# Adaptive Controller Design for Tracking and Disturbance Attenuation in Parametric Strict-Feedback Nonlinear Systems

Zigang Pan, *Member, IEEE,* and Tamer Başar, *Fellow, IEEE*

***Abstract*— The authors develop a systematic procedure for obtaining robust adaptive controllers that achieve asymptotic tracking and disturbance attenuation for a class of nonlinear systems that are described in the parametric strict-feedback form and are subject to additional exogenous disturbance inputs. Their approach to adaptive control is performance-based, where the objective for the controller design is not only to find an adaptive controller, but also to construct an appropriate cost functional, compatible with desired asymptotic tracking and disturbance attenuation specifications, with respect to which the adaptive controller is "worst case optimal." In this respect, they also depart from the standard worst case (robust) controller design paradigm where the performance index is fixed *priori*.**

**Three main ingredients of the paper are the backstepping methodology, worst case identification schemes, and singular perturbations analysis. Under full state measurements, closed-form expressions have been obtained for an adaptive controller and the corresponding value function, where the latter satisfies a Hamilton–Jacobi–Isaacs equation (or inequality) associated with the underlying cost function, thereby leading to satisfaction of a dissipation inequality for the former. An important by-product of the analysis is the finding that the adaptive controllers that meet the dual specifications of asymptotic tracking and disturbance attenuation are generally not certainty-equivalent, but are asymptotically so as the measure quantifying the designer's confidence in the parameter estimate goes to infinity. To illustrate the main results, the authors include a numerical example involving a third-order system.**

***Index Terms*—Adaptive control, backstepping, disturbance attenuation, nonlinear systems, tracking.**

## I. Introduction

The DESIGN of adaptive controllers for parametric uncertain linear or nonlinear systems has been one of the most researched topics in control theory for the past two decades. For linear systems, adaptive controller designs have been centered on the certainty-equivalence principle [1], [2], where the controller structure is borrowed intact from a design with known parameter values and implemented using identified values for these parameters. Many success stories have been reported in achieving global boundedness of internal states and asymptotic performance of the system output using this approach in both stochastic and deterministic (noise-free) settings [3]–[8]. However, establishment of counterparts of these results for general nonlinear systems, using the certainty-equivalence approach, has been quite elusive, with successes reported initially mainly for the case when the nonlinearities satisfy some global linear growth conditions [9]. For nonlinear systems with severe nonlinearities, a breakthrough took place after the much celebrated characterization of the class of *feedback linearizable systems* [10], [11]. A pioneering work in this area has been [12], which presented a systematic design paradigm based on the novel *integrator backstepping* method to globally adaptively stabilize a subclass of feedback linearizable systems described in *parametric strict-feedback form*. This approach was further refined in [13], where over-parameterization was removed and was generalized in [14] to a larger class of nonlinear systems. It has also been applied to decentralized systems [15], as well as to nonholonomic nonlinear systems [16]. For an up-to-date list of references on the development of the backstepping approach, we refer the reader to [17].

Intuitively, an adaptive controller design uses (and generates online) more information on the system uncertainties than non-adaptive designs and therefore should lead to controllers with better robustness properties. In spite of this, many adaptive controllers have been shown to exhibit undesirable robustness properties [18]–[20]. *Nonrobustness* of an adaptive controller could lead to inferior transient behavior and burstiness in the closed-loop system under external disturbance inputs. To overcome these difficulties, various modifications to earlier designs have been proposed to robustify the adaptive controller design, for both linear and nonlinear systems [21]–[23], but these still fall short of addressing directly the disturbance attenuation property for the adaptive controller design.

General objectives of a robust adaptive controller design are (and should be) to improve transient performance, attain a finite (acceptable) level of disturbance attenuation, and sustain unmodeled dynamics. These are precisely the objectives that have motivated the study of the $H^\infty$-optimal control problem for linear systems (with known parameters), which has more recently been extended to the nonlinear framework

Manuscript received April 19, 1996; revised April 25, 1997. Recommended by Associate Editor, A. A. Stoorvogel. This work was supported in part by the U.S. Department of Energy under Grant DOE-DEFG-02-94ER13939 and in part by the National Science Foundation under Grant NSF-ECS-93-12807.

Z. Pan is with the Department of Electrical Engineering, Polytechnic University, Brooklyn, NY 11201 USA.

T. Başar is with the Department of Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801 USA.

Publisher Item Identifier S 0018-9286(98)05811-5.

[24]–[31], motivated by the differential game approach [32]. It would therefore be natural to cast a robust adaptive control problem in the framework of nonlinear $H^\infty$ optimal control, where specific measures of asymptotic tracking, transient behavior, and disturbance attenuation can all be incorporated into a single cost functional. This has in fact been done recently in the context of parameter identification (for linear and nonlinear systems) [33], [34]—a study that has led to a new class of robust identifiers that guarantee desired achievable levels of disturbance attenuation. The structure of these worst case identifiers resembles that of a least squares identifier, except for the presence of additional state estimate dynamics and an extra negative-definite term in the differential equation for the error covariance matrix. The fact that these have been obtained as the result of a *worst case* optimization process, leading to satisfaction of a dissipation inequality, makes them an ideal candidate to use in any cost-optimization-based adaptive controller design—which is what we do here.

Accordingly, this paper studies robust adaptive controller design using the worst case design methodology. To obtain explicit formulas for the controller, we consider the special (but important) class of nonlinear systems that is described in the parametric strict-feedback form, which we further take to be subject to additional affine exogenous disturbance inputs. The design specifications for the robust adaptive controller are asymptotic tracking of a given reference signal and achievement of a desired level of disturbance attenuation over the entire time interval $[0,\infty)$, which would then translate into much improved transient response. We present a systematic design paradigm, which leads to robust adaptive controllers with the following three appealing features: 1) asymptotic convergence to certainty-equivalent controllers as the identification error covariance approaches zero; 2) utilization of robust parameter identification schemes as basic building blocks; 3) attenuation of exogenous disturbance inputs to any desired level of performance over the entire time interval $[0,\infty)$.

Departing from the standard robust control setup, our objective for the controller design includes the characterization of an appropriate cost functional, compatible with the given asymptotic tracking and disturbance attenuation specifications, under which the controller designed satisfies a dissipation inequality, or equivalently, ensures a zero upper value for a particular zero-sum differential game. The cost function includes a positive-definite quadratic weighting on the tracking error (and possibly also weighting on internal states) and a negative-definite quadratic weighting on the exogenous inputs, whose ratio reflects the desired disturbance attenuation level for the closed-loop system. The freedom in the choice of the cost function allows us to extend the backstepping methodology of [12] and apply it to this robust adaptive control problem. We make use of the identifier designs obtained in [33] as a first step to the robust adaptive control design. Because of the two-time-scale structure of the worst case identifier, the controller design is based on the quasi-steady-state dynamics of the identifier dynamics, which are precisely the dynamics under the less restrictive measurement scheme that allows additional access to the derivative of the state variables for feedback. In this case, we show that one of the causes of the nonrobustness of a certainty-equivalent controller design is the fact that, in the closed-loop, the disturbance inputs enter the system through the differentiation of the parameter estimates, also known as the *swapping term* in the adaptive control literature. A singular perturbations analysis is then naturally employed to establish the robustness of the adaptive controller, when the two-time-scale identifier dynamics are utilized. The closed-loop system admits a value function that can be expressed in closed form and satisfies a Hamilton–Jacobi–Isaacs (HJI) equation (or inequality) associated with the underlying cost function, thereby guaranteeing a desired level of performance for the adaptive controller. Three main ingredients of the paper that pervade the derivation and the analyses are the backstepping methodology, worst case identification schemes, and singular perturbations analysis. A numerical example is included in Section IV to illustrate the theoretical findings of the paper. With minor modifications, the design paradigm presented here can be applied to a class of minimum phase parametric uncertain linear systems, as briefly discussed in the conclusions section.

The problem of asymptotic tracking and disturbance attenuation for parametric strict-feedback nonlinear systems has been addressed also in the recent book [35]. However, the approach adopted there is considerably different from the one developed here and is related to the tuning function approach introduced earlier in [13]. Here, we make use of the robust identification schemes derived in [33], and as a result of this combined identification and control design, the adaptive controller obtained is only asymptotically certainty equivalent, and it requires a smaller control magnitude to achieve the same level of disturbance attenuation. On the topic of transient performance, we should mention the two recent references [36] and [37] which have focused on the analysis of existing adaptive control algorithms, rather than on the design of controllers with respect to an optimality criterion, which puts direct weight on the transient performance of the closed-loop system.

The balance of the paper is organized as follows. In the next section, we provide a precise formulation for the nonlinear adaptive tracking and disturbance attenuation problem to be studied in the paper. In Section III, we first present a backstepping design tool for asymptotically tracking and simultaneously disturbance attenuating controllers for a two-level nonlinear system; then, we make repeated use of this backstepping tool to design robust adaptive controllers using the robust identifiers derived earlier in [33] and establish their robustness with respect to exogenous disturbances. A numerical example that involves a third-order nonlinear system with a single unknown parameter is presented in Section IV, which clearly illustrates many appealing features of the designed controller. The paper ends with the concluding remarks of Section V and two Appendixes, the first of which presents some derivations that lead to the identifier dynamics used in the adaptive controller design, based on the results of [33]; the second one provides relevant expressions for a robust adaptive controller designed for a modified version of the main model.

## II. Problem Formulation

Motivated by the results and formulation of [12] (and also [23]), we consider in this paper a class of single-input/single-output (SISO) nonlinear systems, in the following *noise-prone parametric strict-feedback form:*

$$\dot{x}_1 = x_2 + f_1(x_1) + \phi_1'(x_1)\theta_1 + h_1'(x_1)w_1 \tag{1a}$$

$$\vdots \qquad \vdots$$

$$\begin{aligned}\dot{x}_{n-1} = & x_n + f_{n-1}(x_1, \cdots, x_{n-1}) \\ & + \phi_{n-1}'(x_1, \cdots, x_{n-1})\theta_{n-1} \\ & + h_{n-1}'(x_1, \cdots, x_{n-1})w_{n-1}\end{aligned} \tag{1b}$$

$$\begin{aligned}\dot{x}_n = & f_n(x_1, \cdots, x_n) + \phi_n'(x_1, \cdots, x_n)\theta_n \\ & + b(x_1, \cdots, x_n)u + h_n'(x_1, \cdots, x_n)w_n\end{aligned} \tag{1c}$$

$$y = x_1. \tag{1d}$$

Here, $x := (x_1, \cdots, x_n)'$ is the $n$-dimensional state vector, with initial state being $x(0)$; $u$ is the scalar control input; $w := (w_1', \cdots, w_n')'$ is the $q$-dimensional exogenous input (disturbance) where $w_i$ is of dimension $q_i$, $i = 1, \cdots, n$; $y$ is the scalar output; $\theta := (\theta_1', \cdots, \theta_n')'$ is an $r$-dimensional vector of unknown parameters of the system where $\theta_i$ is of dimension $r_i$, $i = 1, \cdots, n$; and the nonlinear functions $f_i$, $\phi_i$, $h_i$, and $b$, $i = 1, \cdots, n$ are known and satisfy the triangular structure depicted in (1). Note that, as compared with the parametric strict-feedback form introduced in [12], the above system further incorporates additional additive disturbance inputs, where the nonlinear functions multiplying the disturbance terms are also in triangular form. We should further note that the above form of the nonlinear system is block diagonal in terms of the disturbance $w$ and the parameter vector $\theta$—this specific structure being essential for the applicability of the backstepping design procedure for the derivation of an adaptive controller when the parameter vector $\theta$ is unknown. In order to bring an original system into the noise-prone parametric strict-feedback form as above, one may have to treat any single parameter that enters (1) at different integration stages as different parameters, which would then clearly lead to overparameterization of the plant; one may also have to treat any single disturbance that enters the dynamics (1) at different integration stages as independent disturbances, which again would lead to an additional level of conservatism.

For the nonlinear system (1), we make the following two basic assumptions as a starting point of our study.

*Assumption A1:* The nonlinear functions $f_i$, $\phi_i$, and $h_i$ are $n-i+1$ times continuously differentiable in all their arguments (or simply are $\mathcal{C}^{n-i+1}$), $i = 1, \cdots, n$. The nonlinear functions $b$ and $1/b$ are $\mathcal{C}^1$ in all their arguments. □

*Assumption A2:* There exists a positive constant $c$, such that $h_i(x)'h_i(x) > c$; $\forall x \in \mathbb{R}^n$, $i = 1, \cdots, n$. □

*Assumption A1* guarantees the smoothness of the nonlinear functions $f_i$, $\phi_i$, and $h_i$, $i = 1, \cdots, n$, which is required for our design procedure. *Assumption A2* requires the disturbance to enter every channel of the nonlinear system (1), which is needed to avoid singularity in the identification of the parameters when full state measurement is available.

Associated with (1), we are given a reference trajectory $y_d(t)$ that the output of the system, $y$, is to track. We make the following smoothness assumptions on this reference trajectory.

*Assumption A3:* The reference trajectory $y_d$ is $n$ times continuously differentiable, where the signal $y_d$ and the derivatives $y_d^{(1)}, \cdots, y_d^{(n)}$ are uniformly bounded, i.e., for some $C_d > 0$, and $\forall i = 0, \cdots, n$

$$|y_d^{(i)}(t)| \le C_d, \qquad \forall t \in [0, \infty)$$

with $y_d^{(0)} = y_d$. The signal $y_d$ and its first $n$ derivatives are available for control design. □

For future reference, we denote the vector $(y_d, y_d^{(1)}, \cdots, y_d^{(n-1)})'$ by $x_d$.

The uncertainty, both intrinsic as well as exogenous to the system, is the triple $(x(0), \theta, w_{[0,\infty)})$, that is the initial state, the true values of the unknown parameters, and the driving disturbance. Since we are interested in the worst case performance, the exogenous input $w_{[0,\infty)}$ can be taken to be any open-loop time function, as in the case of $H^\infty$-optimal control problems. In view of the results of [33], we take $w_{[0,\infty)}$ to belong to some subset of all uniformly bounded $(\mathcal{L}_\infty)$ time functions, and the uncertainty triple $(x(0), \theta, w_{[0,\infty)})$ to belong to $\mathcal{W}$, which is taken as an appropriate subset of $\mathbb{R}^n \times \mathbb{R}^r \times \mathcal{L}_\infty$, to be specified later.

