)\\
&\quad \text{(note that } a^4 \leqslant a^2 + a^8\text{)}\\
&\leqslant 4\gamma(w_0^2 + w_0^8 + w_1^2 + w_1^8)
\end{aligned} \tag{40}$$

It therefore follows from (37) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_x^2\,\mathrm{d}x \leqslant& -k\frac{\mathrm{d}}{\mathrm{d}t}\left(w_0^2 + \frac{1}{4}w_0^8 + w_1^2 + \frac{1}{4}w_1^8\right)\\
&- \frac{\mathrm{d}}{\mathrm{d}t}\left(\eta_0\left(\frac{1}{2}w_0^4 + w_0^2\right) + \eta_1\left(\frac{1}{2}w_1^4 + w_1^2\right)\right)\\
&+ 4\gamma(w_0^2 + w_0^8 + w_1^2 + w_1^8)\\
&+ \frac{2}{\varepsilon}\left(w_0^2 + \int_0^1 u_x^2\,\mathrm{d}x\right)\int_0^1 u_x^2\,\mathrm{d}x
\end{aligned} \tag{41}$$

Denote

$$W = k(w_0^2 + \tfrac{1}{4}w_0^8 + w_1^2 + \tfrac{1}{4}w_1^8)$$

Integrating with respect to t, we deduce from (34) that

$$\|u_x(t)\|^2 + W(t) + \eta_0\left(\frac{1}{2}w_0(t)^4 + w_0(t)^2\right) + \eta_1\left(\frac{1}{2}w_1(t)^4 + w_1(t)^2\right)$$

$$\leqslant W(0) + \eta_0(0)\left(\frac{1}{2}u(0,0)^4 + u(0,0)^2\right) + \eta_1(0)\left(\frac{1}{2}u(1,0)^4 + u(1,0)^2\right) + \|u_x^0\|^2$$

$$+ 4\gamma\int_0^\infty (w_0(s)^2 + w_0(s)^8 + w_1(s)^2 + w_1(s)^8)\,\mathrm{d}s$$

$$+ \frac{2}{\varepsilon}\int_0^t (w_0(s)^2 + \|u_x(s)\|^2)\|u_x(s)\|^2\,\mathrm{d}s$$

$$\left(\text{note that } \eta_i(0) = \tilde{\eta}_i(0) + \frac{1}{6\varepsilon}\right)$$

$$\leqslant C(k,\varepsilon,\gamma)(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^8 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)$$

$$+ C(k,\varepsilon,\gamma)(\|u^0\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2) + \frac{2}{\varepsilon}\int_0^t (w_0(s)^2 + \|u_x(s)\|^2)\|u_x(s)\|^2\,\mathrm{d}s$$

$$= C(k,\varepsilon,\gamma)(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^8 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)$$

$$+ \frac{2}{\varepsilon}\int_0^t (w_0(s)^2 + \|u_x(s)\|^2)\|u_x(s)\|^2\,\mathrm{d}s \tag{42}$$

Since we have assumed that $\eta_0^0, \eta_1^0 \geqslant 0$, we have $\eta_0, \eta_1 \geqslant 0$ and then

$$\|u_x(t)\|^2 + W(t) \leqslant C(k,\varepsilon,\gamma)(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^8 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)$$
$$+ \frac{2}{\varepsilon}\int_0^t (w_0(s)^2 + \|u_x(s)\|^2)(\|u_x(s)\|^2 + W(s))\,\mathrm{d}s \tag{43}$$

By (34) and Gronwall's inequality (see e.g. Reference [12, p. 63]), we deduce that for $t \geqslant 0$

$$\|u_x(t)\|^2 + W(t) \leqslant C(k,\varepsilon,\gamma)(\|u^0\|_{H^1}^2 + \|u^0\|_{H^1}^8 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)$$
$$\times \exp(C(k,\varepsilon,\gamma)(\|u^0\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)) \tag{44}$$

This shows that (17) holds.

