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Adaptive $\mathcal{H}_\infty$ Anti-Synchronization for Time-Delayed Chaotic Neural Networks

Choon Ki Ahn*)

Faculty of the Division of Electronics and Control Engineering, Wonkwang University, 344-2 Shinyong-dong, Iksan 570-749, Korea

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In this paper, an adaptive $\mathcal{H}_\infty$ control scheme is developed to study the anti-synchronization behavior of time-delayed chaotic neural networks with unknown parameters. This adaptive $\mathcal{H}_\infty$ anti-synchronization controller is designed based on Lyapunov-Krasovskii theory and an analytic expression of the controller with its adaptive laws of parameters is shown. The proposed synchronization method guarantees the asymptotical anti-synchronization of drive and response systems. Furthermore, this method reduces the effect of external disturbance to an $\mathcal{H}_\infty$ norm constraint. The proposed controller can be obtained by solving a linear matrix inequality (LMI) problem. An illustrative example is given to demonstrate the effectiveness of the proposed method.

Subject Index: 044, 055

§1. Introduction

Synchronization is a fundamental phenomenon that enables coherent behavior in coupled dynamical systems. Since the discovery of chaos synchronization by Pecora and Carroll,[1] there have been tremendous interests in studying the synchronization of various chaotic systems. It has been widely explored in a variety of fields including physical, chemical and ecological systems.[2] Another interesting phenomenon discovered was the anti-synchronization, which is noticeable in periodic oscillators. The anti-synchronization, which is the vanishing of the sum of the relevant state variables of synchronized systems, has been investigated both experimentally and theoretically in many physical systems.[3)–7)] A recent study of the anti-synchronization phenomenon in non-equilibrium systems suggests that the anti-synchronization could be exploited as a technique for particle separation in a mixture of interacting particles.[7] There have been trials on applying some control methods to anti-synchronize chaotic systems. In 5), a linear controller was constructed for anti-synchronizing coupled identical chaotic systems. Nonlinear anti-synchronization controllers for nonlinear gyros and non-identical chaotic ratchets were proposed in 6) and 7), respectively. Recently, Al-sawalha and Noorania[8] constructed nonlinear controllers for anti-synchronization between two identical and different chaotic systems.

Time-delay often appears in many physical systems such as aircraft, chemical, and biological systems. Unlike ordinary differential equations, time-delayed systems are infinite dimensional in nature and time-delay is, in many cases, a source of instability. The stability issue and the performance of time-delayed systems are, therefore,

*) Tel: +82-63-850-6884, Fax: +82-63-850-7315, E-mail: hironaka@wonkwang.ac.kr

both of theoretical and practical importance. Since Mackey and Glass[9] first found chaos in time-delay system, there has been increasing interest in time-delay chaotic systems.[10],[11] In this regard, some control methods, such as guaranteed cost control,[12],[13] delayed feedback control,[14] neural network approach,[15] and impulsive control,[16],[17] were proposed for synchronizing time-delayed chaotic systems.

In real physical systems, one is faced with model uncertainties and a lack of statistical information on the signals. This had led in recent years to an interest in mini-max control, with the belief that $\mathcal{H}_\infty$ control is more robust and less sensitive to disturbance variances and model uncertainties.[18] In order to reduce the effect of the disturbance, Ahn[19] firstly adopted the $\mathcal{H}_\infty$ control concept[18] for the anti-synchronization problem of chaotic systems. In this work, the $\mathcal{H}_\infty$ anti-synchronization between the drive and response systems is based on the known system parameters. But, in practical engineering situations, parameters are probably unknown and may change from time to time. When chaotic systems have some unknown parameters, it is generally difficult to anti-synchronize the chaotic systems. In this case, it is well known that the adaptive control scheme is an effective method for the chaos anti-synchronization. Thus, knowledge of the adaptive anti-synchronization for chaotic systems with unknown parameters is of considerable practical importance. Recently, controllers for the adaptive anti-synchronization were proposed in 20) and 21). However, these works were restricted to chaotic systems without external disturbances and time-delays. To the best of our knowledge, for the adaptive $\mathcal{H}_\infty$ anti-synchronization of chaotic systems with both external disturbances and time-delays, there is no result in the literature so far, which still remains open and challenging.

Motivated by the above discussion, our main aim in this paper is to shorten this gap by investigating the adaptive $\mathcal{H}_\infty$ anti-synchronization problem of chaotic neural networks with external disturbance and time-delay. A new controller with its adaptive laws of unknown parameters for the adaptive $\mathcal{H}_\infty$ anti-synchronization of time-delayed chaotic neural networks is proposed on the basis of Lyapunov-Krasovskii method and linear matrix inequality (LMI) approach. This controller is a new contribution to the topic of chaos anti-synchronization. The proposed controller can be obtained by solving the LMI problem. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms.[22]

This paper is organized as follows. In §2, we formulate the problem. In §3, an LMI problem for the adaptive $\mathcal{H}_\infty$ anti-synchronization of delayed chaotic neural networks is proposed. In §4, a numerical example is given, and finally, conclusions are presented in §5.