The objective of the controller design is to force the output $y$ to track the reference signal $y_d$ asymptotically, while attenuating the effect of the exogenous input (disturbance) $w$, the initial condition $x(0)$, and the unknown parameter vector $\theta$. A precise statement of this objective is now given below.

*Definition II.1:* A controller $\mu$ is said to be *asymptotically tracking with disturbance attenuation level* $\gamma$ if there exist nonnegative functions $l(t, x_{[0,t]})$ and $l_0(x(0), \theta - \overline{\theta})$ such that for all $t \ge 0$ the following dissipation inequality holds:

$$\begin{aligned}\sup_{(x(0),\theta,w_{[0,\infty)})\in\mathcal{W}} \Bigg\{ \int_0^t & ((y - y_d)^2 + l(\tau, x_{[0,\tau]}) \\ & - \gamma^2|w|^2)\, d\tau - \gamma^2|\theta - \overline{\theta}|_{Q_0}^2 \\ & - l_0(x(0), \theta - \overline{\theta}) \le 0.\end{aligned} \tag{2}$$

Here, $|\cdot|$ denotes the Euclidean norm, $\overline{\theta} := (\overline{\theta}_1', \cdots, \overline{\theta}_n')'$ is the initial guess for the unknown parameters, and the $r \times r$ dimensional matrix $Q_0 > 0$ is the quadratic weighting of error between the true value of $\theta$ and the initial guess $\overline{\theta}$, quantifying our level of confidence in the initial guess.

We will take $Q_0$ to be of block diagonal structure:

$$Q_0 = \text{block diagonal}\{Q_{01}, \cdots, Q_{0n}\}$$

where $Q_{0i} > 0$ is of dimensions $r_i \times r_i$ and corresponds to our level of confidence in the initial guess $\overline{\theta}_i$, $i = 1, \cdots, n$.

An important point to note here is that in the performance function (2), there is no weighting on the control input. Hence, any attenuation level $\gamma > 0$ can be achieved by allowing the magnitude of the control input to increase as $\gamma$ decreases. The smaller the value of $\gamma$ is, the better will be the disturbance rejection property, but at the expense of a larger control effort. Any controller that achieves the above objective has the following property, for any $t \geq 0$:

$$\sup_{(x(0),\theta,w_{[0,\infty)})\in\mathcal{W}} \left\{ \left( \int_0^t ((y-y_d)^2 + l(\tau, x_{[0,\tau]}))\, d\tau \right)^{1/2} \Big/ \left( \int_0^t |w|^2\, d\tau + |\theta - \overline{\theta}|^2_{Q_0} + (1/\gamma^2) l_0(x(0), \theta - \overline{\theta}) \right)^{1/2} \right\} \leq \gamma.$$

Hence, the $\mathcal{L}_2$ norm of the tracking error is always smaller than $\gamma$ times the $\mathcal{L}_2$ norm of the disturbance input plus a constant that depends only on the initial states of the system.

We will consider the class of *state trajectory-feedback* controllers $u(t) = \mu(t, x_{[0,t]})$, where $\mu$ is piecewise continuous in $t$ and Lipschitz continuous in the state trajectory $x_{[0,t]}$. We will refer to this measurement scheme underlying the given controller as the *unknown parameter full state information* (UPFSI), to differentiate it from the one where also the trajectory of the derivative of $x$ is available for control purposes—which we will refer to as the *unknown parameter full state with derivative information* (UPFSDI); we will have occasion to use this measurement scheme at a crucial intermediate step in the derivation of the robust adaptive controller. For a more in-depth discussion of the controller design under these two measurement schemes, as well as a third (more expanded) measurement scheme that allows the controller to have access to the true value of the parameter vector—called *known parameter full state information* (KPFSI)—we refer the reader to [38].

We now conclude this section by introducing some notation and convention that we will adopt throughout the paper. The vector $z := (z_1, \cdots, z_n)'$ will denote some transformed state variables; the vector $\hat{\theta} := (\hat{\theta}'_1, \cdots, \hat{\theta}'_n)'$ will denote the estimate of the parameter vector $\theta$, with $\hat{\theta}_i$ being the estimate of $\theta_i$, $i = 1, \cdots, n$; the vector $\tilde{\theta} := (\tilde{\theta}'_1, \cdots, \tilde{\theta}'_n)'$ is then the estimation error $\theta - \hat{\theta}$; any function symbol with an "over bar" will denote a function defined in terms of the transformed state variables, such as $\overline{\alpha}$ denoting the equivalent form of function $\alpha$ (of $x$) in terms of transformed state variable $z$. For any matrix $M$, the vector $\vec{M}$ is formed by stacking up its column vectors. For any functional $a$, depending on a parameter vector $\zeta$, $a_\zeta$ denotes the partial derivative $\partial a/\partial\zeta$, which is a row vector.

## III. Adaptive Controller Design

We first present a backstepping lemma that will be used repeatedly in the controller design stage. The intuition behind this result is that if a desired disturbance attenuation level $\gamma$ can be achieved through a virtual control input, then the same attenuation level $\gamma$ can be achieved using the actual control law which generates the virtual control input through a first-order dynamics. As compared with the existing backstepping algorithms, the lemma below provides a recursive solution for achieving disturbance attenuation without the necessity of increasing the guaranteed attenuation level at each step of recursion and takes advantage of the achieved disturbance attenuation property in the original design. Therefore, this algorithm leads to a controller with smaller controller gain, while guaranteeing the same level of robustness with respect to the disturbance.

*Lemma III.1:* Consider a noise perturbed nonlinear system given by

$$\dot{x}_o = f_o(x_o) + g_o(x_o)x_a + h_o(x_o)w \tag{3a}$$
$$\dot{x}_a = f_a(x_o, x_a) + g_a(x_o, x_a)u + h_a(x_o, x_a)w \tag{3b}$$

where $x_o$ is $n_1$ dimensional, $x_a$ is scalar, $w$ is $q$ dimensional, and the functions $f_o$, $g_o$, $h_o$, $f_a$, $g_a$, and $h_a$ are smooth with $g_a(x_o, x_a) \neq 0$, for any $(x_o, x_a)$. Suppose that with $x_a$ picked as the virtual control input to the subsystem dynamics (3a), and with $q_o(x_o)$ some arbitrary but fixed nonnegative-definite function, there exists a control law $\alpha_o(x_o)$ such that the following HJI inequality is satisfied by a nonnegative-definite value function $V_o(x_o)$:

$$\frac{\partial V_o}{\partial x_o}(f_o + g_o\alpha_o) + \frac{1}{4\gamma^2}\frac{\partial V_o}{\partial x_o} h_o h'_o \left(\frac{\partial V_o}{\partial x_o}\right)' + q_o \leq 0. \tag{4}$$

Then, there exists a control law $\alpha$ and a nonnegative-definite value function $V(x_o, x_a)$ for the overall system $(x_o, x_a)$, such that the following HJI inequality is satisfied for any desired nonnegative-definite $\overline{\beta}(x_o, z)$, where $z := x_a - \alpha_o(x_o)$

$$0 \geq q_o + \overline{\beta}z^2 + \begin{bmatrix} \left(\frac{\partial V}{\partial x_o}\right)' \\ \frac{\partial V}{\partial x_a} \end{bmatrix}' \begin{bmatrix} f_o + g_o x_a \\ f_a + g_a\alpha \end{bmatrix} + \frac{1}{4\gamma^2} \begin{bmatrix} \left(\frac{\partial V}{\partial x_o}\right)' \\ \frac{\partial V}{\partial x_a} \end{bmatrix}' \begin{bmatrix} h_o \\ h_a \end{bmatrix} \begin{bmatrix} h_o \\ h_a \end{bmatrix}' \begin{bmatrix} \left(\frac{\partial V}{\partial x_o}\right)' \\ \frac{\partial V}{\partial x_a} \end{bmatrix}. \tag{5}$$

*Proof:* First we observe that $V_o$ satisfies the HJI inequality (4) if and only if it satisfies, along with the dynamics (3a), the following inequality:

$$\dot{V}_o \leq \frac{\partial V_o}{\partial x_o} g_o(x_a - \alpha_o) - q_o + \gamma^2|w|^2 - \gamma^2 \left| w - \frac{1}{2\gamma^2} h'_o \left(\frac{\partial V_o}{\partial x_o}\right)' \right|^2.$$

Now, to prove the lemma, we introduce a new disturbance $\overline{w} = w - (1/2\gamma^2)h'_o(\partial V_o/x_o)'$ and a new state variable $z = x_a - \alpha_o(x_o)$. In terms of these quantities, the system dynamics can be expressed as follows:

$$\begin{aligned}\dot{x}_o =& f_o(x_o) + g_o(x_o)\alpha_o + g_o(x_o)z \\ &+ \frac{1}{2\gamma^2}h_o(x_o)h_o'(x_o)\left(\frac{\partial V_o}{\partial x_o}\right)' + h_o(x_o)\overline{w} \\ \dot{z} =& \overline{\tilde{f}}_a(x_o,z) + \overline{g}_a(x_o,z)u + \overline{\tilde{h}}_a(x_o,z)\overline{w} \\ \overline{\tilde{f}}_a(x_o,z) =& \overline{f}_a(x_o,z) - \frac{\partial \alpha_o}{\partial x_o}\bigg(f_o(x_o) + g_o(x_o)\alpha_o + g_o(x_o)z \\ &+ \frac{1}{2\gamma^2}h_o(x_o)h_o'(x_o)\left(\frac{\partial V_o}{\partial x_o}\right)'\bigg) \\ &+ \frac{1}{2\gamma^2}\,\overline{h}_a(x_o,z)h_o'(x_o)\left(\frac{\partial V_o}{\partial x_o}\right)' \\ \overline{\tilde{h}}_a(x_o,z) =& \overline{h}_a x_o,z) - \frac{\partial \alpha_o}{\partial x_o}\,(x_o)h_o(x_o).\end{aligned}$$

Introduce a candidate value function for the overall system

$$\overline{V}(x_o,z) = V_o(x_o) + \tfrac{1}{2}z^2.$$

Differentiation of this function yields the inequality

$$\begin{aligned}\dot{\overline{V}} \le& \frac{\partial V_o}{\partial x_o}g_o z - q_o + \gamma^2|w|^2 - \gamma^2|\overline{w}|^2 \\ &+ z(\overline{\tilde{f}}_a + \overline{g}_a u + \overline{\tilde{h}}_a\overline{w}) \\ =& -q_o - \overline{\beta}z^2 + \gamma^2|w|^2 - \gamma^2\left|\overline{w} - \frac{1}{2\gamma^2}\,\overline{\tilde{h}}_a' z\right|^2\end{aligned}$$

where the control $u$ is given by

$$\begin{aligned}u =& \alpha(x_o,x_a) =: \overline{\alpha}(x_o,z) \\ =& -\frac{1}{\overline{g}_a}\left(\frac{\partial V_o}{\partial x_o}g_o + \overline{\beta}z + \overline{\tilde{f}}_a + \frac{1}{4\gamma^2}\,\overline{\tilde{h}}_a\overline{\tilde{h}}_a' z\right).\end{aligned}$$

By the same observation as at the beginning of the proof above, and some manipulations, we obtain that the inequality for $\overline{V}$ above, evaluated along the full system dynamics, is equivalent to the HJI inequality (5). This completes the proof of the lemma. □

We now proceed to the derivation of the adaptive controller design, by first obtaining a (worst case) parameter identifier for $\theta$ (using the framework and results of [33]) and then developing the control law using a backstepping procedure.

Worst case identifiers for $\theta$ in (1), with guaranteed disturbance attenuation bounds, can be derived by making use of the general approach and results of [33], as outlined in Appendix A. These identifiers are parameterized in terms of $n$ nonnegative-definite matrices $Q_i$, $i = 1,\cdots,n$, where $Q_i$ is of dimensions $r_i \times r_i$ and may depend on the variables $x_1,\cdots,x_i$, $\hat{\theta}_1,\cdots,\hat{\theta}_i$, $\vec{\Sigma}_1,\cdots,\vec{\Sigma}_i$, i.e., $i = 1,\cdots,n$

$$Q_i(x_1,\cdots,x_i,\hat{\theta}_1,\cdots,\hat{\theta}_i,\vec{\Sigma}_1,\cdots,\vec{\Sigma}_i) \ge 0.$$

Note that the dependence of the $Q_i$'s on the internal states of the identifier is in a lower triangular form. Now, in terms of these matrices and a small design parameter $\epsilon$, the $H^\infty$-identifier for $\theta$, to be denoted by $\hat{\theta}$, is generated by the following[1]:

$$\dot{\hat{\theta}}_i = \Sigma_i\phi_i(h_i'h_i)^{-(1/2)}\,\frac{1}{\epsilon}\,(x_i - \hat{x}_i), \qquad \hat{\theta}_i(0) = \overline{\theta}_i \tag{6a}$$
$$\dot{\Sigma}_i = -\Sigma_i(\phi_i(h_i'h_i)^{-1}\phi_i' - Q_i)\Sigma_i, \qquad \Sigma_i(0) = Q_{0i}^{-1} \tag{6b}$$
$$\dot{\hat{x}}_i = \chi_i + \phi_i'\hat{\theta}_i + \frac{1}{\epsilon}\,(h_i'h_i)^{1/2}(x_i - \hat{x}_i), \qquad \hat{x}_i(0) = x_i(0) \tag{6c}$$

where the functions $\chi_i$, $i = 1,\cdots,n$, are defined by

$$\begin{aligned}\chi_i &:= f_i + x_{i+1}, \qquad i = 1,\cdots,n-1 \\ \chi_n &= f_n + bu.\end{aligned} \tag{7}$$

For small values of $\epsilon$, the identifier (6) exhibits a two-time-scale behavior. The coordinate transformation that yields the standard singularly perturbed form is given by $\hat{\theta}_i$, $\Sigma_i$, $e_i := (1/\epsilon)(x_i - \hat{x}_i)$, $i = 1,\cdots,n$. The identifier dynamics in this coordinate system is given by

$$\dot{\hat{\theta}}_i = \Sigma_i\phi_i(h_i'h_i)^{-(1/2)}e_i \tag{8a}$$
$$\dot{\Sigma}_i = -\Sigma_i(\phi_i(h_i'h_i)^{-1}\phi_i' - Q_i)\Sigma_i \tag{8b}$$
$$\epsilon\dot{e}_i = -(h_i'h_i)^{1/2}e_i + \phi_i'\tilde{\theta}_i + h_i'w_i. \tag{8c}$$

For ease of reference, we introduce the notation $\hat{x} := (\hat{x}_1,\cdots,\hat{x}_n)'$ and $e := (e_1,\cdots,e_n)'$. The quasi-steady-state behavior of the identifier is given by the following dynamics, where $i = 1,\cdots,n$[2]:

$$\dot{\hat{\theta}}_i = \Sigma_i\phi_i(h_i'h_i)^{-1}(\dot{x}_i - \chi_i - \phi_i'\hat{\theta}_i), \qquad \hat{\theta}_i(0) = \overline{\theta}_i \tag{9a}$$
$$\dot{\Sigma}_i = -\Sigma_i(\phi_i(h_i'h_i)^{-1}\phi_i' - Q_i)\Sigma_i, \qquad \Sigma_i(0) = Q_{0i}^{-1}. \tag{9b}$$

We note here that the quasi-steady-state dynamics correspond precisely to the dynamics of the identifier obtained under the UPFSDI measurement scheme where the derivative of the state variables are also available for feedback; see Appendix A. The specific structure of the limiting identifier (9) enables us to employ backstepping tools in the derivation of a controller for the overall system that uses derivative information, i.e., a robust adaptive controller under the UPFSDI measurement scheme. As it will become clear shortly, in the actual implementation, we will retain the structure of the robust control law but will replace the identifier part with the UPFSI identifier (6). This leads to the desired UPFSI adaptive control law, whose robustness with respect to exogenous disturbances is then proven using a singular perturbations analysis under additional *Assumptions A4 and A5,* to be introduced shortly.

