ADAPTIVE LOCAL PHASE APPROXIMATIONS AND GLOBAL UNWRAPPING

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ABSTRACT

The paper introduces a new modulo-2π phase denoising algorithm based on local polynomial approximations. The zero and first order approximations of the phase are calculated in sliding windows of varying size. The former is used for pointwise adaptive window size selection, while the latter is used for filtering the phase in the obtained windows. For unwrapping, we input the PUMA unwrapping algorithm [1] with the denoised wrapped phase obtained with the proposed approach. Simulation shows that this technique enables strong noise attenuation while preserving image details.

Index Terms— Fringe techniques, hologram, local polynomial approximation, phase unwrap, speckle interferometry.

1. INTRODUCTION

A variety of holographic imaging systems deal with phase measurements using coherent radiation in order to illuminate objects. The scattered return carries information on the physical and geometrical properties of objects such as shape, deformation, movement and surface's structure. The phase is a key element for both wave-front and wave-field reconstructions in 3D holographic imaging and with important applications in such areas as 3D TV, interferometric aperture radar and sonar, magnetic resonance imaging, adaptive optics, diffraction tomography, nondestructive testing of components, deformation and vibration measurements [2, 3].

Common to these applications is that the observations are periodic functions, defined on the interval $[-\pi, \pi)$, of the true phase, the so-called absolute phase. If the true phase is outside this interval, its observed value is *wrapped* into it, corresponding to an addition or subtraction of an integer number of 2π. Many approaches to absolute phase estimation follow a two-step procedure: in the first step, the wrapped phase is inferred from noisy wrapped observations; in the second step, the absolute phase is inferred from the wrapped estimates.

The latter procedure is known as phase unwrapping. A variety of techniques have been developed for phase unwrapping [2, 4, 1]. A classical approach implements path-dependent local techniques, where a pixel-by-pixel unwrapping is confirmed by local phase congruence tests. A more recent direction formulates the phase unwrapping as an inverse problem leading to path-independent optimization techniques.

Owing to the periodic observation mechanism, phase unwrapping is known to be a hard problem that has since long fostered active research. In fact, if the magnitude of phase variation between neighboring pixels is larger than 2π, *i.e.*, the so-called Ito condition is violated, then the inference of the 2π multiples is an ill-posed problem. These violations may be due to spatial undersampling, discontinuities, or noise. To deal with these difficulties, some sort of a priori information shall be used. One efficient way, for both the local path-dependent and global path-independent methods, is the use of external information in the from of masks and quality maps.

The high levels of observation noise, typical of many holographic imaging modalities, introduce further difficulties in the phase reconstruction, as the phase unwrapping methods developed for noiseless data are very sensitive to noise. One of the first and natural ideas is prefiltering the noisy wrapped data and then using it for further processing, in particular for phase unwrapping. However, a phase fringe pattern is a very delicate object with crucial details easily to be damaged in prefiltering. If the noise level is small, any reasonable filtering is acceptable, However, in the heavy noisy case, the standard approaches often damage data in such a way that further unwrapping becomes impossible.

In this paper, we propose a novel filtering technique based on local polynomial approximation with varying adaptive neighborhood used in reconstruction [5]. This adaptivity is a crucial element: it assumes that for each point (pixel) there is a neighborhood where the polynomial approximation fits well the data. Both elements of this model, the coefficients of the polynomial approximation and the neighborhood size (some times and shape), are subjects of estimation. The adaptiveness of the algorithm trades bias with variance in such a way that the window size stretches in areas where the underlying true phase is smooth and shrinks otherwise, namely in the presence of discontinuities. Simulation studies show that the developed filtering is very efficient, namely in the presence of

discontinuities. We found out, namely, that the phase unwrapping equipped with this sort of prefiltering yields very good accuracy of the phase reconstruction, quite often overcoming the state-of-the-art algorithms developed for noisy phase unwrapping.

The polynomial modeling for the phase unwrap is a popular idea. In particular, the efficiency of the local phase fitting is demonstrated in [6] for two-dimensional magnetic resonance imaging data. In the paper [7] the linear local polynomial approximation of height profiles is used for the surface reconstruction from the multifrequency InSAR data. Using the local polynomial fit in the phase tracking for the phase unwrap is proposed in the paper [8], where it is further exploited for initialization of the global optimization giving the final estimate.