§2. Problem formulation

Consider a class of uncertain time-delayed chaotic neural networks

$$\dot{x}(t) = Ax(t) + \bar{A}x(t-\tau) + Bf(x(t)) + \bar{B}g(x(t-\tau))$$

$$+\sum_{k=1}^{p}\Phi_k(x(t))\theta_k+\sum_{l=1}^{q}\Psi_l(x(t-\tau))\phi_l, \tag{2·1}$$

where $x(t)\in R^n$ is the state vector, $\tau>0$ is the time-delay, $A\in R^{n\times n}$ is the self-feedback matrix, $\bar{A}\in R^{n\times n}$ is the delayed self-feedback matrix, $B\in R^{n\times n}$ is the connection weight matrix, $\bar{B}\in R^{n\times n}$ is the delayed connection weight matrix, $\Phi_k(x(t))$ $(k=1,\ldots,p):R^n\to R^{n\times r}$ and $\Psi_l(x(t))$ $(l=1,\ldots,q):R^n\to R^{n\times s}$ are nonlinear function matrices, and $\theta_k\in R^r$ $(k=1,\ldots,p)$ and $\phi_l\in R^s$ $(l=1,\ldots,q)$ represent the unknown constant parameter vectors. $f(x(t)):R^n\to R^n$ and $g(x(t)):R^n\to R^n$ are activation function vectors satisfying the global Lipschitz conditions with Lipschitz constants $L_f>0$ and $L_g>0$:

$$||f(x(t))-f(y(t))||\le L_f||x(t)-y(t)||,\qquad \forall x(t),y(t)\in R^n, \tag{2·2}$$

$$||g(x(t))-g(y(t))||\le L_g||x(t)-y(t)||,\qquad \forall x(t),y(t)\in R^n. \tag{2·3}$$

The system (2·1) is considered as a drive system. The anti-synchronization problem of system (2·1) is considered by using the drive-response configuration. According to the drive-response concept, the controlled response system is given by

$$\dot{\hat{x}}(t)=A\hat{x}(t)+\bar{A}\hat{x}(t-\tau)+Bf(\hat{x}(t))+\bar{B}g(\hat{x}(t-\tau))+u(t)+Gd(t), \tag{2·4}$$

where $\hat{x}(t)\in R^n$ is the state vector of the response system, $u(t)\in R^n$ is the control input, $d(t)\in R^k$ is the external disturbance, and $G\in R^{n\times k}$ is a known constant matrix. Define the anti-synchronization error $e(t)=\hat{x}(t)+x(t)$. Then we obtain the anti-synchronization error system

$$\begin{aligned}\dot{e}(t)&=Ae(t)+\bar{A}e(t-\tau)+B(f(\hat{x}(t))+f(x(t)))+\bar{B}(g(\hat{x}(t-\tau))+g(x(t-\tau)))\\&\quad+\sum_{k=1}^{p}\Phi_k(x(t))\theta_k+\sum_{l=1}^{q}\Psi_l(x(t-\tau))\phi_l+u(t)+Gd(t).\end{aligned} \tag{2·5}$$

Throughout this paper, we define that $\hat{\theta}_k(t)$ $(k=1,\ldots,p)$ and $\hat{\phi}_l(t)$ $(l=1,\ldots,q)$ are the estimate values of θ_k and ϕ_l, respectively.

Definition 1. *(Adaptive $\mathcal{H}_\infty$ anti-synchronization) With zero initial condition and a given level $\gamma>0$, the error system (2·5) is adaptively $\mathcal{H}_\infty$ anti-synchronized if the anti-synchronization error $e(t)$ satisfies*

$$\int_0^\infty e^T(t)Se(t)dt<\gamma^2\int_0^\infty d^T(t)d(t)dt, \tag{2·6}$$

under the control input $u(t)$ with the adaptive laws $\hat{\theta}_k(t)$ and $\hat{\phi}_l(t)$ $(k=1,\ldots,p,\ l=1,\ldots,q)$, where S is a positive symmetric matrix. The parameter γ is called the $\mathcal{H}_\infty$ norm bound or the disturbance attenuation level.

Definition 2. *(Adaptive asymptotical anti-synchronization) The error system (2·5) is adaptively asymptotically anti-synchronized if the anti-synchronization error $e(t)$ satisfies*

$$\lim_{t\to\infty}e(t)=0 \tag{2·7}$$

under the control input $u(t)$ with the adaptive laws $\hat{\theta}_k(t)$ and $\hat{\phi}_l(t)$ $(k = 1, \ldots, p,\ l = 1, \ldots, q)$.