Using the UPFSDI identifier (9), the identification error $\tilde{\theta}$ obeys the following dynamics:

$$\dot{\tilde{\theta}}_i = -\Sigma_i\phi_i(h_i'h_i)^{-1}h_i'v_i, \qquad i = 1,\cdots,n$$

where

$$v_i := w_i + h_i(h_i'h_i)^{-1}\phi_i'\tilde{\theta}_i, \qquad i = 1,\cdots,n.$$

[1] This identifier was derived in [33] as a limiting case of another full-order identifier for a noise-perturbed measurement scheme, using singular perturbations analysis. It was called there approximate noise-perturbed full-state information (NPFSI) identifier.

[2] This is obtained by setting the LHS of (8c) to *zero*, solving for $e_i$, and substituting it into (8a) by also making use of the original system dynamics.

Introduce a candidate value function associated with the identifier

$$W(\tilde{\theta}, x, \Sigma_1, \cdots, \Sigma_n) := \gamma^2 \sum_{i=1}^{n} |\tilde{\theta}_i|^2_{\Sigma_i^{-1}} \tag{10}$$

whose derivative is given by

$$\dot{W} = -\gamma^2 \sum_{i=1}^{n} |\tilde{\theta}_i|^2_{Q_i} + \gamma^2|w|^2 - \gamma^2|v|^2$$

where $v := (v_1', \cdots, v_n')'$. The left-hand side (LHS) of (2) can then be equivalently written as follows, by adding the identically zero function $W(0) - W(t) + \int_0^t \dot{W}(\tau)\, d\tau$:

$$\begin{aligned}
&\sup_{(x(0),\theta,w_{[0,\infty)})\in\mathcal{W}} \left\{ \int_0^t \left((y-y_d)^2 + l(\tau, x_{[0,\tau]}) - \gamma^2|w|^2\right) d\tau \right. \\
&\qquad \left. - \gamma^2|\theta - \overline{\theta}|^2_{Q_0} - l_0(x(0), \theta - \overline{\theta}) \right\} \\
&= \sup_{(x(0),\theta,w_{[0,\infty)})\in\mathcal{W}} \left\{ \int_0^t \left( (y-y_d)^2 + l(\tau, x_{[0,\tau]}) \right.\right. \\
&\qquad \left. - \gamma^2|v|^2 - \gamma^2 \sum_{i=1}^{n} |\tilde{\theta}_i|^2_{Q_i} \right) d\tau \\
&\qquad - \gamma^2 \sum_{i=1}^{n} |\tilde{\theta}_i(t)|^2_{\Sigma_i^{-1}(t)} \\
&\qquad \left. - l_0(x(0), \theta - \overline{\theta}) \right\}.
\end{aligned}$$

Thus, the attenuation problem with respect to $w$ has been effectively converted to an attenuation problem with respect to the equivalent disturbance $v$. In terms of $v$, the system dynamics (1) and the identifier (9) satisfy, for $i = 1, \cdots, n$

$$\dot{x}_i = \chi_i + \phi_i'\hat{\theta}_i + h_i' v_i \tag{11a}$$
$$\dot{\hat{\theta}}_i = -\dot{\tilde{\theta}}_i = \Sigma_i \phi_i (h_i' h_i)^{-1} h_i' v_i \tag{11b}$$
$$\dot{\Sigma}_i = -\Sigma_i(\phi_i (h_i' h_i)^{-1}\phi_i' - Q_i)\Sigma_i. \tag{11c}$$

Thus settling the issue of design of the robust identifier, we now move on to controller design under *Assumptions A1–A3*, which involves $n$ steps of integrator backstepping by repeated application of Lemma III.1. Let $\gamma > 0$ be any fixed, desired level of disturbance attenuation.

*Step 1:* At this step, we consider the combined states $\hat{\theta}_1, \Sigma_1$ as $x_o$, and state $z_1 := x_1 - y_d$ as $x_a$, in the statement of Lemma III.1. We choose $V_o = 0$ as the value function for $x_o$ under the virtual control input $z_1$, which allows the desired virtual control law to be chosen as zero. For notational consistency, we again let $z_0 \equiv 0$. Following the steps of Lemma III.1, the derivative of $z_1$ then satisfies

$$\dot{z}_1 = x_2 - y_d^{(1)} + \overline{f}_1 + \overline{\phi}_1'\hat{\theta}_1 + \overline{h}_1' v_1.$$

Choose a nonnegative function $\overline{\beta}_1(z_1, y_d, \hat{\theta}_1) \in \mathcal{C}^n$ which is again a design parameter. We introduce

$$\begin{aligned}
z_2 &:= x_2 - y_d^{(1)} + \overline{\alpha}_1 \\
\overline{\alpha}_1(z_1, y_d, \hat{\theta}_1) &:= z_0 + \overline{k}_1 z_1 + \overline{f}_1 + \overline{\phi}_1'\hat{\theta}_1 \\
\overline{k}_1(z_1, y_d, \hat{\theta}_1) &:= 1 + \overline{\beta}_1 + \gamma^2 \overline{\nu}_{11}'\overline{\nu}_{11} \\
\overline{\nu}_{11}(z_1, y_d, \hat{\theta}_1) &:= \frac{1}{2\gamma^2}\, \overline{h}_1.
\end{aligned}$$

Then, the dynamics for $z_1$ can be rewritten as

$$\dot{z}_1 = z_2 - \overline{k}_1 z_1 - z_0 + 2\gamma^2 \overline{\nu}_{11} v_1.$$

Introduce a value function

$$\overline{V}_1(z_1) = \tfrac{1}{2} z_1^2$$

the derivative of which, along the dynamics of $z_1$, is

$$\dot{\overline{V}}_1 = z_1 z_2 - z_1^2 - \overline{\beta}_1 z_1^2 + \gamma^2|v_1|^2 - \gamma^2|v_1 - \overline{\nu}_{11} z_1|^2.$$

This completes the first step of the design.

Note that at this stage, the nonlinear functions $\overline{\beta}_1$, $\overline{k}_1$, and $\overline{\alpha}_1$ do not depend on the covariance states $\Sigma_1$. Hence, if $n = 1$, the adaptive controller is a certainty-equivalent controller. But, for the general case, this certainty equivalence property will only hold asymptotically as $\Sigma_1 \to 0, \cdots, \Sigma_n \to 0$, as we will shortly see.

*Step $i, 2 \le i < n$:* Assume that from the previous steps we have the following structures:

$$\begin{aligned}
z_i = x_i - y_d^{(i-1)} + \overline{\alpha}_{i-1}(&z_1, \cdots, z_{i-1}, y_d, \cdots, y_d^{(i-2)}, \\
&\hat{\theta}_1, \cdots, \hat{\theta}_{i-1}, \Sigma_1, \cdots, \Sigma_{i-2})
\end{aligned} \tag{12a}$$

$$\overline{V}_{i-1} = \tfrac{1}{2} \sum_{l=1}^{i-1} z_l^2 \tag{12b}$$

$$\dot{\overline{V}}_{i-1} = z_{i-1}z_i - z_1^2 - \sum_{l=1}^{i-1} \overline{\beta}_l z_l^2 + \gamma^2 \sum_{l=1}^{i-1} |v_l|^2 \tag{12c}$$

$$- \gamma^2 \sum_{l=1}^{i-1} |v_l - \sum_{m=l}^{i-1} \overline{\nu}_{ml} z_m|^2 \tag{12d}$$

$$\begin{aligned}
\overline{\beta}_l = \overline{\beta}_l(&z_1, \cdots, z_l, y_d, \cdots, y_d^{(l-1)}, \\
&\hat{\theta}_1, \cdots, \hat{\theta}_l, \Sigma_1, \cdots, \Sigma_{l-1}), \\
&l = 1, \cdots, i-1
\end{aligned} \tag{12e}$$

$$\begin{aligned}
\overline{\nu}_{ml} = \overline{\nu}_{ml}(&z_1, \cdots, z_m, y_d, \cdots, y_d^{(m-1)}, \\
&\hat{\theta}_1, \cdots, \hat{\theta}_m, \Sigma_1, \cdots, \Sigma_{m-1}), \\
&l = 1, \cdots, i-1; m = l, \cdots, i-1
\end{aligned} \tag{12f}$$

$$\begin{aligned}
\dot{z}_j = z_{j+1} &- \overline{k}_j z_j - z_{j-1} \\
&+ 2\gamma^2 \sum_{l=1}^{j} \overline{\nu}_{jl}' \left( v_l - \sum_{m=l}^{j-1} \overline{\nu}_{ml} z_m \right), \\
&j = 1, \cdots, i-1
\end{aligned} \tag{12g}$$

$$\begin{aligned}
\overline{k}_l = \overline{k}_l(&z_1, \cdots, z_l, y_d, \cdots, y_d^{(l-1)}, \\
&\hat{\theta}_1, \cdots, \hat{\theta}_l, \Sigma_1, \cdots, \Sigma_{l-1}), \\
&l = 1, \cdots, i-1.
\end{aligned} \tag{12h}$$

At this step, we first evaluate the dynamics of $z_i$

$$\begin{aligned}\dot{z}_i =& x_{i+1} - y_d^{(i)} + \overline{f}_i + \overline{\phi}_i'\hat{\theta}_i + \overline{h}_i' v_i \\ &+ \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial z_j} \left( z_{j+1} - \overline{k}_j z_j - z_{j-1} + 2\gamma^2 \right. \\ &\left. \cdot \sum_{l=1}^{j} \overline{\nu}_{jl}' \left( v_l - \sum_{m=l}^{j-1} \overline{\nu}_{ml} z_m \right)\right) \\ &+ \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial y_d^{(j-1)}} y_d^{(j)} + \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial \hat{\theta}_j} \Sigma_j \overline{\phi}_j (\overline{h}_j'\overline{h}_j)^{-1} \overline{h}_j' v_j \\ &- \sum_{j=1}^{i-2} \frac{\partial \overline{\alpha}_{i-1}}{\partial \vec{\Sigma}_j} \overrightarrow{\Sigma_j(\overline{\phi}_j(\overline{h}_j'\overline{h}_j)^{-1}\overline{\phi}_j' - \overline{Q}_j)\Sigma_j}.\end{aligned}$$

Following the procedure of Lemma III.1, we will construct a value function to include the state $z_i$. Fix a design parameter $\overline{\beta}_i$, which is a nonnegative $\mathcal{C}^{n-i+1}$ function, according to

$$\overline{\beta}_i(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \hat{\theta}_1, \cdots, \hat{\theta}_i, \Sigma_1, \cdots, \Sigma_{i-1}).$$

The dynamics of $z_i$ can be rewritten as

$$\begin{aligned}\dot{z}_i =& z_{i+1} - \overline{k}_i z_i - z_{i-1} \\ &+ 2\gamma^2 \sum_{j=1}^{i} \overline{\nu}_{ij}' \left( v_j - \sum_{m=j}^{i-1} \overline{\nu}_{mj} z_m \right)\end{aligned}$$

where we have introduced the following functions:

$$\overline{\nu}_{ii} = \frac{1}{2\gamma^2} \overline{h}_i \tag{13a}$$

$$\begin{aligned}\overline{\nu}_{ij} =& \sum_{l=j}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial z_l} \overline{\nu}_{lj} + \frac{1}{2\gamma^2} \frac{\partial \overline{\alpha}_{i-1}}{\partial \hat{\theta}_j} \\ &\cdot \Sigma_j \overline{\phi}_j (\overline{h}_j'\overline{h}_j)^{-1} \overline{h}_j', \qquad j = 1, \cdots, i-1\end{aligned} \tag{13b}$$

$$\overline{k}_i = \overline{\beta}_i + \gamma^2 \sum_{l=1}^{i} \overline{\nu}_{il}' \overline{\nu}_{il} \tag{13c}$$

$$\begin{aligned}\overline{\alpha}_i =& \overline{k}_i z_i + z_{i-1} + \overline{f}_i + \overline{\phi}_i'\hat{\theta}_i \\ &+ \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial z_j} (z_{j+1} - \overline{k}_j z_j - z_{j-1}) \\ &+ \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial y_d^{(j-1)}} y_d^{(j)} - \sum_{j=1}^{i-2} \frac{\partial \overline{\alpha}_{i-1}}{\partial \vec{\Sigma}_j} \\ &\cdot \overrightarrow{\Sigma_j(\overline{\phi}_j(\overline{h}_j'\overline{h}_j)^{-1}\overline{\phi}_j' - \overline{Q}_j)\Sigma_j} \\ &+ 2\gamma^2 \sum_{l=1}^{i-1} \sum_{j=l}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial z_j} \overline{\nu}_{jl}' \sum_{m=j}^{i-1} \overline{\nu}_{ml} z_m \\ &+ \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial \hat{\theta}_j} \Sigma_j \overline{\phi}_j (\overline{h}_j'\overline{h}_j)^{-1} \overline{h}_j' \sum_{m=j}^{i-1} \overline{\nu}_{mj} z_m\end{aligned} \tag{13d}$$

$$z_{i+1} = x_{i+1} - y_d^{(i)} + \overline{\alpha}_i. \tag{13e}$$

The functions $\overline{\nu}_i$, $\overline{k}_i$, and $\overline{\alpha}_i$ depend on the variables $(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \hat{\theta}_1, \cdots, \hat{\theta}_i, \Sigma_1, \cdots, \Sigma_{i-1})$—a property that is consistent with the corresponding hypotheses in (12).

