Adaptive Control of Robot Manipulators with Flexible Joints

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Abstract—**This paper presents an adaptive control scheme for flexible joint robot manipulators. Asymptotic stability is insured regardless of the joint flexibility value, i.e., the results are not restricted to weak joint elasticity. Moreover, the joint flexibility is not assumed to be *a priori* known. Joint position and velocity tracking errors are shown to converge to zero with all the signals in the system remaining bounded.**

I. Introduction

ADAPTIVE control of rigid robot manipulators has been thoroughly studied in the last decade [1]–[5]. However, robot manipulators usually have flexible joints due to gear elasticity, shaft wind up, etc. and flexibility has to be taken into account in the design if high performance is to be achieved. In some cases, joint flexibility can even lead to instability when neglected in the control design [6].

A way to understand the effect of joint flexibility is to consider adaptive control schemes designed for rigid robot manipulators and study their behavior in the presence of weak joint elasticity. This has been done in [7] using singular perturbation techniques. Global stability is guaranteed in [7] under the assumption of weak joint flexibility though the maximum flexibility that can be tolerated has not been explicitly given.

In this paper, we present an adaptive control scheme for robot manipulators which takes joint flexibility into account. Asymptotic stability is ensured regardless of the joint flexibility value, i.e. the results are not restricted to weak joint elasticity. Furthermore the joint flexibility value is not assumed to be known. Joint position and velocity tracking errors are shown to converge to zero and all the signals are proved to be bounded. The adaptive control scheme presented here has the attractive feature that it fully exploits the passivity properties of robot manipulators with flexible joints.

Section II presents the control scheme when all the robot manipulator parameters are known. Section III is devoted to the adaptive control scheme and concluding remarks are finally given in section IV.

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II. Control of Flexible Joint Manipulators

Consider a robot manipulator with flexible joints defined by the following equations:

$$D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) + K(q_1 - q_2) = 0 \quad (1)$$

$$J_m\ddot{q}_2 - K(q_1 - q_2) = u \quad (2)$$

where q_1 and q_2 represent the link angles and motor angles, respectively, $K > 0$ represents the joint stiffness, $D(q_1)$ is the inertia matrix, and $C(q_1, \dot{q}_1)\dot{q}_1$ represents the Coriolis and centrifugal terms, $g(q_1)$ represents the gravitational terms, u is the input torque, and J_m the actuators inertia matrix. The dynamic model (1), (2) does not represent the most general dynamic description of this class of arms but is a simplified model introduced in [12].

The control design will be based on the following properties.

P1: The inertia matrix $D(q)$ is positive definite [1].

P2: The constant parameters of interest (i.e., link masses, moments of inertia, etc.) in each of the terms in (1) appear as coefficients of known functions of the generalized coordinates [8], [9].

P3: For $C(q_1, \dot{q}_1)$ chosen as in [1], the matrix $\dot{D}(q_1) - 2C(q_1, \dot{q}_1)$ is skew symmetric.

Furthermore, the following lemma states the passivity of the system in (1), (2).

Lemma 1: Consider the dynamic equations in (1) and (2). The mapping $u \to \dot{q}_2$ is passive, i.e.,

$$\int_0^T u^T\dot{q}_2 \geq -\gamma^2 \qquad \text{for all } T > 0 \text{ and some } \gamma \in R.$$

The proof is given in Appendix A.

In the case of rigid manipulators the robot's passivity property, i.e., passivity of the mapping: $u \to \dot{q}_1$ has been very useful in the design of the adaptive control schemes [1]–[4]. However, in the case of flexible joint manipulators the mapping: $u \to \dot{q}_1$ cannot be proved to be passive but instead the mapping: $u \to \dot{q}_2$ is passive as stated in Lemma 1. The above lemma is given for completeness purposes and although it is not directly used in the adaptive control law design, it was helpful in finding the Lyapunov function to be used in the control law synthesis.

Consider the following Lyapunov function candidate:

$$V = \frac{1}{2}v_1^TD(q)v_1 + \frac{1}{2}v_2^TJ_mv_2 + \frac{1}{2}\int_0^t(v_1 - v_2)^T\,dt\,K\int_0^t(v_1 - v_2)\,dt \quad (3)$$

where

$$v_i = \dot{\tilde{q}}_i + \lambda \tilde{q}_i; \qquad \tilde{q}_i = q_i - q_{id}. \tag{4}$$

$\lambda > 0$ is a positive constant and q_{id} and its derivatives represent desired values for q_i and its derivatives, respectively. We assume that $q_{1d} \in C^4[\boldsymbol{R}^+]$.

The control law is given in the following lemma.

Lemma 2: Consider the system in (1) and (2) and the following control law:

$$u = -v_2 + J_m(\ddot{q}_{2d} - \lambda \dot{\tilde{q}}_2) - v_1 + D(\ddot{q}_{1d} - \lambda \dot{\tilde{q}}_1) + C(\dot{q}_{1d} - \lambda \tilde{q}_1) + g \tag{5}$$

where q_{2d} is given by

$$q_{2d} = K^{-1}\left\{ -v_1 + D(\ddot{q}_{1d} - \lambda \dot{\tilde{q}}_1) + C(\dot{q}_{1d} - \lambda \tilde{q}_1) + g + K\left[q_{1d} + \tilde{q}_1(0) - \tilde{q}_2(0) - \lambda \int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt \right] \right\}. \tag{6}$$

Then all the signals remain bounded, $\tilde{q}_1$ and $\tilde{q}_2$ converge to zero as $t \to \infty$ and $\ddot{q}_{2d}$ can be computed from (5) and (6) noting that $\ddot{q}_1$ and $q_1^{(3)}$ are bounded functions of q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$.