Remark 1. *The $\mathcal{H}_\infty$ norm*[18),19)] *is defined as*

$$\|T_{ed}\|_\infty = \frac{\sqrt{\int_0^\infty e^T(t)Se(t)dt}}{\sqrt{\int_0^\infty d^T(t)d(t)dt}},$$

where T_{ed} is a transfer function matrix from $d(t)$ to $e(t)$. For a given level $\gamma > 0$, $\|T_{ed}\|_\infty < \gamma$ can be restated in the equivalent form (2·6). If we define

$$H(t) = \frac{\int_0^t e^T(\sigma)Se(\sigma)d\sigma}{\int_0^t d^T(\sigma)d(\sigma)d\sigma}, \tag{2·8}$$

the relation (2·6) can be represented by

$$H(\infty) < \gamma^2. \tag{2·9}$$

In §4, through the plot of $H(t)$ versus time, the relation (2·9) is verified.

The purpose of this paper is to design the controller $u(t)$ with the adaptive laws $\hat{\theta}_k(t)$ and $\hat{\phi}_l(t)$ $(k = 1, \ldots, p,\ l = 1, \ldots, q)$ guaranteeing the adaptive $\mathcal{H}_\infty$ anti-synchronization if there exists the external disturbance $d(t)$. In addition, the controller $u(t)$ with the adaptive laws $\hat{\theta}_k(t)$ and $\hat{\phi}_l(t)$ will be shown to guarantee the adaptive asymptotical anti-synchronization when the external disturbance $d(t)$ disappears.

§3. Main results

In this section, we design the adaptive $\mathcal{H}_\infty$ anti-synchronization controller for uncertain time-delayed chaotic neural networks. The following theorem presents the LMI problem for achieving the adaptive $\mathcal{H}_\infty$ anti-synchronization.

Theorem 1. *For given $\gamma > 0$ and $S = S^T > 0$, if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $W = W^T > 0$, and M such that*

$$\begin{bmatrix} [1,1] & P\bar{A} & W & PB & P\bar{B} & PG & I & 0 & I \\ \bar{A}^TP & -R & -W & 0 & 0 & 0 & 0 & I & 0 \\ W & -W & -\frac{1}{\tau}Q & 0 & 0 & 0 & 0 & 0 & 0 \\ B^TP & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ \bar{B}^TP & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ G^TP & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & -\frac{1}{L_f^2}I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & -\frac{1}{L_g^2}I & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0, \tag{3·1}$$

where

$$[1,1] = A^TP + PA + M + M^T + R + \tau Q,$$

then the adaptive $\mathcal{H}_\infty$ anti-synchronization for time-delayed chaotic neural networks is achieved under the control input

$$u(t) = P^{-1}M(\hat{x}(t) + x(t)) - \sum_{k=1}^{p} \Phi_k(x(t))\hat{\theta}_k(t) - \sum_{l=1}^{q} \Psi_l(x(t-\tau))\hat{\phi}_l(t) \tag{3·2}$$

and the adaptive laws

$$\dot{\hat{\theta}}_k(t) = \Gamma\Phi_k^T(x(t))P(\hat{x}(t) + x(t)), \qquad (k = 1, \ldots, p) \tag{3·3}$$

$$\dot{\hat{\phi}}_l(t) = \Upsilon\Psi_l^T(x(t-\tau))P(\hat{x}(t) + x(t)), \qquad (l = 1, \ldots, q) \tag{3·4}$$

where Γ and Υ are positive definite matrices for design.

Proof. The closed-loop anti-synchronization error system with the control input $u(t) = K(\hat{x}(t)+x(t)) - \sum_{k=1}^{p}\Phi_k(x(t))\hat{\theta}_k(t) - \sum_{l=1}^{q}\Psi_l(x(t-\tau))\hat{\phi}_l(t)$, where $K \in R^{n\times n}$ is the gain matrix of the controller, can be written as

$$\begin{aligned}\dot{e}(t) &= (A + K)e(t) + \bar{A}e(t-\tau) + B(f(\hat{x}(t)) + f(x(t))) + \bar{B}(g(\hat{x}(t-\tau)) \\ &\quad + g(x(t-\tau))) - \sum_{k=1}^{p}\Phi_k(x(t))\tilde{\theta}_k(t) - \sum_{l=1}^{q}\Psi_l(x(t-\tau))\tilde{\phi}_l(t) + Gd(t),\end{aligned} \tag{3·5}$$

where $\tilde{\theta}_k(t) = \hat{\theta}_k(t) - \theta_k$ and $\tilde{\phi}_l(t) = \hat{\phi}_l(t) - \phi_l$. Consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned}V(t) &= e^T(t)Pe(t) + \int_{-\tau}^{0}\int_{t+\beta}^{t} e^T(\alpha)Qe(\alpha)d\alpha d\beta + \int_{-\tau}^{0} e^T(t+\sigma)\mathrm{R}e(t+\sigma)d\sigma \\ &\quad + \left[\int_{-\tau}^{0} e(t+\sigma)d\sigma\right]^T W \left[\int_{-\tau}^{0} e(t+\sigma)d\sigma\right] + \sum_{k=1}^{p}\tilde{\theta}_k^T(t)\Gamma^{-1}\tilde{\theta}_k(t) \\ &\quad + \sum_{l=1}^{q}\tilde{\phi}_l^T(t)\Upsilon^{-1}\tilde{\phi}_l(t).\end{aligned} \tag{3·6}$$