Introduce the value function for this step as

$$\overline{V}_i := \overline{V}_{i-1} + \tfrac{1}{2} z_i^2.$$

After some algebraic manipulations, we obtain

$$\begin{aligned}\dot{\overline{V}}_i =& z_i z_{i+1} - z_1^2 - \sum_{l=1}^{i} \overline{\beta}_l z_l^2 + \gamma^2 \sum_{l=1}^{i} |v_l|^2 \\ &- \gamma^2 \sum_{l=1}^{i} \left| v_l - \sum_{m=l}^{i} \overline{\nu}_{ml} z_m \right|^2\end{aligned}$$

and this completes the $i$th design step. Again, the definitions above are consistent with the corresponding induction hypotheses in (12), and the backstepping process can be continued from $i = 2$ to $i = n-1$ using the same methodology. At step $n$, the actual control appears explicitly in the state equation for $z_n$, which allows us to complete the controller design process.

*Remark III.1:* The above derivation quite naturally takes into account the disturbance inputs that enter the dynamics of $z_i$ through differentiation with respect to the parameter estimates $\hat{\theta}_1, \cdots, \hat{\theta}_{i-1}$—known in the adaptive control literature as *swapping terms*. The standard certainty-equivalent controller design, however, does not take the presence of these terms into account. A significance of this fact is that these disturbance inputs are amplified by nonlinear gains $(\partial \overline{\alpha}_{i-1}/\partial \hat{\theta}_j)\Sigma_j \overline{\phi}_j (\overline{h}_j'\overline{h}_j)^{-1}\overline{h}_j'$, $j = 1, \cdots, i-1$, which generally disrupt the disturbance attenuation property of the system when $\Sigma_j$'s are not zero—an observation that sheds light on the poor transient behavior of certainty-equivalent controllers. □

*Step $n$:* First choose a design parameter $\overline{\beta}_n$, which is a nonnegative $\mathcal{C}^1$ function, according to

$$\overline{\beta}_n(z_1, \cdots, z_n, y_d, \cdots, y_d^{(n-1)}, \hat{\theta}_1, \cdots, \hat{\theta}_n, \Sigma_1, \cdots, \Sigma_{n-1}).$$

Define the functions $\overline{\nu}_{nl}$, $l = 1, \cdots, n$, $\overline{k}_n$, and $\overline{\alpha}_n$ as in (13), with the index $i$ set to $n$. Then, the state dynamics for $z_n$ are given by

$$\begin{aligned}\dot{z}_n =& \overline{b}u - y_d^{(n)} + \overline{\alpha}_n - \overline{k}_n z_n - z_{n-1} \\ &+ 2\gamma^2 \sum_{j=1}^{n} \overline{\nu}_{nj}' \left( v_j - \sum_{m=j}^{n-1} \overline{\nu}_{mj} z_m \right).\end{aligned}$$

This can further be rewritten as

$$\dot{z}_n = -\overline{k}_n z_n - z_{n-1} + 2\gamma^2 \sum_{j=1}^{n} \overline{\nu}_{nj}' \left( v_j - \sum_{m=j}^{n-1} \overline{\nu}_{mj} z_m \right)$$

with the control law defined as

$$u = \tilde{\mu}(t, x_{[0,t]}, \dot{x}_{[0,t]}) = \frac{1}{\overline{b}} (y_d^{(n)} - \overline{\alpha}_n). \tag{14}$$

For this final step, the value function for the closed-loop system is given by

$$\overline{V} := \overline{V}_n = \overline{V}_{n-1} + \tfrac{1}{2} z_n^2$$

whose derivative can be evaluated to be

$$\begin{aligned}\dot{\overline{V}} = &-z_1^2 - \sum_{l=1}^{n} \overline{\beta}_l z_l^2 + \gamma^2 \sum_{l=1}^{n} |v_l|^2 \\ &- \gamma^2 \sum_{l=1}^{n} \left| v_l - \sum_{m=l}^{n} \overline{\nu}_{ml} z_m \right|^2.\end{aligned}$$

This completes the backstepping design process for the UPFSDI case. The closed-loop system under the control law (14) and the identifier (9) is now described collectively by the following set of differential equations:

$$\begin{aligned}
\dot{z}_1 &= z_2 - \overline{k}_1 z_1 - z_0 + 2\gamma^2 \overline{\nu}'_{11} \\
&\quad \cdot (w_1 + \overline{h}_1 (\overline{h}'_1 \overline{h}_1)^{-1} \overline{\phi}'_1 \tilde{\theta}_1) && \text{(15a)}\\
\dot{\tilde{\theta}}_1 &= -\dot{\hat{\theta}}_1 = -\Sigma_1 \overline{\phi}_1 (\overline{h}'_1 \overline{h}_1)^{-1} \overline{h}'_1 \\
&\quad \cdot (w_1 + \overline{h}_1 (\overline{h}'_1 \overline{h}_1)^{-1} \overline{\phi}'_1 \tilde{\theta}_1) && \text{(15b)}\\
\dot{\Sigma}_1 &= -\Sigma_1 (\overline{\phi}_1 (\overline{h}'_1 \overline{h}_1)^{-1} \overline{\phi}'_1 - \overline{Q}_1) \Sigma_1 && \text{(15c)}\\
&\;\;\vdots \qquad \vdots \\
\dot{z}_{n-1} &= z_n - \overline{k}_{n-1} z_{n-1} - z_{n-2} + 2\gamma^2 \sum_{l=1}^{n-1} \overline{\nu}'_{n-1\,l} \\
&\quad \cdot \left( w_l + \overline{h}_l (\overline{h}'_l \overline{h}_l)^{-1} \overline{\phi}'_l \tilde{\theta}_l - \sum_{m=l}^{n-2} \overline{\nu}_{ml} z_m \right) && \text{(15d)}\\
\dot{\tilde{\theta}}_{n-1} &= -\dot{\hat{\theta}}_{n-1} = -\Sigma_{n-1} \overline{\phi}_{n-1} (\overline{h}'_{n-1} \overline{h}_{n-1})^{-1} \overline{h}'_{n-1} \\
&\quad \cdot (w_{n-1} + \overline{h}_{n-1} (\overline{h}'_{n-1} \overline{h}_{n-1})^{-1} \overline{\phi}'_{n-1} \tilde{\theta}_{n-1}) && \text{(15e)}\\
\dot{\Sigma}_{n-1} &= -\Sigma_{n-1} (\overline{\phi}_{n-1} (\overline{h}'_{n-1} \overline{h}_{n-1})^{-1} \overline{\phi}'_{n-1} - \overline{Q}_{n-1}) \Sigma_{n-1} && \text{(15f)}\\
\dot{z}_n &= -\overline{k}_n z_n - z_{n-1} + 2\gamma^2 \sum_{l=1}^{n} \overline{\nu}'_{nl} \\
&\quad \cdot \left( w_l + \overline{h}_l (\overline{h}'_l \overline{h}_l)^{-1} \overline{\phi}'_l \tilde{\theta}_l - \sum_{m=l}^{n-1} \overline{\nu}_{ml} z_m \right) && \text{(15g)}\\
\dot{\tilde{\theta}}_n &= -\dot{\hat{\theta}}_n = -\Sigma_n \overline{\phi}_n (\overline{h}'_n \overline{h}_n)^{-1} \overline{h}'_n \\
&\quad \cdot (w_n + \overline{h}_n (\overline{h}'_n \overline{h}_n)^{-1} \overline{\phi}'_n \tilde{\theta}_n) && \text{(15h)}\\
\dot{\Sigma}_n &= -\Sigma_n (\overline{\phi}_n (\overline{h}'_n \overline{h}_n)^{-1} \overline{\phi}'_n - \overline{Q}_n) \Sigma_n && \text{(15i)}
\end{aligned}$$

Introduce the vector

$$\zeta := [z' \quad \tilde{\theta}' \quad \vec{\Sigma}'_1 \quad \cdots \quad \vec{\Sigma}'_n]'. \tag{16}$$

In terms of this notation, the set of (15) can be written in compact form as

$$\dot{\zeta} = \overline{F}\zeta + \overline{H}w \tag{17}$$

where the nonlinear functions $\overline{F}$ and $\overline{H}$ are defined accordingly. The derivative of $\overline{V} + W$ is given by

$$\begin{aligned}\dot{\overline{V}} + \dot{W} = &-z_1^2 - \sum_{l=1}^{n} \overline{\beta}_l z_l^2 - \gamma^2 \sum_{i=1}^{n} |\tilde{\theta}_i|^2_{\overline{Q}_i} \\ &+ \gamma^2 |w|^2 - \gamma^2 \sum_{l=1}^{n} \left| v_l - \sum_{m=l}^{n} \overline{\nu}_{ml} z_m \right|^2\end{aligned}$$

and hence $\overline{V} + W$ satisfies the HJI equation

$$(\overline{V} + W)_\zeta \overline{F}\zeta + \frac{1}{4\gamma^2} (\overline{V} + W)_\zeta \overline{H}\overline{H}' (\overline{V} + W)'_\zeta + \overline{L} = 0 \tag{18}$$

where $\overline{L} = z_1^2 + \Sigma_{l=1}^n \overline{\beta}_l z_l^2 + \gamma^2 \Sigma_{i=1}^n |\tilde{\theta}_i|^2_{\overline{Q}_i}$. This implies that the control law (14) is asymptotically tracking with disturbance attenuation level $\gamma$.

To remove the dependence of the identifier on the derivative information of the state variables $x$, we will supply the controller (14) with the $\hat{\theta}_i$ and $\Sigma_i$, $i = 1, \cdots, n$, that are generated by the UPFSI identifier (6). To guarantee the robustness of the resulting closed-loop system, we invoke the following two assumptions.

*Assumption A4:* There exists a positive constant $\kappa_Q$, such that for some $\check{Q}_i \geq 0$

$$Q_i = \Sigma_i^{-1} \Delta_i \Sigma_i^{-1} + \check{Q}_i, \qquad \Delta_i > \kappa_Q I_{r_i}$$

where the matrices $\Delta_i$'s and $\check{Q}_i$'s are functions of $(x_1, \cdots, x_i, \hat{\theta}_1, \cdots, \hat{\theta}_i, \Sigma_1, \cdots, \Sigma_{i-1})$, and the inequalities hold for all values of these variables. □

*Assumption A5:* There exists a constant $\kappa_\beta > 0$ such that $\overline{\beta}_i > \kappa_\beta$; $i = 1, \cdots, n$, for all values of their arguments. □

Using the special structure of the matrices $Q_i$'s, as given in *A4*, the identifier dynamics can be expressed as

$$\begin{aligned}
\dot{\hat{\theta}}_i &= \Sigma_i \phi_i (h'_i h_i)^{-(1/2)} e_i && \text{(19a)}\\
\dot{\Sigma}_i &= -\Sigma_i (\phi_i (h'_i h_i)^{-1} \phi'_i - \check{Q}_i) \Sigma_i + \Delta_i && \text{(19b)}\\
\epsilon \dot{e}_i &= -(h'_i h_i)^{1/2} e_i + \phi'_i \tilde{\theta}_i + h'_i w_i. && \text{(19c)}
\end{aligned}$$

This then makes the identifier prescribed here correspond to the worst case ($H^\infty$) identifier obtained in [33, Section 7], for the time-varying parameter case.

*Remark III.2: Assumption A4* on the specific form of the design matrices $Q_i$'s is introduced above for two reasons. On the one hand, with this specific structure of $Q_i$'s, the covariance matrices $\Sigma_i$'s are bounded away from zero (from below) uniformly. Therefore, it is no longer necessary to use covariance resetting to enable the identifier to track the slowly time-varying parameters. This benefit is also evident from the fact that the proposed identifier structure corresponds to the worst case identifier for the time-varying parameter case, as in [33]. Furthermore, the convergence rate of the parameter estimates is guaranteed to be exponential as long as the $\Sigma_i$'s are uniformly upper bounded (which is the persistency of excitation condition in this context), which is a critical step in the proof of the main result of this section, as we will shortly see. On the other hand, when the design matrices $Q_i$'s are positive definite, it then mandates persistent excitations in the measurements to keep the $\Sigma_i$'s uniformly bounded from above. In fact, a simple choice of letting $Q_i$'s be time-invariant positive-definite matrices exposes the $\Sigma_i$ dynamics to possible finite escapes in the absence of sufficient excitation. Accordingly, the positive-definite components, $\Delta_i$'s, of $Q_i$'s are scaled by $\Sigma_i^{-1}$'s. Consequently, the covariance matrices will be bounded above for an arbitrary level of excitation, even though the precise upper bounds will in general depend on the level of excitation in the closed-loop system. □

In view of the corresponding result of [33, Th. 9], we consider the following set of admissible uncertainty triples, for an arbitrary positive constant $C$:

$$\mathcal{W}_C := \{(x(0), \theta, w_{[0,\infty)})\colon \Sigma_i(t) \le CI_{r_i}, |x(0)| \le C, \\ |\theta| \le C, |w(t)| \le C, \forall t \in [0,\infty), \forall i = 1, \cdots, n.\} \tag{20}$$

We can employ a singular perturbation analysis to establish the following robustness property of the proposed adaptive controller under the above class of uncertainties.

*Theorem III.1:* Consider the nonlinear system (1), and let $\gamma > 0$ be fixed. Let *Assumptions A1–A5* hold, and the set $\mathcal{W}_C$ be defined as in (20) for some $C > 0$. Then we have the following.