The proof is given in Appendix B.

Remark 1: The control input in (5) involves the computation of $\ddot{q}_{2d}$ which can be obtained by differentiating (6) twice. The expression for $\ddot{q}_{2d}$ involves terms like $\ddot{D}$, $\ddot{C}$ which depends on $\ddot{q}_1$ and $q_1^{(3)}$. The system parameters being known, $\ddot{q}_1$ can be expressed, using (1) and the boundedness of D^{-1}, as a function of q_1, $\dot{q}_1$, q_2, $\dot{q}_2$. The same procedure can be repeated for $q_1^{(3)}$ by differentiating (1) once and noting that $\dot{D}$ and $\dot{C}$ depend only on $\dot{q}_1$ and $\ddot{q}_1$.

Therefore the control input in (5) and (6) involves measurement of q_1, $\dot{q}_1$, q_2, $\dot{q}_2$ but does not require any acceleration measurement.

III. Adaptive Control of Flexible Joint Manipulators

One of the fundamental properties of robot manipulators that allow the use of adaptive systems techniques is the fact that the system parameters appear linearly in the equations as coefficients of known functions of q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$ (see [8] and [9]). Throughout this section we will denote that a given function f is linear in the unknown parameters by "f is LP" (Linear in the parameters).

In the development of the adaptive control presented in this section we will have to solve two main problems. First, we will prove that the control input u is LP and second, that there are no possible singularities in the control law. We also assume that the stiffness matrix K is diagonal.

In order to insure that the control input u is LP we had to slightly modify the Lyapunov function in (3) to the following

$$V = \frac{1}{2} v_1^T D v_1 + \frac{1}{2} v_2^T J_m \det(D) v_2 + \frac{1}{2}\int_0^t (v_1 - v_2)^T\, dt\, K \int_0^t (v_1 - v_2)\, dt + \frac{1}{2}\sigma_p \tilde{q}_1^T \tilde{q}_1 + \frac{1}{2}\tilde{\theta}^T \tilde{\theta} \tag{7}$$

where

$$\sigma_p > 0$$
$$\tilde{\theta} = \hat{\theta} - \theta \tag{8}$$

is the parametric error between the parameter vector θ and its estimate $\hat{\theta}$ to be defined later. The term $\det(D)$ has been included in V in (7) to obtain a control input that is LP.

In Appendix C we show that the nonadaptive control law can be written as

$$\theta_5^T h u + Y_6 \theta_6 = 0 \tag{9}$$

where θ_5, θ_6 are unknown parameter vectors, h is a known vector function, and Y_6 is a known matrix function. A constructive way to obtain Y_6, h is given in Appendix C.

The scalar $\theta_5^T h$ multiplying u in (9) is in fact equal to $\det D$ as will be shown in Appendix C. This means that $\theta_5^T h > 0$. This fact will be exploited to avoid singularities in the adaptive control scheme by adequately projecting the estimates into a region where no singularities (i.e., $\hat{\theta}_5^T h = 0$) occur.

A Convex Domain Λ for θ_5

We assume here that a lower bound αI of D is known.

Since $\theta_5^T h \geq \alpha h$ and h is a known function of q_1, we can define a subspace H spanned by $h(q_1)$ as

$$H\colon \{v\colon v = h(q_1) \text{ for some } q_1\}. \tag{10}$$

We can define Λ as

$$\Lambda\colon \{v\colon v^T h \geq \alpha^n \ \forall h \in H\}. \tag{11}$$

The two main properties of Λ that are essential for projecting the estimates to avoid singularities are as follows:

1) Λ is convex (see Appendix D)
2) $\theta_5 \in \Lambda$.

In the remaining of the paper we will assume that the computation of the convex region Λ has been carried out.

Parameter Adaptation Law

We will consider the following parameter adaptation law

$$\dot{\hat{\theta}}_5 = \begin{cases} h u^T v_2 & \text{if } \hat{\theta}_5 \in \operatorname{int}(\Lambda) \\ P_r^t(h u^T v_2) & \text{if } \hat{\theta}_5 \in \partial(\Lambda) \text{ and } (h u^T v_2)^T \hat{\theta}_5^{\perp} > 0 \end{cases} \tag{12}$$

$$\dot{\hat{\theta}}_6 = Y_6^T v_2 \tag{13}$$

where P_r^t denotes orthogonal projection onto Λ, $\partial(\Lambda)$ denotes the boundary of Λ, and $\hat{\theta}_5^{\perp}$ is the vector normal to $\partial(\Lambda)$ at $\hat{\theta}_5$.

Adaptive Control Law

The adaptive control law is given by (see also (9))

$$\hat{\theta}_5^T h u + Y_6 \hat{\theta}_6 = 0. \tag{14}$$

The adaptive control convergence properties are established in the following lemma.

Lemma 3: Consider the flexible joint manipulator described in (1) and (2) and the adaptive control defined in (13) through (15). Then all the signals remain bounded and the position and velocity tracking errors converge to zero.

The proof is given in Appendix C.

Remark 2: In the proof of Lemma 3 we have implicitly assumed *a priori* knowledge of an upper bound of the system parameters (see (C.24).

The K stiffness matrix is estimated on line as part of the controller parameters. Singularity of the $\hat{K}$ estimated matrix is avoided by using an appropriate parameter estimates projection (see (C.30)).

Conclusions

This paper presented an adaptive control scheme applicable to robot manipulators with flexible joints. In contrast to other approaches, global stability is ensured regardless of the joint flexibility value. The control input is computed using link and motor shaft position and velocity measurements. Link position and velocity tracking errors have been shown to converge to zero with all the signals in the control system remaining bounded.