Step 4: Regulation (18). To prove (18), we first estimate $\|u\|_{H^2}$. Integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\,\mathrm{d}x = 2\int_0^1 u_t(\varepsilon u_{xxt} - u_t u_x - u u_{xt})\,\mathrm{d}x$$
$$= -2\varepsilon k w_{1t}(7w_{1t}w_1^6 + w_{1t})$$
$$- 2\varepsilon w_{1t}(\dot{\eta}_1(w_1^3 + w_1) + \eta_1(3w_{1t}w_1^2 + w_{1t}))$$
$$- 2\varepsilon k w_{0t}(7w_{0t}w_0^6 + w_{0t})$$
$$- 2\varepsilon w_{0t}(\dot{\eta}_0(w_0^3 + w_0) + \eta_0(3w_{0t}w_0^2 + w_{0t}))$$
$$- 2\varepsilon\int_0^1 u_{xt}^2\,\mathrm{d}x - 2\int_0^1 u_t(u_t u_x + u u_{xt})\,\mathrm{d}x \tag{45}$$

Since

$$2\varepsilon w_{1t}\eta_1(3w_{1t}w_1^2 + w_{1t}) \geqslant 0$$
$$2\varepsilon w_{0t}\eta_0(3w_{0t}w_0^2 + w_{0t}) \geqslant 0$$

it therefore follows from (45) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\,\mathrm{d}x \leqslant & -2\varepsilon k(w_{1t}^2(7w_1^6+1) + w_{0t}^2(7w_0^6+1)) \\ & - 2\varepsilon w_{1t}\dot{\eta}_1(w_1^3 + w_1) - 2\varepsilon w_{0t}\dot{\eta}_0(w_0^3 + w_0) \\ & - 2\varepsilon\int_0^1 u_{xt}^2\,\mathrm{d}x - 2\int_0^1 u_t(u_t u_x + u u_{xt})\,\mathrm{d}x \end{aligned} \tag{46}$$

Furthermore, since

$$\begin{aligned} 2\dot{\eta}_i w_{it}(w_i^3 + w_i) &\leqslant \dot{\eta}_i\left(\frac{k}{2\gamma}w_{it}^2(w_i^2+1) + \frac{2\gamma}{k}(w_i^4 + w_i^2)\right) \\ &= \frac{k}{2}w_{it}^2 w_i^2(w_i^2+1)^2 + \frac{2\gamma^2}{k}(w_i^4+w_i^2)^2 \\ &\leqslant k w_{it}^2(w_i^6 + w_i^2) + \frac{4\gamma^2}{k}(w_i^8 + w_i^4) \\ &\qquad (\text{note that } a^2 \leqslant a^6 + 1 \text{ and } a^4 \leqslant a^8 + a^2) \\ &\leqslant 2k w_{it}^2(w_i^6+1) + \frac{8\gamma^2}{k}(w_i^8 + w_i^2), \quad i = 0, 1 \end{aligned} \tag{47}$$

and by (38), we have

$$\begin{aligned} \left|2\int_0^1 u_t(u_t u_x + u u_{xt})\,\mathrm{d}x\right| &\leqslant 2(|w_{0t}| + \|u_{xt}\|)\|u_t\|\,\|u_x\| + 2(|w_0| + \|u_x\|)\|u_t\|\,\|u_{xt}\| \\ &\leqslant \varepsilon k|w_{0t}|^2 + \varepsilon\|u_{xt}\|^2 + C(\varepsilon, k)\|u_t\|^2\|u_x\|^2 \\ &\quad + \frac{1}{\varepsilon}(|w_0|^2 + \|u_x\|^2)\|u_t\|^2 + \varepsilon\|u_{xt}\|^2 \end{aligned} \tag{48}$$

it follows from (46) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\,\mathrm{d}x \leqslant \frac{8\gamma^2\varepsilon}{k}(w_0^8 + w_0^2 + w_1^8 + w_1^2) + C(\varepsilon, k)(w_0^2 + \|u_x\|^2)\|u_t\|^2 \tag{49}$$

We then deduce using (34) that

$$\|u_t(t)\|^2 \leqslant C(k, \varepsilon, \gamma)(\|u^0\|^2 + \tilde{\eta}_0(0)^2) + \tilde{\eta}_1(0)^2) + \|u_t(0)\|^2$$
$$+ C(\varepsilon, k) \int_0^t (w_0(s)^2 + \|u_x(s)\|^2)\|u_t(s)\|^2 \, \mathrm{d}s \tag{50}$$

which, by (34) and Gronwall's inequality (see e.g. Reference [12, p. 63]), implies that for $t \geqslant 0$

$$\|u_t(t)\|^2 \leqslant C(k, \varepsilon, \gamma)(\tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2 + \|u^0\|_{H^2}^2 + \|u^0\|_{H^1}^4)$$
$$\times \exp(C(k, \varepsilon, \gamma)(\|u^0\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2)) \tag{51}$$