1) There exists a positive scalar $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$, the control law defined by (14), with identifier (19), achieves asymptotic tracking with disturbance attenuation level $\gamma$ for any uncertainty triple in the set $\mathcal{W}_C$. Furthermore, the closed-loop signals $z$, $\tilde{\theta}$, and $e$ are uniformly bounded on $[0, \infty)$.
2) For any uncertainty triple in the set $\mathcal{W}_C$ such that $w_{[0,\infty)} \equiv 0$, the expanded state vector $(z', \tilde{\theta}', e')'$ converges to zero as $t \to \infty$ for any $\epsilon \in [0, \epsilon_0]$.

*Proof:* Introduce the value functions $\overline{V}$ and $W$, defined exactly as before. Let $\zeta$ again denote the expanded state vector defined as in (16), and let nonlinear functions $\overline{F}$ and $\overline{H}$ be defined exactly as in (17). In terms of state variables $\zeta$ and $e$, the closed-loop system under the control law (14) and identifier (19) can be expressed as

$$\dot{\zeta} = \overline{F}_{11}(\zeta) + \overline{F}_{12}(\zeta)e + \overline{H}_1(\zeta)w$$
$$\epsilon\dot{e} = \overline{F}_{21}(\zeta) + \overline{F}_{22}(\zeta)e + \overline{H}_2(\zeta)w$$

where

$$\overline{F}_{22}(\zeta) := \begin{bmatrix} -(\overline{h}_1'\overline{h}_1)^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -(\overline{h}_n'\overline{h}_n)^{1/2} \end{bmatrix}$$
$$\overline{H}_2(\zeta) := \begin{bmatrix} \overline{h}_1' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{h}_n' \end{bmatrix}, \qquad \overline{F}_{21}(\zeta) := \begin{bmatrix} \overline{\phi}_1'\tilde{\theta}_1 \\ \vdots \\ \overline{\phi}_n'\tilde{\theta}_n \end{bmatrix}.$$

It is not necessary here to explicitly write down the analytic forms of the functions $\overline{F}_{11}$, $\overline{F}_{12}$, and $\overline{H}_1$, which turn out to be quite complicated. It is important to note, however, that these nonlinear functions all belong to $\mathcal{C}^1$ and further satisfy the following two relationships:

$$\overline{F}\zeta = \overline{F}_{11} - \overline{F}_{12}\overline{F}_{22}^{-1}\overline{F}_{21}, \qquad \overline{H} = \overline{H}_1 - \overline{F}_{12}\overline{F}_{22}^{-1}\overline{H}_2.$$

These two algebraic relationships are obtained using the fact that the slow manifold of the above singularly perturbed dynamics is exactly the dynamics (17). These nonlinear singularly perturbed dynamics are linear in the fast state variable $e$ and the disturbance input $w$. Time-scale decomposition and robust control of this class of systems has been studied extensively in the earlier paper [39]. Motivated by the results of that paper, we introduce here another function, $\overline{\Xi}(\zeta, e)$, associated with the fast dynamics:

$$\overline{\Xi}(\zeta, e) := \gamma^2 \left|e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\,\overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1' \cdot (\overline{V} + W)_\zeta'\right|^2_{(\overline{H}_2\overline{H}_2')^{-(1/2)}}.$$

Let the overall value function be $\overline{\Upsilon} := \overline{V} + W + \epsilon\overline{\Xi}$. Then, the derivative of $\overline{\Upsilon}$ along the closed-loop trajectory can be written as (after some lengthy algebraic manipulations)

$$\begin{aligned}\dot{\overline{\Upsilon}} = & -z_1^2 - \sum_{l=1}^n \overline{\beta}_l z_l^2 - \gamma^2 \sum_{l=1}^n |\tilde{\theta}_l|^2_{Q_l} + \epsilon \sum_{l=1}^n \frac{\partial\overline{\Xi}}{\partial y_d^{(l-1)}} y_d^{(l)} \\ & + \gamma^2|w|^2 - \gamma^2\left|w - \frac{1}{2\gamma^2}\,\overline{\sigma}\right|^2 - \gamma^2\left|e + \overline{F}_{22}^{-1}\overline{F}_{21}\right. \\ & \left. + \frac{1}{2\gamma^2}\,\overline{F}_{22}^{-1}\overline{H}_2\overline{H}'(\overline{V} + W)_\zeta'\right|^2 + \frac{\epsilon^2}{4\gamma^2}\,\overline{\Xi}_\zeta\overline{H}_1\overline{H}_1'\overline{\Xi}_\zeta' \\ & + \frac{\epsilon}{2\gamma^2}\,\overline{\Xi}_\zeta\overline{H}_1(\overline{H}_1'(\overline{V} + W)_\zeta' \\ & + 2\gamma^2\overline{H}_2'(\overline{H}_2\overline{H}_2')^{-(1/2)} \\ & \cdot \left(e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\,\overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1'(\overline{V} + W)_\zeta'\right)\end{aligned}$$

where the partial differential equation (18) has been used in the derivation, and the function $\overline{\sigma}$ is defined as

$$\begin{aligned}\overline{\sigma} := & \overline{H}_1'(\overline{V} + W)_\zeta' \\ & - 2\gamma^2\overline{H}_2'\overline{F}_{22}^{-1}\left(e + \overline{F}_{22}^{-1}\overline{F}_{21}\frac{1}{2\gamma^2}\,\overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1'\right. \\ & \left. + (\overline{V} + W)_\zeta'\right) + \epsilon\overline{H}_1'\overline{\Xi}_\zeta.\end{aligned}$$

Consider the following time-varying set:

$$\Omega(t, M) := \{(\zeta, e)\colon \Sigma_i(t) \le CI_{r_i}, \overline{\Upsilon} \le M, \\ \forall i = 1, \cdots, n\} \tag{21}$$

where the positive scalar $M$ is sufficiently large such that

$$M \ge \frac{2\gamma^2 C^2}{\min\left\{\kappa_\beta, \dfrac{\kappa_Q}{2C}\right\}} \quad \text{and} \quad (\zeta(0), e(0)) \in \Omega(0, M).$$

The significance of the first inequality above will be seen shortly.

The first statement of the theorem can be established by proving the following two statements.

1) If the state of the closed-loop system, $(\zeta(t), e(t))$, belongs to $\Omega(t, M)$ for any $t \in [0, \infty)$, then the controller achieves asymptotic tracking with disturbance attenuation level $\gamma$ for any uncertainty triple in $\mathcal{W}_C$, and $(\zeta(t), e(t))$ is uniformly bounded, for sufficiently small values of $\epsilon$.
2) If the closed-loop system starts in $\Omega(0, M)$, then it belongs to $\Omega(t, M)$ for any $t \ge 0$ and for sufficiently small values of $\epsilon$.

Toward proving statement 1), suppose that $(\zeta(t), e(t)) \in \Omega(t, M)$ for any $t \in [0, \infty)$, which implies that

$$|z| \le \sqrt{2M}, \qquad |\tilde{\theta}| \le \sqrt{CM}/\gamma.$$

Then, there exists a constant $\kappa_x > 0$ such that $|x(t)| \le \kappa_x$ for all $t \in [0, \infty)$, since $x$ can be expressed as a continuous function of $\zeta$ and $x_d$, which themselves are uniformly bounded. Under the working *Assumption A3*, corresponding to any uncertainty triple in $\mathcal{W}_C$, there exists a constant $\tilde{C} > 0$ such that $|e(t)| \le \tilde{C}$. Furthermore, the constant $\tilde{C}$ can be chosen to be independent of the parameter $\epsilon$ because of the diagonal structure of the dynamics for $e$ and *Assumption A2*. Because of the uniform boundedness of $x(\cdot)$, the covariance matrices $\Sigma_i(t)$'s are strictly larger than or equal to, in matrix sense, the solutions of the following Riccati differential equations:

$$\dot{\Pi}_i = -C_r \Pi_i \Pi_i + \kappa_Q I_{r_i}$$
$$\Pi_i(0) = Q_{0i}^{-1}, \qquad i = 1, \cdots, n$$

where $C_r$ is a positive constant such that

$$C_r I_{r_i} \ge \phi_i (h_i' h_i)^{-1} \phi_i' - \check{Q}_i, \qquad \forall \zeta, i.$$

Hence, there exists a positive constant $\check{C} > 0$, which generally depends on the value of the constant $C$, such that

$$\Sigma_i(t) \ge \check{C} I_{r_i} \qquad \forall t \in [0, \infty), \qquad i = 1, \cdots, n.$$

The covariance matrices $\Sigma_i$'s are then upper and lower bounded in the following fashion:

$$\frac{1}{\check{C}} I_{r_i} \ge \Sigma_i^{-1} \ge \frac{1}{C} I_{r_i}$$

over the entire time interval $[0, \infty)$.

Utilizing the above bounds on $(\zeta, e)$ and $\Sigma_i$'s, there further exist positive scalar constants $C_1, C_2, \cdots$, such that the following growth conditions are satisfied on $\Omega(t, M)$ for any $t \in [0, \infty)$ and any uncertainty triple in $\mathcal{W}_C$:

$$|(\overline{V} + W)_\zeta| \le C_1(|z| + |\tilde{\theta}|)$$
$$|\Xi_\zeta| \le C_2 \left( \left| e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}' \cdot (\overline{V} + W)_\zeta' \right| + |z| + |\tilde{\theta}| \right)$$

and, for each $i = 1, \cdots, n$,

$$\left| \frac{\partial}{\partial y_d^{(i-1)}} \, (\overline{F}_{22}^{-1}\overline{F}_{21}) \right| = |(*) \cdot \tilde{\theta}| \le C_3 |\tilde{\theta}|$$

$$\left| \frac{\partial}{\partial y_d^{(i-1)}} \left( \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1'(\overline{V} + W)_\zeta' \right) \right| = |(*) \cdot z + (*) \cdot \tilde{\theta}| \le C_4(|z| + |\tilde{\theta}|)$$

$$\begin{aligned} \left| \frac{\partial \Xi}{\partial y_d^{(i-1)}} \right| = & \Big| 2\gamma^2 (e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1' \\ & \cdot (\overline{V} + W)_\zeta')'(\overline{H}_2\overline{H}_2')^{-(1/2)} \frac{\partial}{\partial y_d^{(i-1)}} \\ & \cdot \left( \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1'(\overline{V} + W)_\zeta' \right) \\ & + \gamma^2 \Big| e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1' \\ & \cdot\ (\overline{V} + W)_\zeta' \Big|^2_{\partial(\overline{H}_2\overline{H}_2')^{-(1/2)}/\partial y_d^{(i-1)}} \Big| \\ \le & C_5 \left| e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}'(\overline{V} + W)_\zeta' \right|^2 \\ & + C_5(|z|^2 + |\tilde{\theta}|^2) + C_6 |e + \overline{F}_{22}^{-1}\overline{F}_{21} \\ & + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}_1'(\overline{V} + W)_\zeta')|^2 \\ \le & C_7 |e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2} \overline{F}_{22}^{-1}\overline{H}_2\overline{H}'(\overline{V} + W)_\zeta'|^2 \\ & + C_8(|z|^2 + |\tilde{\theta}|^2) \end{aligned}$$

where $(*)$ denotes any continuous nonlinear function, of no direct interest to us in this derivation. These inequalities imply that, for sufficiently small values of $\epsilon > 0$

$$\begin{aligned} \dot{\overline{\Upsilon}} \le & -z_1^2 - \frac{1}{2} \sum_{l=1}^{n} \overline{\beta}_l z_l^2 - \frac{\gamma^2}{2} \sum_{l=1}^{n} |\tilde{\theta}_l|^2_{Q_l} + \gamma^2 |w|^2 \\ & - \gamma^2 \left| w - \frac{1}{2\gamma^2}\, \overline{\sigma} \right|^2 - \frac{\gamma^2}{2} \Big| e + \overline{F}_{22}^{-1}\overline{F}_{21} \\ & + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}'(\overline{V} + W)_\zeta' \Big|^2 \end{aligned}$$

for any uncertainty triple in the set $\mathcal{W}_C$.

Let functions $l$ and $l_0$ be such that

$$\begin{aligned} \bar{l} = & \frac{1}{2} \sum_{l=1}^{n} \overline{\beta}_l z_l^2 + \frac{\gamma^2}{2} \sum_{l=1}^{n} |\tilde{\theta}_l|^2_{Q_l} \\ & + \frac{\gamma^2}{2} \left| e + \overline{F}_{22}^{-1}\overline{F}_{21} + \frac{1}{2\gamma^2}\, \overline{F}_{22}^{-1}\overline{H}_2\overline{H}'(\overline{V} + W)_\zeta' \right|^2 \\ \bar{l}_0 = & \overline{V}(0) + \epsilon \overline{\Xi}(0). \end{aligned}$$

Then the performance inequality (2) is satisfied for $\epsilon > 0$ sufficiently small, and this establishes statement 1).

For statement 2), let us fix any $t \in [0, \infty)$ and any $(\zeta, e) \in \Omega$ such that $\overline{\Upsilon}(\zeta, e) = M$. Then the result will follow if we can prove that $\dot{\overline{\Upsilon}} \le 0$ for any uncertainty triple in $\mathcal{W}_C$. By the definition of the function $\overline{\Upsilon}$ and the fact that $\overline{\Upsilon}(\zeta, e) = M$, we have either $\overline{V} + W \ge M/2$ or $\epsilon \overline{\Xi} \ge M/2$. In the former case, we have the following set of inequalities, under the working

*Assumptions A4 and A5*:

$$\begin{aligned}\dot{\overline{\Upsilon}} &\leq -\frac{\kappa_\beta}{2}|z|^2 - \frac{\gamma^2\kappa_Q}{2}\sum_{l=1}^{n}|\Sigma_i^{-1}\tilde{\theta}|^2 + \gamma^2 C^2 \\ &\leq -\min\left\{\kappa_\beta, \frac{\kappa_Q}{2C}\right\}\left(\frac{1}{2}|z|^2 + \gamma^2 C\sum_{l=1}^{n}|\Sigma_i^{-1}\tilde{\theta}|^2\right) + \gamma^2 C^2 \\ &\leq -\min\left\{\kappa_\beta, \frac{\kappa_Q}{2C}\right\}\left(\frac{1}{2}|z|^2 + \gamma^2\sum_{l=1}^{n}|\Sigma_i^{-1}\tilde{\theta}|^2_{\Sigma_i}\right) + \gamma^2 C^2 \\ &\leq -\min\left\{\kappa_\beta, \frac{\kappa_Q}{2C}\right\}M/2 + \gamma^2 C^2 \leq 0\end{aligned}$$

which yields the nonpositiveness of the derivative of $\overline{\Upsilon}$. In the latter case, the uniform bounds on $\zeta$ and $x$ imply that $|e| = O(\epsilon^{-(1/2)})$, and further $\dot{\overline{\Upsilon}} \leq 0$ for sufficiently small values of $\epsilon$. This establishes statement 2) and therefore the first statement of the theorem.