Possible singularities in the adaptive control algorithm have been avoided by projecting the estimates into a convex region in the parameter space.

Appendix A

Proof of Lemma 1: Equations (1) and (2) can be derived via the so-called Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau \tag{A.1}$$

where $q^T = [q_1^T, q_2^T]$, τ is the generalized force vector $\tau = [0 \;\; u^T]$, and L is the Lagrangian of the system given by

$$L(q, \dot{q}_1, q_2, \dot{q}_2) = T_1(q_1, \dot{q}_1) + T_2(\dot{q}_2) - V(q_1, q_2) \tag{A.2}$$

where T_1 is the kinetic energy of the rigid links, T_2 is the kinetic energy of the actuators where the effect of $\dot{q}_1$ has been neglected assuming a high-gear ratio [12], and V is the total potential energy considering both gravity and elasticity.

From (A.1) we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \tag{A.3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2}. \tag{A.4}$$

Consider the adjoint Lagrangian function

$$L^* = p^T\dot{q} - L \tag{A.5}$$

where

$$p = \frac{\partial L}{\partial \dot{q}} \tag{A.6}$$

is the generalized momentum. It is well known that in the case of scleronomic systems (i.e., holonomic systems having a Lagrangian function independent of time) L^* is equal to the total energy in the system, i.e.,

$$L^* = T_1 + T_2 + V. \tag{A.7}$$

From (A.5) we have

$$\frac{dL^*}{dt} = \frac{dp^T}{dt}\dot{q} + p^T\ddot{q} - \frac{\partial L^T}{\partial q}\dot{q} - \frac{\partial L^T}{\partial \dot{q}}\ddot{q}. \tag{A.8}$$

From (A.3), (A.4), and (A.6) it follows

$$\begin{aligned}\frac{dp^T}{dt}\dot{q} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)^T\dot{q}_1 + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)^T\dot{q}_2 \\ &= \frac{\partial L}{\partial q_1}\dot{q}_1 + \left[u + \frac{\partial L}{\partial q_2}\right]^T\dot{q}_2.\end{aligned} \tag{A.9}$$

From (A.8) and (A.9) it finally follows that

$$\frac{dL^*}{dt} = \dot{q}_2^T u. \tag{A.10}$$

Integrating the above equation and noting that L^* is positive definite we conclude the proof. ∇

Appendix B

Proof of Lemma 2: For simplicity of notation we will omit the arguments of $D(q_1)$ and $C(q_1, \dot{q}_1)$.

Differentiating V in (3) and using property P3 we get

$$\begin{aligned}\dot{V} &= v_1^T\left[D\dot{v}_1 + \frac{1}{2}\dot{D}v_1\right] + v_2^T J_m \dot{v}_2 \\ &\quad + (v_1 - v_2)^T K\int_0^t (v_1 - v_2)\,dt \\ &= v_1^T\left[D\dot{v}_1 + Cv_1 + K\int_0^t (v_1 - v_2)\,dt\right] \\ &\quad + v_2^T\left[J_m\dot{v}_2 - K\int_0^t (v_1 - v_2)\,dt\right].\end{aligned} \tag{B.1}$$

The first term in brackets in the (B.1) can also be rewritten using (4) as

$$\begin{aligned}&D\dot{v}_1 + Cv_1 + K\int_0^t (v_1 - v_2)\,dt \\ &\quad = D[\ddot{q}_1 - \ddot{q}_{1d} + \lambda\dot{\tilde{q}}_1] + C[\dot{q} + \lambda\tilde{q}_1] \\ &\qquad + K\Big[\tilde{q}_1 - \tilde{q}_2 - \tilde{q}_1(0) + \tilde{q}_2(0) \\ &\qquad + \lambda\int_0^t (\tilde{q}_1 - \tilde{q}_2)\,dt\Big] = w_1 + Kq_{2d}\end{aligned} \tag{B.2}$$

with (see also (1)):

$$\begin{aligned}w_1 &= D[-\ddot{q}_{1d} + \lambda\dot{\tilde{q}}_1] + C[-\dot{q}_{1d} + \lambda\tilde{q}_1] - g \\ &\quad + K\left[-q_{1d} - \tilde{q}_1(0) + \tilde{q}_2(0) + \lambda\int_0^t (\tilde{q}_1 - \tilde{q}_2)\,dt\right].\end{aligned} \tag{B.3}$$

Let us define q_{2d} as follows:

$$Kq_{2d} = -v_1 - w_1. \tag{B.4}$$

Note that (6) and (B.4) are equivalent. Introducing (B.2) through (B.4) into (B.1) we get

$$\dot{V} = -v_1^T v_1 + v_2^T w_2 \tag{B.5}$$

with

$$w_2 = J_m \dot{v}_2 - K \int_0^t (v_1 - v_2)\, dt \tag{B.6}$$

which can also be written using (2) and (4) as

$$\begin{aligned} w_2 &= J_m(\ddot{q}_2 - \ddot{q}_{2d} + \lambda \dot{\tilde{q}}_2) \\ &\quad - K\left[\tilde{q}_1 - \tilde{q}_2 - \tilde{q}_1(0) + \tilde{q}_2(0) + \lambda \int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt\right] \\ &= u + w_3 \end{aligned} \tag{B.7}$$

where

$$\begin{aligned} w_3 &= J_m(-\ddot{q}_{2d} + \lambda \dot{\tilde{q}}_2) \\ &- K\left[-q_{1d} + q_{2d} - \tilde{q}_1(0) + \tilde{q}_2(0) + \lambda \int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt\right]. \end{aligned} \tag{B.8}$$