Further, since

$$\|u_{xx}\|^2 \leqslant C(\varepsilon)(\|u_t\|^2 + \|u_x\|^4) \tag{52}$$

it follows from (33), (44) and (51) that

$$\|u(t)\|_{H^2}^2 \leqslant C(\tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2 + \tilde{\eta}_0(0)^4 + \tilde{\eta}_1(0)^4 + \|u^0\|_{H^2}^2 + \|u^0\|_{H^2}^{16})$$
$$\times \exp(C(\tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2 + \|u^0\|^2)), \quad \forall t \geqslant 0 \tag{53}$$

In order to prove (18), we argue by contradiction. Suppose that (18) is not true. Then there exists a positive constant $\delta_0 > 0$ and $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\|u(t_n)\|_{H^s} \geqslant \delta_0, \quad n = 1, 2, \ldots \tag{54}$$

On the other hand, it follows from (53) and the compact imbedding theorem (see, e.g. Reference [11, vol. 1, p. 99]) that there exists a subsequence $\{u(x, t_{n_i})\}$ such that $u(x, t_{n_i})$ converges to a function $w(x)$ in $H^s(0, 1)$ and, of course, also in $L^2(0, 1)$ as $i \to \infty$. Since we have proved that $u(x, t_{n_i})$ converges to 0 in $L^2(0, 1)$ (recall (16)), we have $w \equiv 0$, which is in contradiction with (54). □

4. ANALYSIS OF EXISTENCE AND UNIQUENESS

In this section we analyse the existence and uniqueness of a solution of problem (14). By (14), we first have

$$\eta_0 = \eta_0^0 + \gamma \int_0^t [w_0(s)^4 + w_0(s)^2] \, \mathrm{d}s \tag{55}$$

$$\eta_1 = \eta_1^0 + \gamma \int_0^t [w_1(s)^4 + w_1(s)^2] \, \mathrm{d}s \tag{56}$$

Substituting η_0, η_1 into the boundary condition of (14), the problem (14) becomes a standard Neumann boundary value problem

$$\begin{aligned} &u_t - \varepsilon u_{xx} + u u_x = 0 \\ &u_x(0,t) = g(w_0, \eta_0^0)(t) \\ &u_x(1,t) = -\, g(w_1, \eta_1^0)(t) \\ &u(x,0) = u^0(x) \end{aligned} \tag{57}$$

where $g(w, r)(t)$ is defined by

$$g(w,r)(t) = k[w(t) + w(t)^7] + \left(r + \gamma \int_0^t [w(s)^4 + w(s)^2]\,\mathrm{d}s\right)[w(t)^3 + w(t)] \tag{58}$$

for any function $w = w(t)$ and $r \in \mathbb{R}$. Once we solve problem (57), we obtain η_0, η_1 through (55) and (56). Therefore, it suffices to prove that problem (57) has a unique solution. It is well-known (see, e.g. Reference [13, Theorem 19.3.5, p. 339]) that the problem (57) is equivalent to the following integral equation

$$\begin{aligned} u(x,t) &= G(u(x,t)) \\ &= \int_0^1 [\theta(x-y,t) + \theta(x+y,t)]u^0(y)\,\mathrm{d}y \\ &\quad - \int_0^t \int_0^1 [\theta(x-y,t-\tau) + \theta(x+y,t-\tau)]u(y,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad - 2\int_0^t \theta(x-1,t-\tau) g(w_1,\eta_1^0)(\tau)\,\mathrm{d}\tau \\ &\quad - 2\int_0^t \theta(x,t-\tau) g(w_0,\eta_0^0)(\tau)\,\mathrm{d}\tau \\ &= \int_0^1 [\theta(x-y,t) + \theta(x+y,t)]u^0(y)\,\mathrm{d}y \\ &\quad + \frac{1}{2}\int_0^t \int_0^1 [\theta_x(x+y,t-\tau) - \theta_x(x-y,t-\tau)]u(y,\tau)u(y,\tau)^2\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad - \int_0^t \theta(x-1,t-\tau)[2g(w_1,\eta_1^0)(\tau) + w_1(\tau)^2]\,\mathrm{d}\tau \\ &\quad - \int_0^t \theta(x,t-\tau)[2g(w_0,\eta_0^0)(\tau) - w_0(\tau)^2]\,\mathrm{d}\tau \end{aligned} \tag{59}$$

where the theta function θ is defined by (see, e.g. References [14, p. 86; 13, p. 59])

$$\theta(x, t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(x+2n)^2}{4\varepsilon t}\right), \quad t > 0 \tag{60}$$