For any uncertainty triple in $\mathcal{W}_C$ such that $w_{[0,\infty)} \equiv 0$, the signals $z, \tilde{\theta}$, and $e+\overline{F}_{22}^{-1}\overline{F}_{21}+(1/2\gamma^2)\overline{F}_{22}^{-1}\overline{H}_2\overline{H}'(\overline{V}+W)'_\zeta$ all belong to $\mathcal{L}_2$, for sufficiently small values of $\epsilon$. Furthermore, their derivatives are uniformly bounded for any fixed value of $\epsilon$. Hence, these signals converge to zero asymptotically. This verifies the second statement of the theorem, and thus completes its proof. □

*Remark III.3:* We note that the controller (14) depends on the covariance information of the identifier and therefore is not a certainty-equivalent controller. Yet, it is asymptotically equivalent to a controller designed with fixed parameters, as $\Sigma_1 \to 0, \cdots, \Sigma_n \to 0$. On the other hand, with the design parameter matrices $Q_i$'s chosen as in *A4*, the error covariance matrices $\Sigma_i$'s become uniformly bounded from below by a constant positive-definite matrix when the system state $x$ is uniformly bounded from above. The controller (14) will not converge to a certainty equivalent controller in this case. When the covariance matrices $\Sigma_i$'s, as well as their derivatives, are small (in the spectral radius sense), the behavior of the controller is close to that of the certainty-equivalent one. □

*Remark III.4:* Consider the class of nonlinear systems

$$\dot{x}_1 = x_2 + f_1(x_1) + \phi_1'(x_1)\theta + h_1'(x_1)w \tag{22a}$$

$$\vdots \quad \vdots$$

$$\dot{x}_{n-1} = x_n + f_{n-1}(x_1,\cdots,x_{n-1}) + \phi_{n-1}'(x_1)\theta + h_{n-1}'(x_1)w \tag{22b}$$

$$\dot{x}_n = f_n(x_1,\cdots,x_n) + \phi_n'(x_1)\theta + b(x_1,\cdots,x_n)u + h_n'(x_1)w \tag{22c}$$

$$y = x_1 \tag{22d}$$

where the same disturbance $w$ enters all subsystems, but the nonlinear functions $h_i$ and $\phi_i$ have only $x_1$ as their arguments. For the results of this section to be directly applicable to this class of systems, overparameterization and overconservativeness have to be introduced in order to transform the system into the noise-prone parametric strict-feedback from (1). This can be avoided, however, by utilizing the special structure of the system, where the functions $\phi_i$, and $h_i$, $i = 1,\cdots,n$, depend only on $x_1$ and are $\mathcal{C}^{n+1}$. To avoid singularity in identification, we must assume that the set of vectors $(h_1,\cdots,h_n)$ are linearly independent for all $x_1 \in \mathbb{R}$. The parameter identifier for this class of systems depends only on the variable $x_1$, which then allows for a backstepping procedure similar to the one described in Lemma III.1 to be employed in the design of an asymptotically tracking and disturbance attenuating controller. See Appendix B for the design equations for this class of systems, and Section IV for a numerical example involving this type of a nonlinear system. □

## IV. An Example

To illustrate the design procedure developed in the previous sections, we consider a third-order nonlinear system with an unknown scalar parameter $\theta$. To further corroborate the statement of Remark III.4, the system is taken to be of the special form (22), for which the relevant design equations can be found in Appendix B

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta x_1^2 + [1 \quad x_1 \quad 0]w \\ \dot{x}_2 &= x_3 + \theta x_1 + [0 \quad 1 \quad 0]w \\ \dot{x}_3 &= u + [1+x_1 \quad -x_1 \quad 1]w \\ y &= x_1.\end{aligned}$$

For this system, the reference signal $y_d$ is generated by a third-order internal model with transfer function

$$\frac{1}{(s+1)^3}.$$

In state space, the reference model can be represented by

$$\begin{aligned}\dot{x}_{d1} &= x_{d2} \\ \dot{x}_{d2} &= x_{d3} \\ \dot{x}_{d3} &= d - x_{d1} - 3x_{d2} - 3x_{d3} \\ y_d &= x_{d1}\end{aligned}$$

where $d$ is a command signal, taken to be a step function, $d \equiv 1$.

The initial states for the plant and reference model are

$$x = [0 \quad 0 \quad 0]', \qquad x_d = [0 \quad 0 \quad 0]'.$$

The true value of the parameter $\theta$ is taken to be 1. The desired attenuation level with respect to the disturbance $w$ is taken to be 3.

*KPFSI Case:* As a benchmark for comparison, we first designed a robust disturbance attenuating controller $\mu(x, x_d, \theta, d)$, which uses full state measurements and full knowledge of the parameter $\theta$. This was done by employing recursively the backstepping procedure described in Lemma III.1, and by picking the design parameter $\gamma = 3$, which is the guaranteed level of disturbance attenuation. Due to its complexity, we are not including here the explicit form of the controller.

*UPFSDI Case:* In this case, the parameter $\theta$ is unknown, but the derivative of the state is available. We took the initial guess to be $\overline{\theta} = -1$, with a confidence level of $Q_0 = 1/5$. The design parameter $Q$ was taken to be zero. Note that this choice of the parameter $Q$ corresponds to a standard least squares identifier. The identifier for the system in this case is given by

$$\begin{aligned}\dot{\hat{\theta}} =& \Sigma \begin{bmatrix} x_1^2 \\ x_1 \\ 0 \end{bmatrix}' \left( \begin{bmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 1+x_1 & -x_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1+x_1 \\ x_1 & 1 & -x_1 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \\ &\cdot \left( \dot{x} - \begin{bmatrix} x_2 + \hat{\theta} x_1^2 \\ x_3 + \hat{\theta} x_1 \\ u \end{bmatrix} \right) \\ \dot{\Sigma} =& -\Sigma \begin{bmatrix} x_1^2 \\ x_1 \\ 0 \end{bmatrix}' \left( \begin{bmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 1+x_1 & -x_1 & 1 \end{bmatrix} \right. \\ &\left. \cdot \begin{bmatrix} 1 & 0 & 1+x_1 \\ x_1 & 1 & -x_1 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} x_1^2 \\ x_1 \\ 0 \end{bmatrix} \Sigma\end{aligned}$$

where the initial states are set to $\hat{\theta}(0) = \overline{\theta}$ and $\Sigma(0) = 5$.

Using the expressions given in Appendix B, we arrived at an asymptotically tracking controller $\mu$ with disturbance attenuation level 3.

*UPFSI Case:* The parameter $\theta$ is again unknown. The only measurement available is the state history $x_{[0,t]}$. In this case, the design parameter $Q$ was chosen, according to *Assumption A4*, to be $0.1 + 0.1\Sigma^{-1}\Sigma^{-1}$, where the parameters $\Delta$ and $\check{Q}$ were both fixed at 0.1. Since the system output $y = x_1$ is to asymptotically track 1, there is enough excitation in the system to keep $\Sigma$ from escaping to infinity for a large class of disturbance signals. When the output $y$ tracks 1 exactly, the covariance $\Sigma$ will converge to $1/\sqrt{19} \approx 0.2294$, but not zero. We took the controller $\mu$ as designed in the UPFSDI case. The additional design parameter $\epsilon$ was fixed at 0.05. In this case, the identifier is given by, after some algebraic manipulations

$$\begin{aligned}\dot{\hat{\theta}} =& \Sigma \begin{bmatrix} x_1^2 \\ x_1 \\ 0 \end{bmatrix}' \begin{bmatrix} 2+2x_1+x_1^2 & -x_1(3+3x_1+x_1^2) & -1-x_1 \\ -x_1(3+3x_1+x_1^2) & 1+5x_1^2+4x_1^3+x_1^4 & x_1(2+x_1) \\ -1-x_1 & x_1(2+x_1) & 1 \end{bmatrix}^{1/2} \frac{1}{\epsilon}(x-\hat{x}) \\ \dot{\Sigma} =& -\Sigma(x_1^4+x_1^2-0.1)\Sigma + 0.1 \\ \dot{\hat{x}} =& \begin{bmatrix} x_2 + \hat{\theta} x_1^2 \\ x_3 + \hat{\theta} x_1 \\ u \end{bmatrix} \\ &+ \begin{bmatrix} 1+x_1^2 & x_1 & 1+x_1-x_1^2 \\ x_1 & 1 & -x_1 \\ 1+x_1-x_1^2 & -x_1 & 2(1+x_1+x_1^2) \end{bmatrix}^{1/2} \\ &\cdot \frac{1}{\epsilon}(x-\hat{x})\end{aligned}$$

where the initial conditions are $\hat{\theta}(0) = \overline{\theta}$, $\Sigma(0) = 5$, and $\hat{x}(0) = x(0)$.

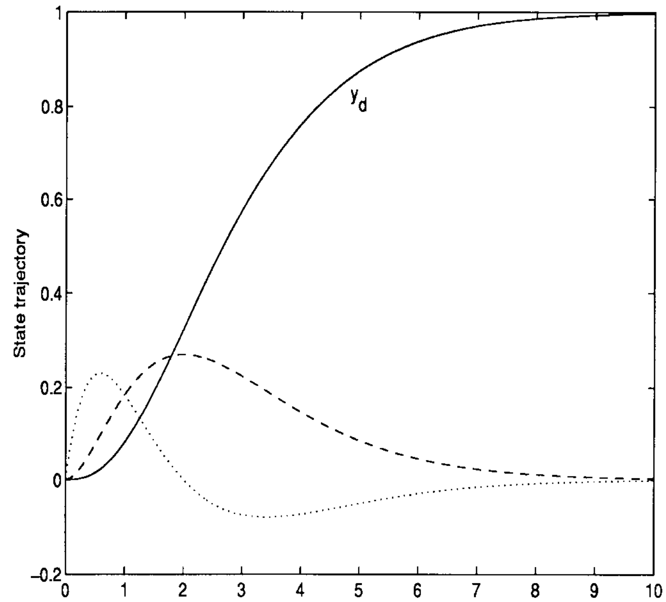

Fig. 1. State trajectory of the reference model.

The design process for the above three controllers consisted of programming in Mathematica, to arrive at the controller formula, followed by implementation in Matlab codes, and finally simulation of the closed-loop system using Simulink. The state trajectory of the reference model can be seen in Fig. 1.

We simulated the closed-loop system, under each of the three controllers, against two sets of disturbance inputs. First, the disturbances were set to zero, to demonstrate the command following property of the controller. Next, the disturbance inputs were picked as follows:

$$\begin{aligned} w_1(t) =& 0.1\sin(2\pi t/5), \qquad w_3(t) = 0.1\cos(\pi t) \\ w_2(t) =& \text{Band limited white noise signal} \\ & \text{with power 0.01 and sample rate 5 Hz.}\end{aligned}$$

The system response under the KPFSI controller is depicted in Fig. 2. When the disturbances are set to zero, the output $y$ tracks the reference signal exactly (the difference is in the order of $10^{-8}$). Under sinusoidal and white noise disturbance inputs, the system output still follows the reference signal, with the tracking error bounded by 0.4.

The system response under the UPFSDI controller is depicted in Fig. 3. Despite the large initial error of the parameter estimate, the system has a fair transient response which settles in less than 5 s. The parameter estimate $\hat{\theta}$ converges to the true value faster in the noisy case than in the noise-free case. This is due to the increased level of excitation for the noisy system. After the transient, the performance of the controller is similar to that in the KPFSI case for both sets of disturbances. The transient performance depicted here appears to be worse than that of the UPFSI case. This is because of the relatively slow convergence rate of the least squares identifier. As will be pointed out later, in the UPFSI case the controller makes use of the additional *a priori* excitation information about the system, which leads to performance improvement.

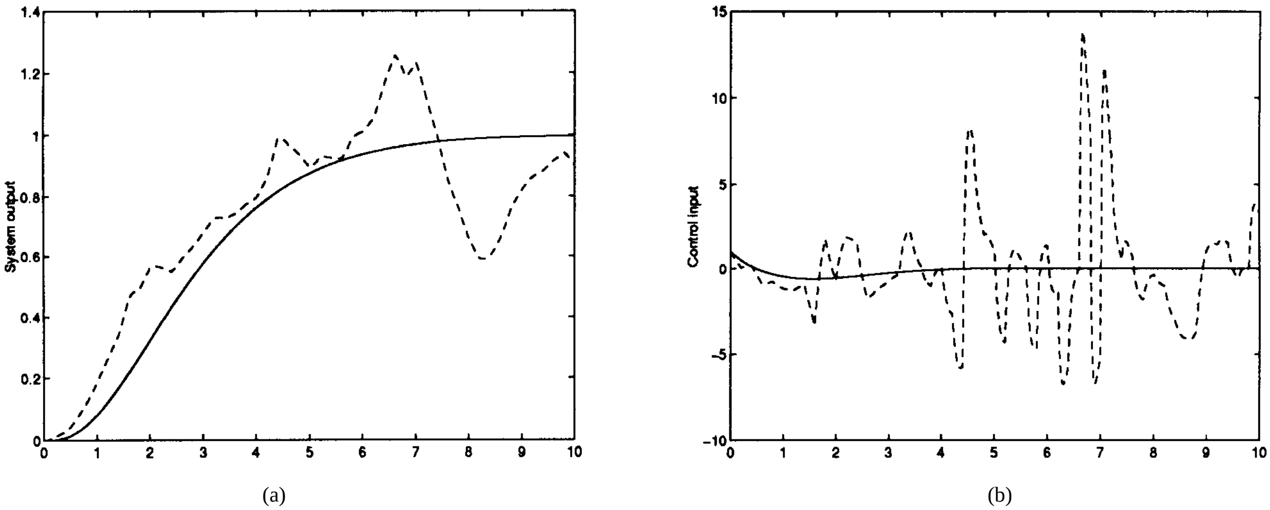


Fig. 2. System response under the KPFSI controller. (a) System output $y$ and (b) control input $u$. Solid line for noise-free case; dash line for noisy case.