Using the control input in (5) it follows from (6), (B.5), (B.7), and (B.8) that

$$\dot{V} = -v_1^T v_1 - v_2^T v_2. \tag{B.9}$$

Therefore from (3) and (B.9) it follows that

$$v_1, v_2 \in L_2 \cap L_\infty \tag{B.10}$$

and

$$\int_0^t (v_1 - v_2)\, dt \in L_\infty. \tag{B.11}$$

From (4) and (B.10) it follows [11, p. 59] that $\tilde{q}_1, \tilde{q}_2 \in L_2 \cap L_\infty$, $\dot{\tilde{q}}_1, \dot{\tilde{q}}_2 \in L_2 \cap L_\infty$, and $\tilde{q}_1, \tilde{q}_2 \to 0$ as $t \to \infty$. In view of (B.11) and the fact that $\tilde{q}_1, \tilde{q}_2 \in L_\infty$ it follows that

$$\int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt \in L_\infty. \tag{B.12}$$

From (B.3) and the above we conclude that q_{2d} in (B.4) is bounded.

In order to prove boundedness of u in (5), (6) it suffices to prove that $\ddot{q}_{2d}$ is bounded. For that purpose let us differentiate q_{2d} in (B.4) which gives

$$K\ddot{q}_{2d} = -\ddot{v}_1 - w_4 + K[-\ddot{q}_{1d} + \lambda(\dot{\tilde{q}}_1 - \dot{\tilde{q}}_2)] \tag{B.13}$$

with

$$\begin{aligned} w_4 = \frac{d^2}{dt^2}[&D(-\ddot{q}_{1d} + \lambda \dot{\tilde{q}}_1) \\ &+ C(-\dot{q}_{1d} + \lambda \tilde{q}_1) - g]. \end{aligned} \tag{B.14}$$

Since q_{2d} in (6) is bounded and $\tilde{q}_2 \in L_\infty$ it follows that $q_2 \in L_\infty$.

Let us note that in view of (1) and the fact that q_1, $\dot{q}_1$, and $D^{-1}(q_1)$ are bounded it follows that $\ddot{q}_1 \in L_\infty$. Therefore, differentiating (6) we conclude that $\dot{q}_{2d}$ is bounded and since $\dot{\tilde{q}}_2 \in L_\infty$ then $\dot{q}_2 \in L_\infty$. Differentiating (1) we can also conclude in the same manner that $q_1^{(3)} \in L_\infty$ since $\dot{D}$ and $\dot{C}$ are functions of q_1, $\dot{q}_1$, and $\ddot{q}_1$ which are also bounded. Thus w_4 in (B.14) is bounded and so is $\ddot{q}_{2d}$ in (B.13) which implies that u in (5) is bounded. Let us also note that in the case when D, C, and g are known, $\ddot{q}_1$ and $q_1^{(3)}$ can be computed and are bounded. $\dot{\tilde{q}}_1, \dot{\tilde{q}}_2$ being bounded and $\tilde{q}_1, \tilde{q}_2 \in L_2$ it follows that $\tilde{q}_1, \tilde{q}_2 \to 0$. ∇

Appendix C

Proof of Lemma 3: We will omit the arguments in $D(q_1)$ and $C(q_1, \dot{q}_1)$ whenever the arguments are q_1 and $\dot{q}_1$ otherwise they will be indicated.

Differentiating (7) and using property P3 we obtain the following:

$$\begin{aligned} \dot{V} = v_1^T&\left[D\dot{v}_1 + Cv_1 + K\int_0^t (v_1 - v_2)\, dt\right] + \sigma_p \tilde{q}_1^T \dot{\tilde{q}}_1 \\ &+ v_2^T\left[J_m \det(D)\dot{v}_2 + \frac{J_m}{2}\frac{d}{dt}(\det(D))v_2\right. \\ &\left.-K\int_0^t (v_1 - v_2)\, dt\right] + \tilde{\theta}^T \dot{\tilde{\theta}}. \end{aligned} \tag{C.1}$$

Let us manipulate the first term in brackets in the RHS of (C.1). Using (1) and (4) we obtain

$$\begin{aligned} T_1 &= v_1^T\left[D\dot{v}_1 + Cv_1 + K\int_0^t (v_1 - v_2)\, dt\right] \\ &= v_1^T\Big\{D(\ddot{q}_1 - \ddot{q}_{1d} + \lambda \dot{\tilde{q}}_1) + C(\dot{\tilde{q}}_1 + \lambda \tilde{q}_1) \\ &\qquad + K\Big[\tilde{q}_1 - \tilde{q}_1(0) - \tilde{q}_2 + \tilde{q}_2(0) \\ &\qquad + \lambda\int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt\Big]\Big\} \\ &= v_1^T\{\Delta_1 + \Delta_2 + K[q_{2d} + w_5]\} \end{aligned} \tag{C.2}$$

where

$$\begin{aligned} \Delta_1 = (D(q_{1d}) - D)\ddot{q}_{1d} &+ (C(q_{1d}, \dot{q}_{1d}) - C)\dot{q}_{1d} \\ &+ g(q_{1d}) - g + \lambda(D\dot{\tilde{q}}_1 + C\tilde{q}_1) \end{aligned} \tag{C.3}$$

$$\Delta_2 = -D(q_{1d})\ddot{q}_{1d} - C(q_{1d}, \dot{q}_{1d})\dot{q}_{1d} - g(q_{1d}) \tag{C.4}$$

$$w_5 = -q_{1d} - \tilde{q}_1(0) + \tilde{q}_2(0) + \lambda \int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt. \tag{C.5}$$