By Theorem 4.1 of Reference [14, p. 90], the function θ can be expressed as

$$\theta(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \cos(n\pi x) \mathrm{e}^{-\varepsilon n^2 \pi^2 t}$$

By definition, we call the solution of integral equation (59) as a weak solution of problem (14).

We now show that if T is sufficiently small then (59) has a unique solution in the following function space:

$$\mathscr{C} = C([0, T]; C[0, 1])$$

with the norm

$$\|u\|_{\mathscr{C}} = \max_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant t \leqslant T}} |u(x, t)|$$

To prove this, it suffices to prove that the mapping G defined by (59) has a unique fixed point in $\mathscr{C}$. We employ the Banach fixed point theorem to prove this. In what follows, we denote by $C = C(\varepsilon, \gamma, k, T)$ a generic positive constant depending on $\varepsilon, \gamma, k, T$, which is a non-decreasing function of T and may vary from line to line. Using (A2) and (A3) from Lemma A.1 below, we obtain (note that the following $x - \xi$ and $x + \xi$ vary from -1 to 1 and 0 to 2, respectively)

$$\begin{aligned}
&\left| \int_0^1 [\theta(x-\xi, t) + \theta(x+\xi, t)] u^0(\xi) \, \mathrm{d}\xi \right| \\
&\quad \leqslant C \max_{0 \leqslant x \leqslant 1} |u^0(x)| \int_0^1 \frac{1}{\sqrt{t}} \left[\exp\left(-\frac{(x-\xi)^2}{4\varepsilon t}\right) \right. \\
&\qquad \left. + \exp\left(-\frac{(x+\xi)^2}{4\varepsilon t}\right) + \exp\left(-\frac{(x+\xi-2)^2}{4\varepsilon t}\right) \right] \mathrm{d}\xi \\
&\quad \leqslant C \|u^0\|_{H^1} \int_{-\infty}^{\infty} \mathrm{e}^{-\xi^2} \, \mathrm{d}\xi \\
&\quad \leqslant C \|u^0\|_{H^1}
\end{aligned} \tag{61}$$

Using (A4) and (A5) from Lemma A.1 below, we obtain

$$\begin{aligned}
&\left|\int_0^t\int_0^1 [\theta_x(x+\xi,t-\tau)-\theta_x(x-\xi,t-\tau)]u(\xi,\tau)^2\,\mathrm{d}\xi\,\mathrm{d}\tau\right| \\
&\quad\leqslant C\,|u|_{\mathscr{C}}^2\int_0^t\int_0^1\frac{1}{t-\tau}\left[\exp\left(-\frac{(x-\xi)^2}{8\varepsilon(t-\tau)}\right)\right. \\
&\qquad\left.+\exp\left(-\frac{(x+\xi)^2}{8\varepsilon(t-\tau)}\right)+\exp\left(-\frac{(x+\xi-2)^2}{8\varepsilon(t-\tau)}\right)\right]\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad\leqslant C\,\|u\|_{\mathscr{C}}^2\int_0^t\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi \\
&\quad\leqslant C\sqrt{T}\,\|u\|_{\mathscr{C}}^2 \qquad (62)
\end{aligned}$$

To estimate the last two terms of (59), we first note that

$$\max_{0\leqslant t\leqslant T}|g(w_i,\eta_i^0)(t)|\leqslant C(|\eta_i^0|+\|u\|_{\mathscr{C}}+\|u\|_{\mathscr{C}}^7),\quad i=0,1$$