Its time derivative along the trajectory of (3·5) is

$$\begin{aligned}\dot{V}(t) &= \dot{e}(t)^TPe(t) + e^T(t)P\dot{e}(t) + \tau e^T(t)Qe(t) - \int_{t-\tau}^{t} e^T(\sigma)Qe(\sigma)d\sigma + e(t)^T\mathrm{R}e(t) \\ &\quad - e^T(t-\tau)\mathrm{R}e(t-\tau) + [e(t) - e(t-\tau)]^TW\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right] + \left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]^T \\ &\quad \times W[e(t) - e(t-\tau)] + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)\Gamma^{-1}\dot{\hat{\theta}}_k(t) + 2\sum_{l=1}^{q}\tilde{\phi}_l^T(t)\Upsilon^{-1}\dot{\hat{\phi}}_l(t) \\ &= e^T(t)[A^TP + PA + PK + K^TP]e(t) + e^T(t)P\bar{A}e(t-\tau) + e^T(t-\tau)\bar{A}^TPe(t) \\ &\quad + e(t)^TPGd(t) + d^T(t)G^TPe(t) + e^T(t)PB(f(\hat{x}(t)) + f(x(t))) + (f(\hat{x}(t)) \\ &\quad + f(x(t)))^TB^TPe(t) + e^T(t)P\bar{B}(g(\hat{x}(t-\tau)) + g(x(t-\tau))) + (g(\hat{x}(t-\tau))\end{aligned}$$

$$+ g(x(t-\tau)))^T \bar{B}^T Pe(t) - 2\sum_{k=1}^{p} \tilde{\theta}_k^T(t)\Phi_k^T(x(t))Pe(t) - 2\sum_{l=1}^{q} \tilde{\phi}_l^T(t)\Psi_l^T(x(t-\tau))$$

$$\times Pe(t) + \tau e^T(t)Qe(t) - \int_{t-\tau}^{t} e^T(\sigma)Qe(\sigma)d\sigma + e(t)^T \mathrm{R}e(t) - e^T(t-\tau)\mathrm{R}e(t-\tau)$$

$$+ [e(t)-e(t-\tau)]^T W \left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right] + \left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]^T W[e(t)-e(t-\tau)]$$

$$+ 2\sum_{k=1}^{p} \tilde{\theta}_k^T(t)\Gamma^{-1}\dot{\hat{\theta}}_k(t) + 2\sum_{l=1}^{q} \tilde{\phi}_l^T(t)\Upsilon^{-1}\dot{\hat{\phi}}_l(t). \qquad (3{\cdot}7)$$

Since the activation function vectors $f(x(t))$ and $g(x(t))$ of the Hopfield neural networks and the cellular neural networks are odd functions, for each $x, y \in R^n$, it is easy to have

$$\|f(x(t)) + f(y(t))\| \le L_f\|x(t) + y(t)\|,$$
$$\|g(x(t)) + g(y(t))\| \le L_g\|x(t) + y(t)\|, \qquad (3{\cdot}8)$$

from the Lipschitz conditions (2·2)–(2·3) of $f(x(t))$ and $g(x(t))$. If we use the inequality $X^TY + Y^TX \le X^T\Lambda X + Y^T\Lambda^{-1}Y$, which is valid for any matrices $X \in R^{n\times m}$, $Y \in R^{n\times m}$, $\Lambda = \Lambda^T > 0$, $\Lambda \in R^{n\times n}$, we have

$$\begin{aligned}
&e^T(t)PB(f(\hat{x}(t)) + f(x(t))) + (f(\hat{x}(t)) + f(x(t)))^T B^T Pe(t)\\
&\le (f(\hat{x}(t)) + f(x(t)))^T(f(\hat{x}(t)) + f(x(t))) + e^T(t)PBB^TPe(t)\\
&\le L_f^2(\hat{x}(t) + x(t))^T(\hat{x}(t) + x(t)) + e^T(t)PBB^TPe(t)\\
&= e^T(t)(L_f^2 I + PBB^TP)e(t),
\end{aligned} \qquad (3{\cdot}9)$$

$$\begin{aligned}
&e^T(t)P\bar{B}(g(\hat{x}(t-\tau)) + g(x(t-\tau))) + (g(\hat{x}(t-\tau)) + g(x(t-\tau)))^T\bar{B}^TPe(t)\\
&\le (g(\hat{x}(t-\tau)) + g(x(t-\tau)))^T(g(\hat{x}(t-\tau)) + g(x(t-\tau))) + e^T(t)P\bar{B}\bar{B}^TPe(t)\\
&\le L_g^2(\hat{x}(t-\tau) + x(t-\tau))^T(\hat{x}(t-\tau) + x(t-\tau)) + e^T(t)P\bar{B}\bar{B}^TPe(t)\\
&= L_g^2e^T(t-\tau)e(t-\tau) + e^T(t)P\bar{B}\bar{B}^TPe(t),
\end{aligned} \qquad (3{\cdot}10)$$

and

$$e(t)^TPGd(t) + d^T(t)G^TPe(t) \le \gamma^2 d^T(t)d(t) + \frac{1}{\gamma^2}e(t)^TPGG^TPe(t). \qquad (3{\cdot}11)$$