Using Lemma 1 in [4] we can bound $v_1^T \Delta_1$ in (C.2) as follows:

$$\begin{aligned} v_1^T \Delta_1 \le v_1^T(\lambda D + b_1 I)v_1 &+ v_1^T(-\lambda^2 D + b_2 I)\tilde{q}_1 \\ &+ b_3(\|v_1\|^2 \|\tilde{q}_1\| + \lambda \|v_1\|\, \|\tilde{q}_1\|^2) \end{aligned} \tag{C.6}$$

where b_1, b_2, and b_3 are positive bounded functions of q_{1d},

$\dot{q}_{1d}$, $\ddot{q}_{1d}$, and the dynamic model parameters. Note that

$$\begin{aligned}\|v_1\|^2\|\tilde{q}_1\| &+ \lambda\|v_1\|\,\|\tilde{q}_1\|^2\\ &= \frac{\|v_1\|^2}{4} + \frac{\lambda\|\tilde{q}_1\|^2}{4} - \|\nu_1\|^2\left[\frac{1}{2} - \|\tilde{q}_1\|\right]^2\\ &\quad - \lambda\|\tilde{q}_1\|^2\left[\frac{1}{2} - \|v_1\|\right]^2 + (1+\lambda)\|v_1\|^2\|\tilde{q}_1\|^2\\ &\le \frac{\|v_1\|^2}{4} + \frac{\lambda\|\tilde{q}_1\|^2}{4} + (1+\lambda)\|v_1\|^2\|\tilde{q}_1\|^2. \quad (C.7)\end{aligned}$$

Introducing (C.7) into (C.6) we have

$$v_1^T\Delta_1 \le a_1 v_1^T v_1 + a_2\tilde{q}_1^T\tilde{q}_1 + a_3\|v_1\|^2\|\tilde{q}_1\|^2 \quad (C.8)$$

where a_1, a_2, and a_3 are positive bounded functions of q_{1d}, $\dot{q}_{1d}$, $\ddot{q}_{1d}$, and the dynamic model parameters.

From (C.4) and property P2, Δ_2 can be expressed as

$$\Delta_2 = Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d})\theta_1 \quad (C.9)$$

where Y_d is a matrix of appropriate dimension and θ_1 is a vector of parameters. Since K is a diagonal matrix we can rewrite the last term in (C.2) as follows:

$$v_1^T K[q_{2d} + w_5] = \theta_K^T \operatorname{diag}(v_1^i)[q_{2d} + w_5] \quad (C.10)$$

$$\text{with: } \theta_K^T = [k_{11}, k_{22}, \cdots k_{nn}] \quad (C.11)$$

where k_{ii} represents the ith diagonal element of K and $\operatorname{diag}(v_1^i)$ is a diagonal matrix whose diagonal entries are equal to the elements of v_1. Therefore, from (C.10) we have

$$\begin{aligned}v_1^T K[q_{2d} + w_5] &= \theta_K^T \operatorname{diag}(v_1^i)[q_{2d} + w_5]\\ &= Y_{2d}(q_1, \dot{q}_1, w_5, q_{2d})\theta_K \quad (C.12)\end{aligned}$$

where

$$Y_{2d}(q_1, \dot{q}_1, w_5, q_{2d}) = [q_{2d} + w_5]^T \operatorname{diag}(v_1^i). \quad (C.13)$$

Introducing (C.12) into (C.2) we have

$$\begin{aligned}T_1 = v_1^T(\Delta_1 + \Delta_2) &- Y_{2d}\tilde{\theta}_K + Y_{2d}\hat{\theta}_K\\ &\pm(\sigma_v + \sigma_n\|\tilde{q}_1\|^2)v_1^T D v_1 \quad (C.14)\end{aligned}$$

where

$$\tilde{\theta}_K = \hat{\theta}_K - \theta_K. \quad (C.15)$$

The last term in (C.14) will be used to compensate the term $v_1^T\Delta_1$. On the other hand, using (1) and (4) we have

$$\begin{aligned}D\dot{q}_1 &= \int_0^t \frac{d}{dt}(D\dot{q}_1)\,dt + D(q_1(0))\dot{q}_1(0)\\ &= \int_0^t(\dot{D}\dot{q}_1 - C\dot{q}_1 - g - Kq_1)\,dt + K\int_0^t q_2\,dt + \theta_2\\ &= \left[\int_0^t Y_3(q_1, \dot{q}_1)\,dt\right]\theta_3 + K\int_0^t q_2\,dt + \theta_2 \quad (C.16)\end{aligned}$$

where

$$\theta_2 = D(q_1(0))\dot{q}_1(0) \quad (C.17)$$

and we have used property P3 to define a function Y_3 and a vector θ_3 which depends on the system parameters.

Therefore

$$\begin{aligned}Dv_1 &= D\dot{q}_1 - D\dot{q}_{1d} + \lambda D\tilde{q}_1\\ &= Y_4\left(\int_0^t Y_3(q_1, \dot{q}_1)\,dt, \int_0^t q_2\,dt, q_1\right)\theta_4 \quad (C.18)\end{aligned}$$

where we have used (C.16) and the linearity with respect to the parameters to define a new function Y_4 whose partition depends on $\int_0^t Y_3\,dt$, $\int_0^t q_2\,dt$ and q_1, respectively.