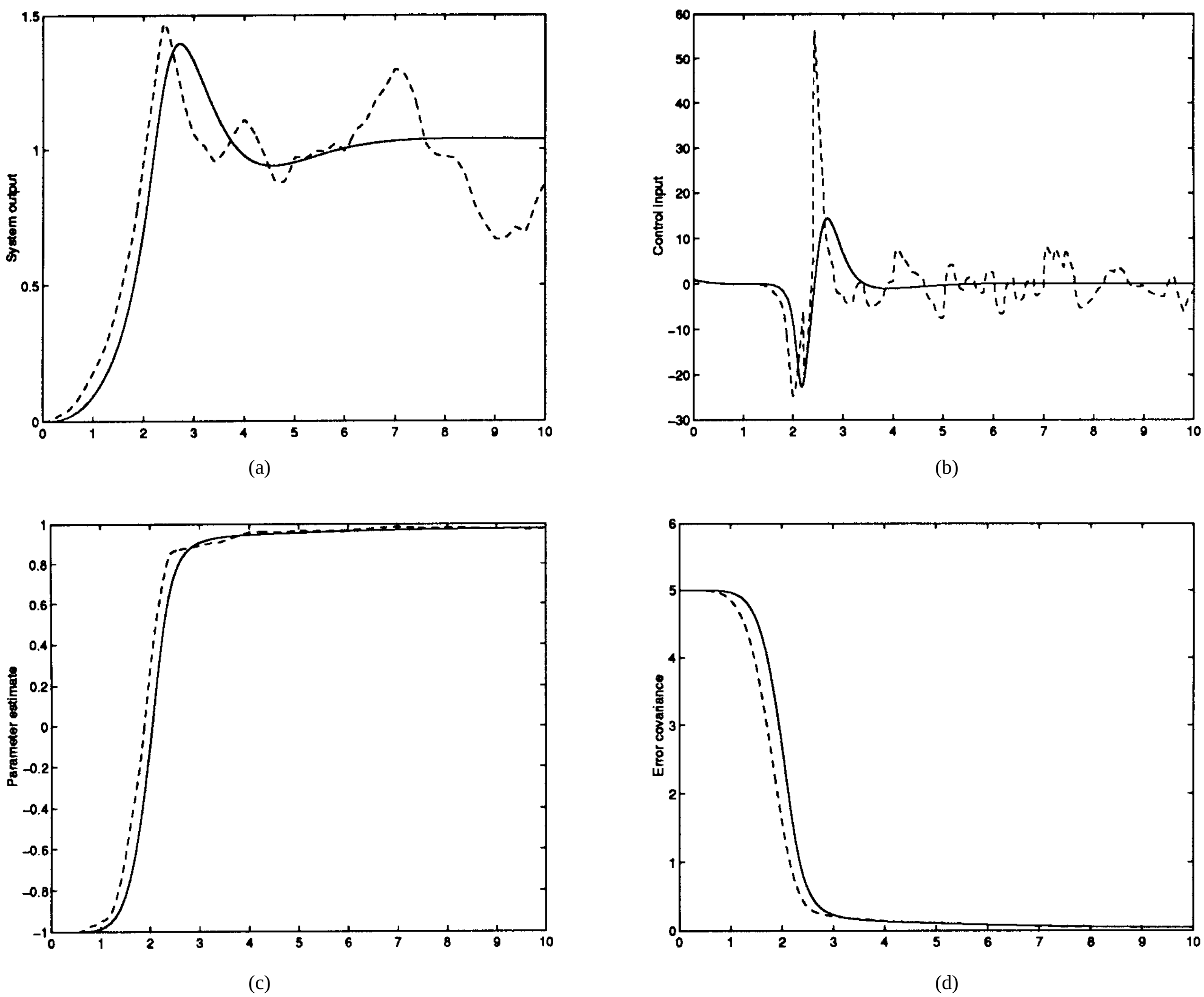


Fig. 3. System response under the UPFSDI controller. (a) System output $y$; (b) control input $u$; (c) parameter estimate $\hat{\theta}$; and (d) error covariance $\Sigma$. Solid line for noise-free case; dash line for noisy case.

The system response under the UPFSI controller is depicted in Fig. 4. The closed-loop system has a very good transient performance because of the fast convergence rate and robustness of the identifier. It is important to note that this desirable transient performance is made possible by utilizing the *a priori* information of the excitation level of the system. The choice of the parameter $Q$ also limits the set of admissible uncertainties. The performance of the controller is very similar to that in

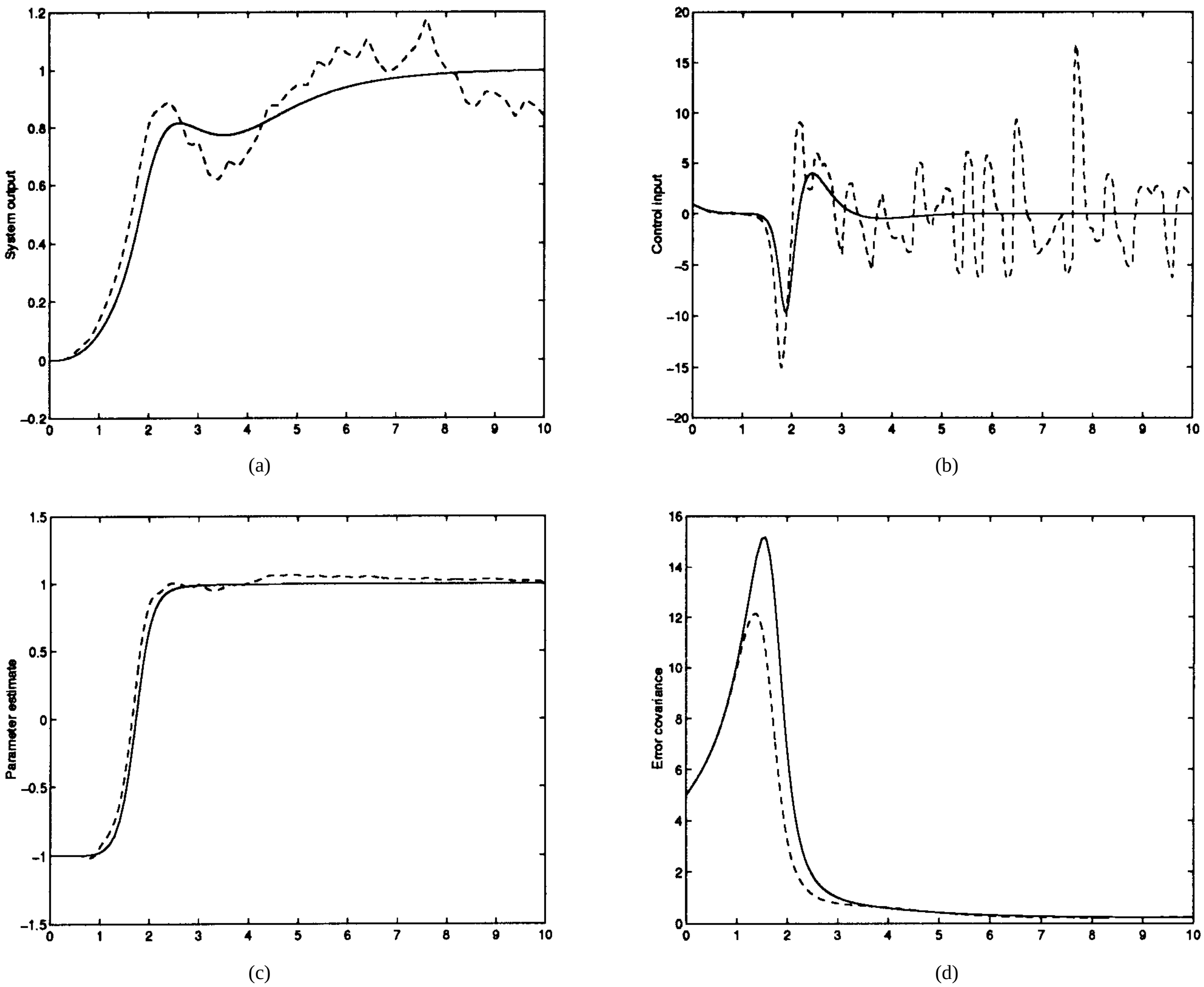


Fig. 4. System response under the UPFSI controller. (a) System output $y$; (b) control input $u$; (c) parameter estimate $\hat{\theta}$; and (d) error covariance $\Sigma$. Solid line for noise-free case; dash line for noisy case.

the UPFSDI case in steady state, which validates the singular perturbations analysis of Theorem III.1.

This example clearly illustrates the effectiveness of the controller design tool developed in this paper. In the face of an open-loop unstable plant (with possibility of a finite escape), the controller designed achieves both asymptotic tracking and disturbance attenuation with a moderate control effort. □

## V. Conclusions

For a class of SISO nonlinear systems described in noise-prone parametric strict-feedback form, we have developed design tools that lead to an explicit construction for a class of (robust adaptive) controllers that asymptotically track a given reference signal and achieve prespecified disturbance attenuation levels with respect to exogenous system inputs. We have presented an explicit design paradigm that leads to robust adaptive controllers with the following three appealing features:

1) asymptotic convergence to certainty-equivalent controllers as the identification error covariance approaches zero (in the UPFSDI case);
2) utilization of robust parameter identification schemes as basic building blocks;
3) attenuation of exogenous disturbance inputs to desired performance levels over the time interval $[0, \infty)$.

The design procedure developed is based on worst case identification, the integrator backstepping methodology, and singular perturbations analysis. The closed-loop system is shown to admit a closed-form value function that satisfies an associated HJI inequality, thereby guaranteeing a desired level of performance for the adaptive controller. We have shown that the certainty-equivalence principle holds in the strict sense only for first-order systems, whereas for higher order nonlinear systems it holds only asymptotically, as the confidence in the parameter estimates reaches infinity. A numerical example, included in Section IV, clearly demonstrates the superior performance of the controller designed.

Viewed as an $H^{\infty}$-control problem, we have here one of the rare situations where there exists an explicit solution to a genuine nonlinear problem with partial information (note that, the parameters, which also constitute state, are not directly measurable). The controllers are nonlinear and are parametrically defined with one of these parameters being the prespecified level of disturbance attenuation.

An immediate extension of the results developed here would be to the class of input–output linearizable systems whose zero dynamics are bounded-input/bounded-state stable with respect to control, disturbance, and the state variables $x_1, \cdots, x_r$, where $r$ is the relative degree. This includes, in particular, the minimum phase parametric uncertain linear systems.

The general results of this paper can also be immediately extended to the time-varying parameter case. In this case, the parameter vector $\theta$ would, for example, be generated by dynamics such as

$$\dot{\theta} = \delta, \qquad \theta(0) = \theta_0$$

where $\delta$ is an $r$ dimensional unknown exogenous input to the system. The performance criterion (2) would then include a negative cost penalty associated with the disturbance $\delta$:

$$-\gamma^2 \int_0^t |\delta(\tau)|^2 \, d\tau.$$

A parameter identifier appropriate for this case can be found in [33], which again facilitates a backstepping design for an asymptotically tracking and disturbance attenuating controller for the system.

Future research on this topic lies in several directions. One of these is the generalization of these results to the output feedback measurements scheme. Two possible subcases are:

1) only the output trajectory is available for feedback;
2) only noisy measurements are available.

Another direction would be the investigation of the case when there is an additional prescribed weighting with respect to the control input in the performance criterion (2), as in [40] but with the special nonlinear structure of the model adopted here. Yet a further direction of study would be to study robustness of these controllers to unmodeled dynamics.

## Appendix A
## Derivation of Worst Case Identifiers

In this Appendix, we present the derivations that lead to the UPFSI and UPFSDI identifiers used in Section III, as well as those used in the next section (Appendix B).

First, we derive the identifier in the UPFSDI case, that is when both state and derivative information are available. For the nonlinear system (1), we consider the following cost function (to be minimized):

$$\sup_{(x(0),\theta,w_{[0,\infty)})\in\mathcal{W}} \left\{ \int_0^t \left( \gamma^2 \sum_{i=1}^n |\theta_i - \hat{\theta}_i|^2_{Q_i} - \gamma^2 |w|^2 \right) d\tau - \gamma^2 |\theta - \overline{\theta}|^2_{Q_0} \right\} \tag{23}$$

where $\hat{\theta}_i$'s are the parameter estimates to be designed, $\gamma$ is a positive scalar, and $Q_i$'s are the positive-definite design parameter matrices.

When both the state and its derivative are available for feedback, we view $\dot{\theta} = 0$ as the dynamic equation and (1) as the set of measurements. Then following the *cost-to-come* function analysis that led to [33, Th. 1], it is rather straightforward to derive (9) as the worst case identifier minimizing (23). It should be noted that the worst case covariance matrices $\Sigma_i$'s here correspond to $\Sigma^{-1}$ in the notation of [33]. We also note that, in the present case, we have the following counterpart of [33, eq. (11)] as the cost-to-come function:

$$\begin{aligned} \check{W}(t, \theta, &x_{[0,t]}, \dot{x}_{[0,t]}, \hat{\theta}_{[0,t]}, u_{[0,t]}) \\ &= -\gamma^2 \sum_{i=1}^n |\theta - \check{\theta}|^2_{\Sigma_i^{-1}(t)} + m(t) \\ \dot{\check{\theta}} &= \Sigma_i^{-1} \phi_i (h_i' h_i)^{-1} (\dot{x}_i - \phi_i' \check{\theta}_i - \chi_i) + \Sigma_i Q_i (\check{\theta}_i - \hat{\theta}_i) \\ \check{\theta}(0) &= \overline{\theta}_i, \qquad i = 1, \cdots, n \\ \dot{m} &= \gamma^2 \sum_{i=1}^n |\check{\theta}_i - \hat{\theta}_i|^2_{Q_i} - \gamma^2 \sum_{i=1}^n |\dot{x}_i - \phi_i' \check{\theta}_i - \chi_i|^2_{(h_i' h_i)^{-1}} \end{aligned}$$

where $\chi_i$'s are defined by (7), and $\Sigma_i$'s satisfy the differential equations (9a). The choice of $\hat{\theta}_i = \check{\theta}_i$ then yields the identifier (9).