Using (3·9), (3·10) and (3·11), we obtain

$$\dot{V}(t) \le e^T(t)\Bigg[A^TP + PA + PK + K^TP + L_f^2I + PBB^TP + P\bar{B}\bar{B}^TP + \frac{1}{\gamma^2}PGG^TP$$

$$+ \tau Q + R\Bigg]e(t) + e^T(t)P\bar{A}e(t-\tau) + e^T(t-\tau)\bar{A}^TPe(t) + e^T(t-\tau)\left[L_g^2I - R\right]$$

$$\times e(t-\tau) - \int_{t-\tau}^{t} e^T(\sigma)Qe(\sigma)d\sigma + [e(t)-e(t-\tau)]^TW\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]$$

$$+\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]^T W[e(t)-e(t-\tau)] + \gamma^2 d^T(t)d(t) + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)\Gamma^{-1}\Big[\dot{\hat{\theta}}_k(t) - \Gamma\Phi_k^T(x(t))Pe(t)\Big] + 2\sum_{l=1}^{q}\tilde{\phi}_l^T(t)\Upsilon^{-1}\left[\dot{\hat{\phi}}_l(t) - \Upsilon\Psi_l^T(x(t-\tau))Pe(t)\right].$$

Using the inequality[23)]

$$\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]^T Q\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right] \le \tau\int_{t-\tau}^{t} e(\sigma)^T Q e(\sigma)d\sigma, \qquad (3\cdot12)$$

we have

$$\begin{aligned}
\dot{V}(t) \le\; & e^T(t)\Big[A^TP + PA + PK + K^TP + L_f^2 I + PBB^TP + P\bar{B}\bar{B}^TP + \frac{1}{\gamma^2}PGG^TP \\
& + \tau Q + R\Big]e(t) + e^T(t)P\bar{A}e(t-\tau) + e^T(t-\tau)\bar{A}^TPe(t) + e^T(t-\tau)\left[L_g^2 I - R\right] \\
& \times e(t-\tau) - \frac{1}{\tau}\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]^T Q\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right] + [e(t)-e(t-\tau)]^T W \\
& \times\left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right] + \left[\int_{t-\tau}^{t} e(\sigma)d\sigma\right]^T W[e(t)-e(t-\tau)] + \gamma^2 d^T(t)d(t) + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t) \\
& \times\Gamma^{-1}\left[\dot{\hat{\theta}}_k(t) - \Gamma\Phi_k^T(x(t))Pe(t)\right] + 2\sum_{l=1}^{q}\tilde{\phi}_l^T(t)\Upsilon^{-1}\left[\dot{\hat{\phi}}_l(t) - \Upsilon\Psi_l^T(x(t-\tau))Pe(t)\right] \\
=\; & \begin{bmatrix} e(t) \\ e(t-\tau) \\ \int_{t-\tau}^{t} e(\sigma)d\sigma \end{bmatrix}^T \begin{bmatrix} (1,1) & P\bar{A} & W \\ \bar{A}^TP & (2,2) & -W \\ W & -W & -\frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau) \\ \int_{t-\tau}^{t} e(\sigma)d\sigma \end{bmatrix} - e^T(t)Se(t) \\
& + \gamma^2 d^T(t)d(t) + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)\Gamma^{-1}\left[\dot{\hat{\theta}}_k(t) - \Gamma\Phi_k^T(x(t))Pe(t)\right] + 2\sum_{l=1}^{q}\tilde{\phi}_l^T(t)\Upsilon^{-1}\Big[\dot{\hat{\phi}}_l(t) \\
& - \Upsilon\Psi_l^T(x(t-\tau))Pe(t)\Big],
\end{aligned}$$

where

$$\begin{aligned}
(1,1) &= A^TP + PA + PK + K^TP + L_f^2 I + PBB^TP + P\bar{B}\bar{B}^TP + \frac{1}{\gamma^2}PGG^TP \\
&\quad + S + \tau Q + R, \\
(2,2) &= L_g^2 I - R.
\end{aligned}$$

If the adaptive laws (3·3)–(3·4) are used and the following matrix inequality is satisfied

$$\begin{bmatrix} (1,1) & P\bar{A} & W \\ \bar{A}^TP & (2,2) & -W \\ W & -W & -\frac{1}{\tau}Q \end{bmatrix} < 0, \qquad (3\cdot13)$$

we have

$$\dot{V}(t) < -e^T(t)Se(t) + \gamma^2 d^T(t)d(t). \qquad (3\cdot14)$$

Integrating both sides of (3·14) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty e^T(t)Se(t)dt + \gamma^2 \int_0^\infty d^T(t)d(t)dt. \qquad (3\cdot15)$$