Introducing (C.18) into (C.14) we get

$$\begin{aligned}T_1 = v_1^T(\Delta_1 + \Delta_2) &- Y_{2d}\tilde{\theta}_K + Y_{2k}\hat{\theta}_K\\ &+(\sigma_v + \sigma_n\|\tilde{q}_1\|^2)v_1^T Y_4\theta_4\\ &-(\sigma_v + \sigma_n\|\tilde{q}_1\theta^2)v_1^T D v_1. \quad (C.19)\end{aligned}$$

Provided the elements of $\hat{\theta}_K$ are all strictly positive we can define q_{2d} by the following equation:

$$\hat{K}[q_{2d} + w_5] = -(\sigma_v + \sigma_n\|\tilde{q}_1\|^2)Y_4\hat{\theta}_4 - Y_d\hat{\theta}_1 - \sigma_p\tilde{q}_1 \quad (C.20)$$

where $\hat{K}$ is a diagonal matrix whose entries are given by the elements of $\hat{\theta}_K$. Introducing (C.20) into (C.13) we have

$$\begin{aligned}Y_{2d}\hat{\theta}_K &= \hat{\theta}_K^T \operatorname{diag}(v_1^i)[q_{2d} + w_5] = v_1^T\hat{K}[q_{2d} + w_5]\\ &= -(\sigma_v + \sigma_n\|\tilde{q}_1\|^2)v_1^T Y_4\hat{\theta}_4 - v_1^T Y_d\hat{\theta}_1\\ &\quad - \sigma_p v_1^T\tilde{q}_1 \quad (C.21)\end{aligned}$$

where $\sigma_p > 0$ will be defined later. Introducing (C.21) and (C.9) into (C.19) we obtain

$$\begin{aligned}T_1 = v_1^T\Delta_1 &- v_1^T Y_d\tilde{\theta}_1 - Y_{2d}\tilde{\theta}_K - (\sigma_v + \sigma_n\|\tilde{q}_1\|^2)\\ &\cdot(v_1^T Y_4\tilde{\theta}_4 + v_1^T D v_1)\\ &- \sigma_p v_1^T\tilde{q}_1. \quad (C.22)\end{aligned}$$

Furthermore from (C.8) we have that

$$\begin{aligned}v_1^T\Delta_1 &- (\sigma_v + \sigma_n\|\tilde{q}_1\|^2)v_1^T D v_1 - \lambda\sigma_p\|\tilde{q}_1\|^2\\ &\le -v_1^T v_1(\lambda_{\min} D\sigma_v - a_1) - \tilde{q}^T\tilde{q}_1(\lambda\sigma_p - a_2)\\ &\quad - \|v_1\|^2\|\tilde{q}_1\|^2(\lambda_{\min} D\sigma_n - a_3) \quad (C.23)\end{aligned}$$

and if σ_v, σ_p, σ_n are chosen large enough such that

$$\begin{aligned}\lambda_{\min} D\sigma_v - a_1 &\ge \delta_0 > 0\\ \lambda\sigma_p - a_2 &\ge \delta_1 > 0\\ \lambda_{\min} D\sigma_n - a_3 &\ge 0. \quad (C.24)\end{aligned}$$

We obtain

$$\begin{aligned}T_1 \le -\delta_0 v_1^T v_1 &- \delta_1\tilde{q}_1^T\tilde{q}_1 - \sigma_p\dot{\tilde{q}}_1^T\tilde{q}_1 - v_1^T Y_d\tilde{\theta}_1\\ &- Y_{2d}\tilde{\theta}_K - (\sigma_v + \sigma_n\|\tilde{q}_1\|^2)v_1^T Y_4\tilde{\theta}_4. \quad (C.25)\end{aligned}$$

Combining (C.1), (C.2), and (C.25) we obtain

$$\begin{aligned}\dot{V} \le -\delta_0 v_1^T v_1 &- \delta_1\tilde{q}_1^T\tilde{q}_1 - v_1^T Y_d\tilde{\theta}_1 - Y_{2d}\tilde{\theta}_K\\ &- (\sigma_v + \sigma_n\|\tilde{q}_1\|^2)v_1^T Y_4\tilde{\theta}_4 + \tilde{\theta}^T\dot{\tilde{\theta}} + v_2^T T_2 \quad (C.26)\end{aligned}$$

where

$$T_2 = J_m \det(D)\dot{v}_2 + \frac{J_m}{2}\frac{d}{dt}(\det(D))v_2 - K\int_0^t (v_1 - v_2)\,dt. \quad (C.27)$$

Let us define

$$\theta^T = \left[\theta_K^T, \theta_1^T, \theta_4^T, \theta_5^T, \theta_6^T\right] \quad (C.28)$$

where θ_5 and θ_6 will be defined later. Consider the following parameter estimation algorithms:

$$\dot{\hat{\theta}}_1 = Y_d^T v_1 \quad (C.29a)$$

$$\dot{\hat{\theta}}_4 = \left(\sigma_v + \sigma_n \|\tilde{q}_1\|^2\right) Y_4^T v_1 \quad (C.29b)$$

Since K is a diagonal matrix with strictly positive entries and we require $\hat{K}$ to be invertible in (C.20) we will use the following parameter adaptation law

$$\dot{\hat{\theta}}_K^i = \begin{cases} x^i & \text{if } \hat{\theta}_K^i \geq \delta_k \\ x^i & \text{if } \hat{\theta}_K^i \geq \frac{\delta_k}{2} \text{ and } x^i \geq 0 \\ \left[f(\hat{\theta}_K^i)\right]^{-x^i} x^i & \text{if } \frac{\delta_k}{2} \leq \hat{\theta}_K^i \leq \delta_k \text{ and } x^i \leq 0 \end{cases} \quad (C.30a)$$

where

$$x^i = Y_{2d}^i \text{ and } 0 < \delta_k \leq \min \theta_K^i \quad (C.30b)$$

$\hat{\theta}_K^i$ denotes the ith element of $\hat{\theta}_K$.