It therefore follows from (A2) that

$$\begin{aligned}
&\left|\int_0^t\theta(x-1,t-\tau)\,[2g(w_1,\eta_1^0)(\tau)+w_1(\tau)^2]\,\mathrm{d}\tau\right| \\
&\quad\leqslant C(|\eta_1^0|+\|u\|_{\mathscr{C}}+\|u\|_{\mathscr{C}}^7)\int_0^t\frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{(x-1)^2}{4\varepsilon(t-\tau)}\right)\mathrm{d}\tau \\
&\quad\leqslant C\sqrt{T}(|\eta_1^0|+\|u\|_{\mathscr{C}}+\|u\|_{\mathscr{C}}^7) \qquad (63)
\end{aligned}$$

and

$$\left|\int_0^t\theta(x,t-\tau)\,[2g(w_0,\eta_0^0)(\tau)-w_0(\tau)^2]\,\mathrm{d}\tau\right|\leqslant C\sqrt{T}(|\eta_0^0|+\|u\|_{\mathscr{C}}+\|u\|_{\mathscr{C}}^7) \qquad (64)$$

By (61)–(64), we conclude that

$$\|G(u)\|_{\mathscr{C}}\leqslant C\,\|u^0\|_{H^1}+C\sqrt{T}(|\eta_0^0|+|\eta_1^0|)+C\sqrt{T}(\|u\|_{\mathscr{C}}+\|u\|_{\mathscr{C}}^7) \qquad (65)$$

Set

$$R=\max_{0\leqslant T\leqslant 1}\{C(T)\|u^0\|_{H^1}+C(T)\sqrt{T}(|\eta_0^0|+|\eta_1^0|)\}$$

and

$$B(0,2R)=\{u\in\mathscr{C}:\|u\|_{\mathscr{C}}\leqslant 2R\}$$

It therefore follows from (65) that G maps $B(0, 2R)$ into itself if T small enough. On the other hand, as in (62)–(64), we have

$$\left| \int_0^t \int_0^1 [\theta_x(x+\xi, t-\tau) - \theta_x(x-\xi, t-\tau)][u_1(\xi,\tau)^2 - u_2(\xi,\tau)^2]\, d\xi\, d\tau \right|$$
$$\leqslant CR \|u_1 - u_2\|_{\mathscr{C}} \int_0^t \frac{1}{\sqrt{t-\tau}}\, d\tau \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi$$
$$\leqslant CR\sqrt{T}\|u_1 - u_2\|_{\mathscr{C}} \tag{66}$$

$$\left| \int_0^t \theta(x-1, t-\tau)[2(g(w_{11}, \eta_1^0)(\tau) - g(w_{12}, \eta_1^0)(\tau)) + w_{11}(\tau)^2 - w_{12}(\tau)^2]\, d\tau \right|$$
$$\leqslant C(1+R^6)\|u_1 - u_2\|_{\mathscr{C}} \int_0^t \frac{1}{\sqrt{t-\tau}} \exp\left(-\frac{(x-1)^2}{4\varepsilon(t-\tau)} \right) d\tau$$
$$\leqslant C(1+R^6)\sqrt{T}\|u_1 - u_2\|_{\mathscr{C}} \tag{67}$$

and

$$\left| \int_0^t \theta(x, t-\tau)[2(g(w_{01}, \eta_0^0)(\tau) - g(w_{02}, \eta_0^0)(\tau)) - w_{01}(\tau)^2 + w_{02}(\tau)^2]\, d\tau \right|$$
$$\leqslant C(1+R^6)\sqrt{T}\|u_1 - u_2\|_{\mathscr{C}} \tag{68}$$

where

$$w_{ij} = u_j|_{x=i}, \quad i = 0, 1;\ j = 1, 2$$

It therefore follows from (66)–(68) that

$$\|G(u_2) - G(u_1)\|_{\mathscr{C}} \leqslant C(1+R^6)\sqrt{T}\|u_2 - u_1\|_{\mathscr{C}} \tag{69}$$

Thus G is contractive if T is small enough. By Banach contraction fixed point theorem, G has a unique fixed point u and then the problem (59) has a unique solution $u \in C([0, T]; C[0, 1])$ if T is small enough.

To claim that u is a classical solution of (14), we have to analyse every improper integral on the right-hand side of (59). This time we are not able to do so and hopefully finish it in future.

5. CONCLUSIONS

We have solved the problem of stabilization of the Burgers equation with unknown viscosity. Adaptation of a gain related to the reciprocal of viscosity achieves stability without a lower bound on viscosity. The control law is strengthened to ensure not only energy boundedness but also boundedness pointwise, including boundedness in the absence of adaptation.