Since $V(\infty) \geq 0$ and $V(0) = 0$, we have the relation (2·6). From Schur complement, the matrix inequality (3·13) is equivalent to

$$\begin{bmatrix} \{1,1\} & P\bar{A} & W & PB & P\bar{B} & PG & I & 0 & I \\ \bar{A}^T P & -R & -W & 0 & 0 & 0 & 0 & I & 0 \\ W & -W & -\frac{1}{\tau}Q & 0 & 0 & 0 & 0 & 0 & 0 \\ B^T P & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ \bar{B}^T P & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ G^T P & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & -\frac{1}{L_f^2}I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & -\frac{1}{L_g^2}I & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0, \qquad (3\cdot16)$$

where

$$\{1,1\} = A^T P + PA + PK + K^T P + R + \tau Q.$$

If we let $M = PK$, (3·16) is equivalently changed into the LMI (3·1). Then the gain matrix of the control input $u(t)$ is given by $K = P^{-1}M$. This completes the proof. ∎

Remark 2. *Various efficient convex optimization algorithms can be used to check whether the LMI (3·1) is feasible. In this paper, in order to solve the LMI, we utilize MATLAB LMI Control Toolbox,*[24] *which implements state-of- the-art interior-point algorithms.*

Corollary 1. *Without the external disturbance, if we use the control input (3·2) and the adaptive laws (3·3)–(3·4), the adaptive asymptotical anti-synchronization is obtained.*

Proof. When $d(t) = 0$, we obtain

$$\dot{V}(t) < -e^T(t)Se(t) \leq 0 \qquad (3\cdot17)$$

from (3·14). This guarantees

$$\lim_{t\to\infty} e(t) = 0 \qquad (3\cdot18)$$

from Lyapunov-Krasovskii theory. This completes the proof. ∎

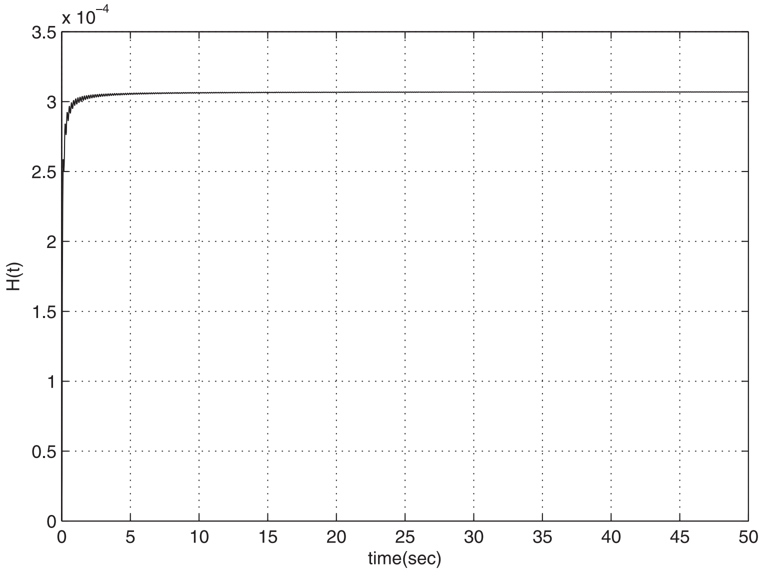

Fig. 1. The plot of $H(t)$ versus time.

§4. Numerical example

Consider the following time-delayed chaotic Hopfield neural network:[11)]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 & -\kappa_1 \\ -5 & 1.5 \end{bmatrix} \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix} + \begin{bmatrix} -1.5 & -0.1 \\ -\kappa_2 & -1 \end{bmatrix} \begin{bmatrix} \tanh(x_1(t-1)) \\ \tanh(x_2(t-1)) \end{bmatrix}, \qquad (4\cdot1)$$

where $x_i(t)$ $(i = 1, 2)$ is the state variable of the neural network (4·1). Parameters κ_1 and κ_2 are assumed unknown. The neural network (4·1) is rewritten as

$$\dot{x}(t) = Ax(t) + \bar{A}x(t-1) + Bf(x(t)) + \bar{B}g(x(t-1)) + \mathit{\Phi}_1(x(t))\theta_1 + \mathit{\Psi}_1(x(t-1))\phi_1, \qquad (4\cdot2)$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ -5 & 1.5 \end{bmatrix}, \bar{B} = \begin{bmatrix} -1.5 & -0.1 \\ 0 & -1 \end{bmatrix},$$

$$f(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix}, g(x(t-1)) = \begin{bmatrix} \tanh(x_1(t-1)) \\ \tanh(x_2(t-1)) \end{bmatrix}, \theta_1 = \kappa_1, \phi_1 = \kappa_2,$$

$$\mathit{\Phi}_1(x(t)) = \begin{bmatrix} -\tanh(x_2(t)) \\ 0 \end{bmatrix}, \mathit{\Psi}_1(x(t-1)) = \begin{bmatrix} 0 \\ -\tanh(x_1(t-1)) \end{bmatrix}.$$