A suitable choice of a smooth function $0 \leq f(\hat{\theta}_K^i) \leq 1$ with $f(\delta_k/2) = 0$ and $f(\delta_k) = 1$ implies that the parameter projection in (C.30) has smoothness properties that will be used later in the proof. This estimation algorithm guarantees that $\hat{\theta}_K^i \geq \delta_k/2$ for all i.

Introducing (C.28) through (C.30) into (C.26) we obtain

$$\dot{V} \leq -\delta_0 v_1^T v_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T \dot{\hat{\theta}}_5 + \tilde{\theta}_6^T \dot{\hat{\theta}}_6 + v_2^T T_2 \quad (C.31)$$

where we have used the fact (see (C.30)) that

$$\left(\dot{\hat{\theta}}_K - Y_{2d}\right)^T \tilde{\theta}_K = \sum_i \left(\dot{\hat{\theta}}_K^i - Y_{2d}^i\right)\left(\hat{\theta}_K^i - \theta_K^i\right) \leq 0.$$

Consider now the following lemma.

Lemma C.1: T_2 in (C.27) can be expressed as

$$T_2 = \theta_5^T h(q_1) u + Y_6(q_1, \dot{q}_1, q_2, \dot{q}_2)\theta_6 \quad (C.32)$$

with

$$\det(D) = \theta_5^T h(q_1) > \alpha^n \quad \text{for some } \alpha \text{ and all } q_1. \quad (C.33)$$

θ_5 and θ_6 are unknown parameter vectors and $h(q_1)$, Y_6 are a known vector and a known matrix, respectively.

Proof: From (2), (4), and (C.27) we have

$$T_2 = \det(D)\left[u + K(q_1 - q_2)\right] + J_m \det(D)\left(-\ddot{q}_{2d} + \lambda\dot{\tilde{q}}_2\right) + \frac{J_m}{2}\frac{d}{dt}\det(D)v_2 - K\int_0^t (v_1 - v_2)\,dt. \quad (C.34)$$

Since D is positive definite, then $\det(D)$ is strictly positive and LP (see property P3) and therefore it can be written as in (C.33).

Since the model is LP (see property P3), terms obtained by differentiation of model terms or by sum of them are indeed still LP, while terms obtained as products can be given as LP structure, eventually in terms of a new extended set of parameters. Therefore premultiplying (1) by the adjoint matrix of D we conclude that $\det(D)\ddot{q}_1$ is LP.

We prove next that $\det(D)\ddot{q}_{2d}$ can be expressed as a LP function of q_1, $\dot{q}_1$, q_2, $\dot{q}_2$. Taking the second derivative of (C.20) we see that $\ddot{q}_{2d}$ is a function of q_1, $\dot{q}_1$, $\ddot{q}_1$, and the following functions: w_5 (C.5), Y_4 ((C.16), and (C.18)), Y_d (C.9), $\hat{K}$ ((C.13), and (C.30)), $\hat{\theta}_1$ and $\hat{\theta}_4$ (C.29) and their first and second derivatives.

Now that in view of (C.5), (C.13), (C.18), (C.20), and (C.30), $\dot{q}_{2d}$ is a measurable signal. Furthermore $\ddot{q}_{2d}$ is a linear function of measurable signals and $\ddot{q}_1$. However, $\det(D)\ddot{q}_1$ is an LP function of q_1, $\dot{q}_1$, and q_2. Therefore we conclude that $\det(D)\ddot{q}_{2d}$ is an LP function of measurable variables and thus (C.34) can be written as in (C.32). ▽

Introducing (C.32) into (C.31) we get

$$\dot{V} \leq -\delta_0 v_1^T v_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T \dot{\hat{\theta}}_5 + \tilde{\theta}_6^T \dot{\hat{\theta}}_6 + v_2^T \theta_5^T h u + v_2^T Y_6 \theta_6 = -\delta_0 v_1^T v_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T \dot{\hat{\theta}}_5 - v_2^T h^T \tilde{\theta}_5^T u + v_2^T h^T \hat{\theta}_5^T u + v_2^T Y_6 \theta_6 \quad (C.35)$$

where we have defined (see (13))

$$\dot{\hat{\theta}}_6 = Y_6^T v_2. \quad (C.36)$$

Introducing the parameter adaptation law in (12) and the adaptive control law in (14) into (C.35) we finally obtain

$$\dot{V} \leq -\delta_0 v_1^T v_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T\left[\dot{\hat{\theta}}_5 - h u^T v_2\right]. \quad (C.37)$$

The term $hu^T v_2$ can be decomposed as

$$hu^T v_2 = P_r^t(hu^T v_2) + P_r^\perp(hu^T v_2) \quad (C.38)$$

where $P_r^t(z)$ denotes the orthogonal projection on the hyperplane tangent to $\partial(\Lambda)$ at z and $P_r^\perp(z)$ is the component of z perpendicular to this hyperplane at z. Therefore using (12) we have

$$\tilde{\theta}_5^T\left(\dot{\hat{\theta}}_5 - hu^T v_2\right) = \begin{cases} 0 & \text{if } \hat{\theta}_5 \in \operatorname{int}(\Lambda) \\ -\tilde{\theta}_5^T P_r^\perp(hu^T v_2) \leq 0 & \text{if } \hat{\theta}_5 \in \partial(\Lambda) \text{ and } \left(hu^T v_2\right)^T \hat{\theta}_5^\perp > 0 \end{cases} \quad (C.39)$$

and then finally

$$\dot{V} \leq -\delta_0 v_1^T v_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1. \tag{C.40}$$