In the UPFSI case, on the other hand, the identifier (6) is obtained by the following process. First, we study the identification design under noise perturbed full state measurements—a measurement scheme that is informationally inferior to the UPFSI measurement. In other words, we first assume that we have the measurement equation

$$y_i = x_i + \epsilon \check{w}_i, \qquad i = 1, \cdots, n$$

where $\check{w}_i$ is a scalar measurement noise affecting the $i$th state variable. Under the UPFSI measurement scheme, we simply have $\check{w}_i \equiv 0$, $i = 1, \cdots, n$. This observation then allows us to rewrite the system dynamics as

$$\begin{aligned} \dot{x}_1 &= y_2 + f_1(y_1) + \phi_1'(y_1)\theta_1 + h_1'(y_1) w_1 \\ &\vdots \quad \vdots \\ \dot{x}_{n-1} &= y_n + f_{n-1}(y_1, \cdots, y_{n-1}) \\ &\quad + \phi_{n-1}'(y_1, \cdots, y_{n-1})\theta_{n-1} \\ &\quad + h_{n-1}'(y_1, \cdots, y_{n-1}) w_{n-1} \\ \dot{x}_n &= f_n(y_1, \cdots, y_n) + \phi_n'(y_1, \cdots, y_n)\theta_n \\ &\quad + b(y_1, \cdots, y_n) u + h_n'(y_1, \cdots, y_n) w_n \end{aligned}$$

where we have replaced $x_i$'s by $y_i$'s on the right-hand side (RHS) of the state equation (1). We associate with this system the cost function

$$\sup_{(x(0),\theta,w_{[0,\infty)})\in\mathcal{W}} \left\{ \int_0^t \left( \gamma^2 \sum_{i=1}^n |\theta_i - \hat{\theta}_i|^2_{Q_i} - \gamma^2 |w|^2 - \gamma^2 |\check{w}|^2 \right) d\tau - \gamma^2 |\theta - \overline{\theta}|^2_{Q_0} \right\}$$

where $\breve{w} := (\breve{w}_1, \cdots, \breve{w}_n)'$, and the weighting matrix $Q_i$ depends on $y_1, \cdots, y_i$ rather than $x_1, \cdots, x_i$, $i = 1, \cdots, n$.

The solution to this worst case identification problem can be obtained from [33], as a special case of Remark 4 of that reference, under the correspondence

$$F \hookrightarrow 0_{n\times n}, \qquad A \hookrightarrow \begin{bmatrix} \phi_1' \\ \vdots \\ \phi_n' \end{bmatrix}, \qquad b \hookrightarrow \begin{bmatrix} y_2 + f_1 \\ \vdots \\ y_n + f_{n-1} \\ f_n + bu \end{bmatrix}$$

$$C \hookrightarrow I_n, \qquad Q \hookrightarrow \gamma^2 Q, \qquad D \hookrightarrow \begin{bmatrix} h_1' \\ \vdots \\ h_n' \end{bmatrix}.$$

This is referred to in [33] as the full-order identifier. To arrive at a reduced-order identifier, we follow the development of [33] and replace the worst case covariance matrix with its limiting solution as $\epsilon \to 0$. We work here with the inverse of the worst case covariance matrix. First partition it into $2 \times 2$ subblocks, compatible with the partitioning of $(\theta', x')'$, to reveal the two-time-scale property as in [33, eq. (37)]. Setting $\epsilon = 0$ in the resulting equations yields the quasi-steady-state dynamics for the worst case covariance matrix. Using this quasi-steady-state dynamics with the state and parameter estimate dynamics yields the identifier (6).

A line of reasoning similar to the above yields the identifiers (24) and (25), under UPFSI and UPFSDI measurements, respectively, for the special class of nonlinear systems considered in Remark III.4.

## Appendix B
## Derivation Under Correlated Disturbances Without Overparameterization

In this appendix, we present explicit design equations for the special class of nonlinear systems considered in Remark III.4. We start by noting that in view of [33, Th. 7], and the discussion of Appendix A, a relevant identifier in this case is given by

$$\dot{\hat{\theta}} = \Sigma\phi(h'h)^{-(1/2)}\frac{1}{\epsilon}\,(x - \hat{x}), \qquad \hat{\theta}(0) = \overline{\theta} \tag{24a}$$

$$\dot{\Sigma} = -\Sigma(\phi(h'h)^{-1}\phi' - Q)\Sigma, \qquad \Sigma(0) = Q_0^{-1} \tag{24b}$$

$$\dot{\hat{x}} = \chi + \phi'\hat{\theta} + \frac{1}{\epsilon}\,(h'h)^{1/2}(x - \hat{x})$$
$$\hat{x}(0) = x(0) \tag{24c}$$

where

$$\phi(x_1) := [\phi_1(x_1) \quad \cdots \quad \phi_n(x_1)]'$$
$$h(x_1) := [h_1(x_1) \quad \cdots \quad h_n(x_1)]'$$
$$\chi(x, u) := [f_1 + x_2 \quad \cdots \quad f_n + bu]'.$$

$Q$ is an $r \times r$ dimensional positive-definite matrix to be chosen by the designer, and $\epsilon > 0$ is a small design parameter.

The quasi-steady-state dynamics of the identifier, as $\epsilon \to 0^+$, are given by

$$\dot{\hat{\theta}} = \Sigma\phi(h'h)^{-1}(\dot{x} - \chi - \phi'\hat{\theta}), \qquad \hat{\theta}(0) = \overline{\theta} \tag{25a}$$

$$\dot{\Sigma} = -\Sigma(\phi(h'h)^{-1}\phi' - Q)\Sigma, \qquad \Sigma(0) = Q_0^{-1} \tag{25b}$$

which is exactly the identifier under the UPFSDI measurement scheme.

Because of the simplified structure of the above quasi-steady-state dynamics of the identifier, the controller design will be based on these dynamics. After such a controller is obtained, the estimate $\hat{\theta}$ and the covariance $\Sigma$ are replaced by those generated by the UPFSI identifier (24), to form an implementable UPFSI adaptive controller. The performance and robustness of this UPFSI adaptive controller can then be established by using a singular perturbations analysis.

Under the UPFSDI identifier, the identification error $\tilde{\theta}$ satisfies the following dynamics:

$$\dot{\tilde{\theta}} = -\Sigma\phi(h'h)^{-1}h'v$$

where $v := w + h(h'h)^{-1}\phi'\tilde{\theta}$.

Introduce a candidate value function associated with the identifier

$$W(\tilde{\theta}, \Sigma) := \gamma^2|\tilde{\theta}|^2_{\Sigma^{-1}}$$

the derivative of which is given by

$$\dot{W} = -\gamma^2|\tilde{\theta}|^2_Q + \gamma^2|w|^2 - \gamma^2|v|^2.$$

Using this, we can (as in Section III) convert the attenuation problem with respect to $w$ to one with respect to $v$. In terms of $v$, dynamics (22) can be rewritten as

$$\dot{x} = \chi + \phi'\hat{\theta} + h'v.$$

Hence, a backstepping design process, which is similar to the one introduced in Section III, can be carried out here without the overparameterization and overconservativeness associated with the standard form (1). This again involves repeated use of Lemma III.1.

Fix $n$ nonnegative functions $\overline{\beta}_i$, $i = 1, \cdots, n$, which are the design parameters:

$$\overline{\beta}_i(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \hat{\theta}, \Sigma), \qquad i = 1, \cdots, n.$$

We can then define the following functions recursively, for $i = 1, \cdots, n$, where $z_0 = 0$ and $\overline{\alpha}_0 \equiv 0$ are introduced for notational consistency

$$z_i := x_i - y_d^{(i-1)} + \overline{\alpha}_{i-1}$$

$$\overline{\nu}_i(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \hat{\theta}, \Sigma)$$
$$:= \frac{1}{2\gamma^2}\,\overline{h}_i + \sum_{j=1}^{i-1}\frac{\partial\overline{\alpha}_{i-1}}{\partial z_j}\,\overline{\nu}_j + \frac{1}{2\gamma^2}\,\frac{\partial\overline{\alpha}_{i-1}}{\partial\hat{\theta}}\Sigma\overline{\phi}(\overline{h}'\overline{h})^{-1}\overline{h}'$$

$$\overline{k}_1(z_1, y_d, \hat{\theta}) := 1 + \overline{\beta}_1 + \gamma^2\overline{\nu}_1'\overline{\nu}_1$$

$$\overline{k}_i(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \hat{\theta}, \Sigma) := \overline{\beta}_i + \gamma^2\overline{\nu}_i'\overline{\nu}_i; \qquad i \geq 2$$

$$
\begin{aligned}
&\overline{\alpha}_i(z_1, \cdots, z_i, y_d, \cdots, y_d^{(i-1)}, \hat{\theta}, \Sigma) \\
&\quad := \overline{k}_i z_i + z_{i-1} + \overline{f}_i + \overline{\phi}_i' \hat{\theta} + \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial z_j} \\
&\qquad \cdot (z_{j+1} - \overline{k}_j z_j - z_{j-1}) + \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial y_d^{(j-1)}} y_d^{(j)} \\
&\qquad + 2\gamma^2 \sum_{j=1}^{i-1} \frac{\partial \overline{\alpha}_{i-1}}{\partial z_j} \overline{\nu}_j' \sum_{l=j}^{i-1} \overline{\nu}_l z_l - \frac{\partial \overline{\alpha}_{i-1}}{\partial \vec{\Sigma}} \\
&\qquad \cdot \overrightarrow{\Sigma(\overline{\phi}(\overline{h}'\overline{h})^{-1}\overline{\phi}' - \overline{Q})\Sigma} \\
&\qquad + \left( h_i' + \frac{\partial \overline{\alpha}_{i-1}}{\partial \hat{\theta}} \Sigma \overline{\phi} (\overline{h}'\overline{h})^{-1} \overline{h}' \right) \sum_{j=1}^{i-1} \overline{\nu}_j z_j.
\end{aligned}
$$

Then, the adaptive controller is given by

$$u = \tilde{\mu}(t, x_{[0,t]}, \dot{x}_{[0,t]}) = \frac{1}{b} \left(y_d^{(n)} - \overline{\alpha}_n\right) \tag{26}$$

and the value function for the closed-loop system becomes

$$W + \tfrac{1}{2} \sum_{i=1}^{n} z_i^2$$

which satisfies a corresponding HJI equality.

The implementable adaptive controller is formed by combining the control law (26) with the UPFSI identifier (24). As in Section III, we will make some structural assumptions on the weighting matrix $Q$ to guarantee the robustness of the closed-loop adaptive system. The parameter matrix $Q(x_1, \hat{\theta}, \Sigma)$ is selected to be of the form

$$Q(x_1, \hat{\theta}, \Sigma) = \Sigma^{-1} \Delta(x_1, \hat{\theta}) \Sigma^{-1} + \check{Q}(x_1, \hat{\theta})$$

where $\Delta(x_1, \hat{\theta}) \geq \kappa_Q I_r$.

Using this structure, the dynamics for the covariance matrix $\Sigma$ can be rewritten as

$$\dot{\Sigma} = -\Sigma(\phi(h'h)^{-1}\phi' - \check{Q})\Sigma + \Delta$$

which is again recognized to be the identifier for the time-varying parameter case (see [33, Th. 9]).

The admissible uncertainty set is then defined as follows, for an arbitrary constant $C > 0$:

$$
\begin{aligned}
\mathcal{W}_C := \{&(x(0), \theta, w_{[0,\infty)}) \colon |\Sigma(t)| \leq C, |x(0)| \leq C, \\
&|\theta| \leq C, |w(t)| \leq C, \forall t \in [0, \infty), \forall i = 1, \cdots, n\}.
\end{aligned}
$$

The counterpart of Theorem III.1 here can be established for this controller (as in the earlier case) under additional smoothness assumptions on the nonlinear functions $\phi$ and $h$, as delineated in Remark III.4.

This completes the derivation of the disturbance-attenuating adaptive controllers for the class of nonlinear systems (22).

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**Zigang Pan** (S'92–M'96) received the B.S. degree in automatic control from Shanghai Jiao Tong University in 1990 and the M.S. and Ph.D. degrees in electrical engineering from University of Illinois, Urbana-Champaign, in 1992 and 1996, respectively.

In 1996, he was a Research Engineer at the Center for Control Engineering and Computation at the University of California, Santa Barbara. In the same year, he joined the Polytechnic University, Brooklyn, NY, as an Assistant Professor in the Department of Electrical Engineering. His current research interests include perturbation theory for robust control systems, multivariable optimality-guided robust parameter identification, multivariable optimality-guided robust adaptive control, and applications of adaptive control systems.

Dr. Pan was a coauthor of a paper (with T. Başar) that received the 1995 George Axelby Best Paper Award. He is a member of SIAM, the AMS, and Phi Kappa Phi Honor Society.

**Tamer Başar** (S'71–M'73–SM'79–F'83) received the B.S.E.E. degree from Robert College in 1969, and the M.S., M.Phil., and Ph.D. degrees in engineering and applied science from Yale University, New Haven, CT, in 1972.

After being at Harvard University, Marmara Research Institute, and Bogaziçi University, he joined the University of Illinois, Urbana-Champaign, in 1981, where he is currently the Fredric and Elizabeth Nearing Professor of Electrical and Computer Engineering. He has spent sabbatical years at Twente University of Technology, the Netherlands, 1978–1979, and INRIA, France, 1987–1988 and 1994–1995. He has authored or coauthored over 150 journal articles and book chapters and also numerous conference publications, in the general areas of optimal, robust and nonlinear control, dynamic games, stochastic control, estimation theory, stochastic processes, information theory, and mathematical economics. He is coauthor of the texts *Dynamic Noncooperative Game Theory* (New York: Academic, 1982, 1995) and *$H^\infty$-Optimal Control and Related Minimax Design Problems* (Boston, MA: Birkhäuser, 1991, 1995).

Dr. Başar carries memberships in several scientific organizations, among which are SIAM, SEDC, and ISDG. Currently, he is the Deputy Editor-in-Chief of the IFAC journal *Automatica*, the Managing Editor of the *Annals* of the International Society of Dynamic Games, and Associate Editor of various journals, including the *Journal of Economic Dynamics and Control* and *Systems & Control Letters*. He is also a Vice-President of the IEEE Control Systems Society and a member of the IEEE Publications Board. Among some of the recent awards and recognitions he has received are: the Medal of Science of Turkey (1993), Distinguished Member Award of the IEEE Control Systems Society (1993), and the George S. Axelby Outstanding Paper Award of the same society (1995) for a 1994 paper he coauthored (with Z. Pan).