For the numerical simulation, we use the following parameters:

$$\kappa_1 = 0.1,\ \kappa_2 = 0.2,\ \mathit{\Gamma} = 50,\ \mathit{\Upsilon} = 200,\ L_f = L_g = 1,\ G = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

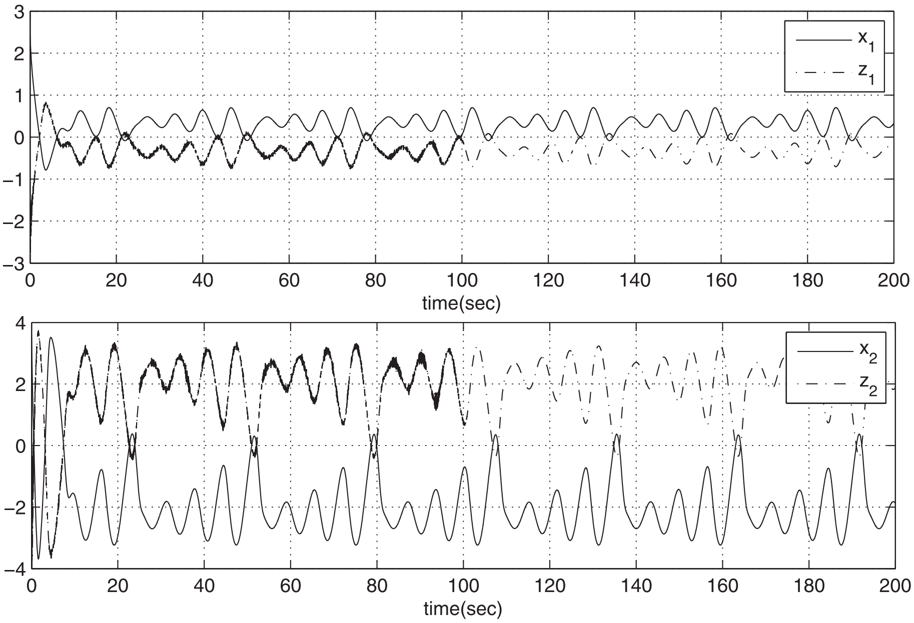

Fig. 2. State trajectories.

For the design objective (2·6), let the $\mathcal{H}_\infty$ performance be specified by $\gamma = 0.2$. Solving the LMI (3·1) by the convex optimization technique of MATLAB software gives

$$P = \begin{bmatrix} 0.7503 & -0.0248 \\ -0.0248 & 0.5179 \end{bmatrix}, \qquad M = \begin{bmatrix} -49.1440 & -2.5296 \\ -0.9283 & -47.5180 \end{bmatrix}.$$

Figure 1 shows the plot of $H(t)$ versus time when $d(t) = \sin(20t)$. Figure 1 verifies $H(\infty) < \gamma^2 = 0.04$. This means that the $\mathcal{H}_\infty$ norm from the external disturbance $d(t)$ to the anti-synchronization error $e(t)$ is reduced within the $\mathcal{H}_\infty$ norm bound γ. Figure 2 shows state trajectories for drive and response systems when the initial conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2.4 \\ 3.6 \end{bmatrix}, \quad \begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \end{bmatrix} = \begin{bmatrix} -1.6 \\ -0.8 \end{bmatrix}, \quad \hat{\theta}_1(0) = 1, \quad \hat{\phi}_1(0) = 0.5, \qquad (4·3)$$

and the external disturbance $d(t)$ is given by

$$d(t) = \begin{cases} w(t), & 0 \le t \le 100, \\ 0, & \text{otherwise,} \end{cases}$$

where $w(t)$ means a Gaussian noise with mean 0 and variance 1. The anti-synchronization error between drive and response systems is given in Fig. 3. It shows that the proposed method reduces the effect of the external disturbance $d(t)$ on the anti-synchronization error $e(t)$. In addition, it is shown that the anti-synchronization error $e(t)$ goes to zero after the external disturbance $d(t)$ disappears. The estimates