From (7) and (C.40) we conclude that (see also Appendix B)

$$\begin{gathered} v_1 \in L_2 \cap L_\infty \Rightarrow \tilde{q}_1 \in L_2 \cap L_\infty,\ \dot{\tilde{q}} \in L_2 \cap L_\infty,\ \tilde{q}_1 \to 0 \\ v_2 \in L_\infty \Rightarrow \tilde{q}_2 \in L_\infty,\ \dot{\tilde{q}}_2 \in L_\infty \\ \int_0^t (v_1 - v_2)\, dt \in L_\infty \Rightarrow \int_0^t (\tilde{q}_1 - \tilde{q}_2)\, dt \in L_\infty \\ \hat{\theta} \in L_\infty. \end{gathered} \tag{C.41}$$

What remains to be proved is that q_2, $\dot{q}_2$, and u remain bounded. This, in turn, will imply that $\ddot{q}_1$ and $\ddot{q}_2$ are also bounded (see (1) and (2)).

Boundedness of q_2: Since $\tilde{q}_2 \in L_\infty$, it suffices to prove that $q_{2d} \in L_\infty$ to conclude that $q_2 \in L_\infty$. Note from (C.20) that since $\hat{K}^{-1}$ is bounded (see (C.30)) and in view of (C.41) we conclude that q_{2d} is bounded by Y_4, i.e., we have (see (C.18))

$$q_{2d} = B(t) + B(t) \int_0^t Y_3(q_1, \dot{q}_1)\, dt + B(t) \int_0^t q_2\, dt \tag{C.42}$$

where $B(t)$ denotes a generic bounded function. On the other hand, using (1) we have

$$\begin{aligned} \int_0^t q_2\, dt &= K^{-1} \int_0^t (D\ddot{q}_1 + C\dot{q}_1 + g + Kq_1)\, dt \\ &= K^{-1} \int_0^t D\ddot{q}_1\, dt + K^{-1} \int_0^t (C\dot{q}_1 + g + Kq_1)\, dt. \end{aligned} \tag{C.43}$$

Integrating by parts we obtain

$$\int_0^t D\ddot{q}_1\, dt = D\dot{q}_1 - D(q_1(0))\dot{q}_1(0) - \int_0^t \dot{D}\dot{q}_1\, dt. \tag{C.44}$$

Combining (C.42) through (C.44) we can see that q_{2d} can be expressed as

$$q_{2d} = B(t) + B(t) \int_0^t B(t)\, dt. \tag{C.45}$$

Therefore q_{2d} cannot grow (or decrease) faster than ct for some constant c. If q_{2d} grows unbounded so will q_2 (see (C.41)). This implies that $\ddot{q}_1$ will grow unbounded at the same rate (see (1)). But $\dot{q}_1 = \int_0^t \ddot{q}_1\, dt + \dot{q}_1(0)$ and then if $\ddot{q}_1$ grows unbounded at a rate no faster than ct, this implies that $\dot{q}_1$ will also grow unbounded. Since this contradicts (C.41) we conclude that q_{2d}, q_2, and $\ddot{q}_1$ remain bounded.

Boundedness of $\dot{q}_2$: Proceeding as before we can bound $\dot{q}_{2d}$ (C.20) as follows:

$$\dot{q}_{2d} = B(t) + B(t) \int_0^t B(t)\, dt + B(t) \left[\int_0^t B(t)\, dt \right]^2 \tag{C.46}$$

where the last term comes from the product $Y_4 \dot{\hat{\theta}}_4$. Using a similar argument we conclude that $\dot{q}_2$, $\dot{q}_{2d}$ remained bounded.

Boundedness of u: From the control law (14) we see that u is bounded by $\| Y_6 \|$ since $\hat{\theta}_5^T h \geq \alpha^n > 0$. On the other hand, Y_6 was defined to express (C.27) as in (C.32). In other words, Y_6 is a function of bounded signals as $\int_0^t (v_1 - v_2)\, dt$, q_1, $\dot{q}_1$, v_2, $\tilde{q}_2$ and the other known functions used to parametrize $\det(D)\ddot{q}_{2d}$.

Note that in view of the smoothness of the parameter projection in (C.30) $\hat{K}$ is well defined for all finite time. Therefore, differentiating (C.20) twice and premultiplying by $\det(D)$ we see that $\det(D)\ddot{q}_{2d}$ is parametrized with known functions such that u is bounded as follows:

$$u = B(t) + B(t) \int_0^t B(t)\, dt + B(t) \left[\int_0^t B(t)\, dt \right]^2. \tag{C.47}$$

If u grows unbounded so will $\ddot{q}_2$ (see (2)). But $\dot{q}_2$ is bounded so that using similar arguments as above we conclude that u remains bounded. ∇

Appendix D

Proof of the Convexity of Λ in (10), (11): Let x_1 and $x_2 \in \Lambda$, therefore $x_1^T h \geq \alpha^n$ and $x_2^T h \geq \alpha^n$ for any $h \in H$. Consider now

$$x = \lambda x_1 + (1 - \lambda) x_2 \qquad \lambda \in [0, 1].$$

It is clear that $x^T h \geq \alpha^n$ for any $h \in H$ and therefore Λ is convex. ∇

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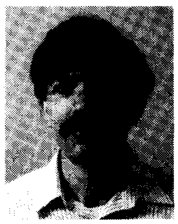

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