APPENDIX A: TECHNICAL LEMMAS

First we present the proof of Lemma 3.1.

Proof. If $M \leqslant 0$, then $f(t)$ is non-increasing. Hence conditions (i) and (iii) immediately imply (32).

We now suppose that $M > 0$. We argue by contradication. If (32) is not true, then there exist a positive constant δ and a sequence $\{t_n\}$ $(n = 1, 2, \dots)$ with $t_n \to \infty$ as $n \to \infty$ such that

$$f(t_n) \geqslant \delta, \quad n = 1, 2, \dots$$

Let

$$F(t) = f(t) - M(t - t_n) - f(t_n)$$

By condition (ii), we have

$$F'(t) = f'(t) - M \leqslant 0$$

Therefore, we reduce

$$F(t) \geqslant F(t_n) = 0, \quad \forall 0 \leqslant t \leqslant t_n$$

that is,

$$f(t) \geqslant M(t - t_n) + f(t_n), \quad \forall 0 \leqslant t \leqslant t_n$$

Thus we have

$$\begin{aligned} \int_{t_n - \delta/M}^{t_n} f(t)\,\mathrm{d}t &\geqslant \int_{t_n - \delta/M}^{t_n} [M(t - t_n) + f(t_n)]\,\mathrm{d}t \\ &= \frac{\delta f(t_n)}{M} - \frac{\delta^2}{2M} \\ &\geqslant \frac{\delta^2}{2M}, \quad n = 1, 2, \dots \end{aligned} \tag{A1}$$

which is in contradiction with condition (iii). □

Although it seems that the following properties of the theta function defined by (60) should have been well-known in the literature, we could not find them in some standard reference books such as References [13,14]. Therefore, for completeness, we append them here with complete proofs.

Since, in the above, the variable x of the theta function $\theta(x, t)$ is required to vary from -1 to 2, we present the estimates for θ on the interval $[-1, 3]$.

Lemma A.1

Consider the theta function θ defined by (60). Then for any given $T > 0$, there exists a constant $C = C(\varepsilon, T) > 0$ such that

$$\theta(x, t) \leqslant \frac{C}{\sqrt{t}} e^{-x^2/4\varepsilon t}, \quad -1 \leqslant x \leqslant 1, \ 0 < t \leqslant T \tag{A2}$$

$$\theta(x, t) \leqslant \frac{C}{\sqrt{t}} e^{-(x-2)^2/4\varepsilon t}, \quad 1 \leqslant x \leqslant 3, \ 0 < t \leqslant T \tag{A3}$$

$$\left|\frac{\partial}{\partial x} \theta(x, t)\right| \leqslant \frac{C}{t} e^{-x^2/8\varepsilon t}, \quad -1 \leqslant x \leqslant 1, \ 0 < t \leqslant T \tag{A4}$$

$$\left|\frac{\partial}{\partial x} \theta(x, t)\right| \leqslant \frac{C}{t} e^{-(x-2)^2/8\varepsilon t}, \quad 1 \leqslant x \leqslant 3, \ 0 < t \leqslant T \tag{A5}$$

Proof. Since for $-1 \leqslant x \leqslant 1$ and $0 < t \leqslant T$

$$\begin{aligned} \theta(x, t) &= \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-x^2/4\varepsilon t} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n(x+n)}{\varepsilon t}\right) \\ &= \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-x^2/4\varepsilon t} \left[1 + \sum_{n=1}^{\infty} \left(\exp\left(-\frac{n(n+x)}{\varepsilon t}\right) + \exp\left(-\frac{n(n-x)}{\varepsilon t}\right)\right)\right] \\ &= \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-x^2/4\varepsilon t} \left[1 + \sum_{n=1}^{\infty} \left(\exp\left(-\frac{n(n+x)}{\varepsilon T}\right) + \exp\left(-\frac{n(n-x)}{\varepsilon T}\right)\right)\right] \end{aligned} \tag{A6}$$

we deduce (A2) with

$$C(\varepsilon, T) = \frac{1}{\sqrt{4\pi\varepsilon}} \left[1 + \max_{-1 \leqslant x \leqslant 1} \left\{ \sum_{n=1}^{\infty} \left(\exp\left(-\frac{n(n+x)}{\varepsilon T}\right) + \exp\left(-\frac{n(n-x)}{\varepsilon T}\right)\right)\right\}\right] \tag{A7}$$

Then (A3) follows from the periodicity of θ.