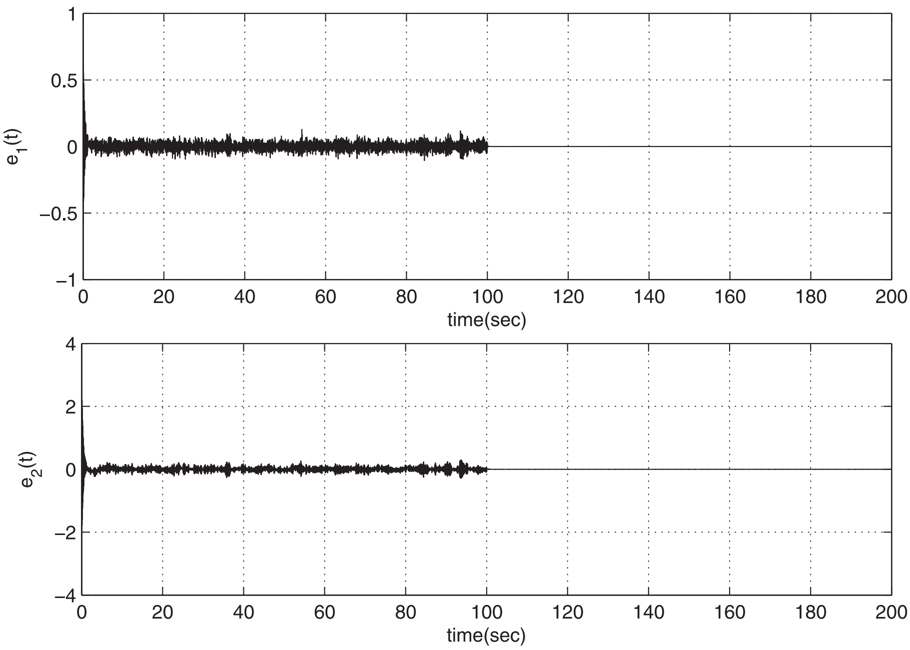

Fig. 3. Anti-synchronization errors.

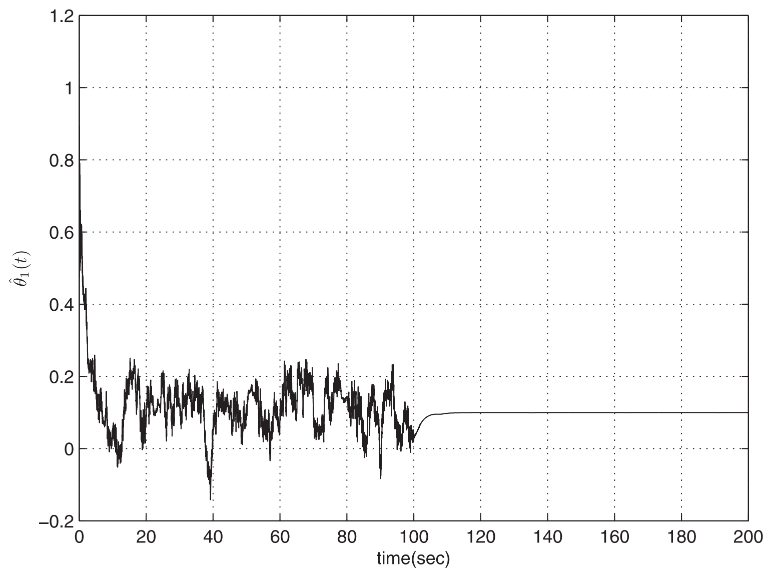

Fig. 4. The estimate value $\hat{\theta}_1(t)$ of parameter θ_1.

$\hat{\theta}_1(t)$ and $\hat{\phi}_1(t)$ of the unknown parameters θ_1 and ϕ_1 are illustrated in Figs. 4 and 5, respectively. These figures show that the estimates $\hat{\theta}_1(t)$ and $\hat{\phi}_1(t)$ approach rapidly to the target values of 0.1 and 0.2, respectively.

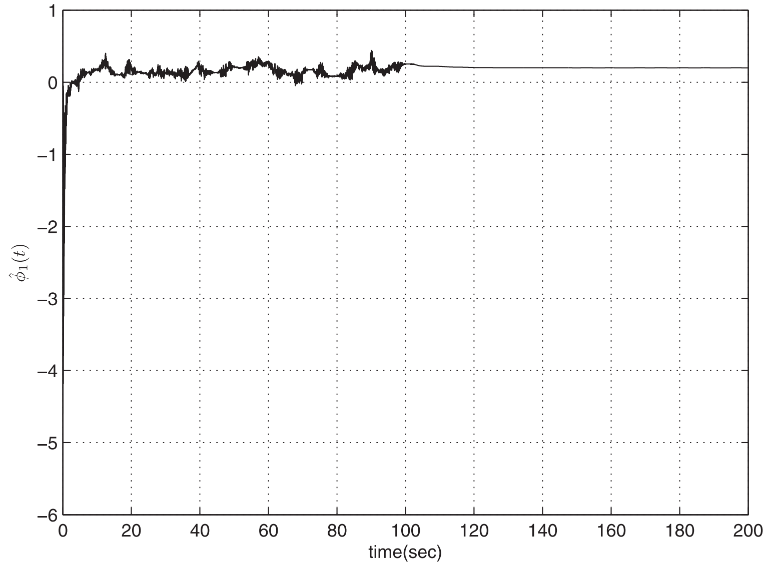

Fig. 5. The estimate value $\hat{\phi}_1(t)$ of parameter ϕ_1.

§5. Conclusion

In this paper, a new adaptive $\mathcal{H}_\infty$ anti-synchronization scheme for uncertain time-delayed chaotic neural networks is proposed. On the basis of Lyapunov-Krasovskii theory and LMI approach, the proposed controller is designed and an analytic expression of the controller with its adaptive laws of unknown parameters is shown. A simulation example is given to show the effectiveness of the proposed method. It is expected that the proposed method can be extended to studying adaptive $\mathcal{H}_\infty$ anti-synchronization problems for chaotic neural networks with time-varying and distributed delays.

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