For (A4), we first have

$$-\frac{\partial}{\partial x}\theta(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\sum_{n=-\infty}^{\infty}\frac{x+2n}{2\varepsilon t}\exp\left(-\frac{(x+2n)^2}{4\varepsilon t}\right)$$

$$= \frac{1}{\sqrt{4\pi\varepsilon t}}\,\mathrm{e}^{-x^2/8\varepsilon t}\sum_{n=-\infty}^{\infty}\frac{x+2n}{2\varepsilon t}\,\mathrm{e}^{-x^2/8\varepsilon t}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right)$$

$$= \frac{1}{\sqrt{4\pi\varepsilon t}}\,\mathrm{e}^{-x^2/8\varepsilon t}\,\frac{x}{2\varepsilon t}\,\mathrm{e}^{-x^2/8\varepsilon t}$$

$$+ \frac{1}{\sqrt{4\pi\varepsilon t}}\,\mathrm{e}^{-x^2/8\varepsilon t}\sum_{n=1}^{\infty}\frac{x+2n}{2\varepsilon t}\,\mathrm{e}^{-x^2/8\varepsilon t}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right)$$

$$+ \frac{1}{\sqrt{4\pi\varepsilon t}}\,\mathrm{e}^{-x^2/8\varepsilon t}\sum_{n=1}^{\infty}\frac{x-2n}{2\varepsilon t}\,\mathrm{e}^{-x^2/8\varepsilon t}\exp\left(-\frac{n(n-x)}{\varepsilon t}\right) \tag{A8}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x+2n}{\varepsilon t}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right)\right) = \frac{x+2n}{\varepsilon t^2}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right)\left(\frac{n(x+n)}{\varepsilon t}-1\right)$$

there exists $N>0$ such that for $-1\leqslant x\leqslant 1$ and $0<t\leqslant T$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x+2n}{\varepsilon t}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right)\right) > 0, \quad n\geqslant N$$

Hence we have for $-1\leqslant x\leqslant 1$ and $0<t\leqslant T$

$$\frac{x+2n}{\varepsilon t}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right) \leqslant \frac{x+2n}{\varepsilon T}\exp\left(-\frac{n(x+n)}{\varepsilon T}\right), \quad n\geqslant N$$

This shows that there exists a positive constant $C(\varepsilon, T)$ such that for $-1\leqslant x\leqslant 1$ and $0<t\leqslant T$

$$\sum_{n=1}^{\infty}\frac{x+2n}{2\varepsilon t}\,\mathrm{e}^{-x^2/8\varepsilon t}\exp\left(-\frac{n(x+n)}{\varepsilon t}\right) \leqslant C(\varepsilon, T) \tag{A9}$$

Similarly, we have for $-1\leqslant x\leqslant 1$ and $0<t\leqslant T$

$$\sum_{n=1}^{\infty}\frac{2n-x}{2\varepsilon t}\,\mathrm{e}^{-x^2/8\varepsilon t}\exp\left(-\frac{n(n-x)}{\varepsilon t}\right) \leqslant C(\varepsilon, T) \tag{A10}$$

In addition, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\sqrt{\varepsilon t}}\exp\left(-\frac{x^2}{8\varepsilon t}\right)\right) = \frac{x}{2\sqrt{\varepsilon}t^{3/2}}\exp\left(-\frac{x^2}{8\varepsilon t}\right)\left(\frac{x^2}{4\varepsilon t}-1\right)$$

the function $x/\sqrt{\varepsilon t}\,\exp(-x^2/8\varepsilon t)$ attains the maximum $2\mathrm{e}^{-1/2}$ at $t = x^2/4\varepsilon$. Therefore, we have for $-1 \leqslant x \leqslant 1$ and $t > 0$

$$\frac{x}{\sqrt{\varepsilon t}}\mathrm{e}^{-x^2/8\varepsilon t} \leqslant 2\mathrm{e}^{-1/2} \tag{A11}$$

Hence (A4) follows from (A9), (A10) and (A11) while (A5) follows from the periodicity of θ. □

Remark A.1

It can be seen from (A7) that the constant $C(\varepsilon, T)$ tends to infinity as $T \to \infty$.

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