The algorithm proposed in this paper is based on accurate least square fitting of the linear polynomial models. In our previous works [9], [10], this fitting is approximate using the local minimum of the local least square criterion and used for the phase unwrap in a phase tracking procedure. In this paper we are focused mainly on filtering of the wrapped phase as the prefiltering procedure for the forthcoming unwrapping.

2. LPA PHASE-APPROXIMATION

The standard formulation of the phase unwrapping starts from the observation model in the form $\phi = W(\varphi + n_\varphi)$, where φ is the true phase and n_φ is a random noise. Here W is a wrapping operator transforming the noisy phase to the interval $[-\pi, \pi)$. For $n_\varphi = 0$, there is an obvious link between the wrapped ϕ and non-wrapped absolute phase φ, $\varphi = \phi + 2\pi k$, $\phi \in [-\pi, \pi)$, where k is an integer. The basic unwrapping problem is to reconstruct $\varphi(x, y)$, $x, y \in X \subset \mathbb{Z}^2$, from the observations $\phi(x, y)$. There is no one-to-one relation between the wrapped and absolute phase. The approach developed in this paper is based on two independent ideas: *local approximation* for design of nonlinear filters (estimators) and *adaptation* of these filters to unknown smoothness of the varying phase. As flexible universal tools, we use *local polynomial approximation* (LPA) for *approximation* and *intersection of confidence intervals* (ICI) for *adaptation [5]*. The LPA is applied for filter design using a polynomial fit in a sliding window. The window size as well as the order of the polynomial define a desirable filter. The *window size* is considered as a *varying adaptation parameter* of the filter. The ICI is an *adaptation scheme* that searches for a largest local window size where the variance and the bias of the estimates are balanced. It is shown that the ICI adaptive LPA yields nearly optimal minimum mean squared error estimates.

The observation model assumed in this paper is of the form

$$z = A \exp(j\varphi) + n, \qquad (1)$$

where $n = n_I + jn_Q$ is complex-valued zero-mean circular white noise of variance $2\sigma^2$ (*i.e.*, n_I and n_Q are zero-mean independent Gaussian random variables with variance σ^2).

Assume that in some neighborhood of the point (x, y) the phase $\varphi(x, y)$ can be represented in the form

$$\tilde{\varphi}(x_s, y_s | \mathbf{c}) = p^T(x_s, y_s)\mathbf{c}, \qquad (2)$$

where $p = (p_1, p_2, p_3)$ is a vector of the first order polynomials $p_1 = 1$, $p_2 = x$, $p_2 = y$, and $\mathbf{c} = (c_1, c_2, c_3)^T$ is a vector of unknown parameters. To infer this vector, we compute

$$\hat{\mathbf{c}} = \arg\min_{\mathbf{c}} L_h(\mathbf{c}),$$

where $L_h(\mathbf{c})$ is a measure of the data misfit defined as

$$L_h(\mathbf{c}) = \frac{1}{2} \sum_s w_{h,s} |z_\phi(x + x_s, y + y_s) - \exp(j\tilde{\varphi}(x_s, y_s | \mathbf{c}))|^2 = \sum_s w_{h,s} \left\{ 1 - \cos[\phi(x + x_s, y + y_s) - \tilde{\varphi}(x_s, y_s | \mathbf{c})] \right\},$$

where $z_\phi \equiv z/|z| = e^{j\phi}$. It can be verified by routine calculations that for the considered linear phase model the optimal solution $\hat{\mathbf{c}}$ is of the form

$$(\hat{c}_2, \hat{c}_3) = \arg\max_{c_2, c_3} |F_h(c_2, c_3)| \qquad (3)$$

$$\hat{c}_1 = \text{angle } F_h(\hat{c}_2, \hat{c}_3), \qquad (4)$$

where $F_h(c_2, c_3)$ is the windowed Fourier transform of the normalized data $z_\phi = e^{j\phi}$ with the arguments c_2, c_3, thus meaning the frequency components of the exponential signal $e^{j\phi}$ at spatial frequency (c_2, c_3); *i.e.*,

$$F_h(c_2, c_3) = \sum_s w_{h,s} z_\phi(x + x_s, y + y_s) e^{-j(c_2 x_s + c_3 y_s)}.$$

Then, the phase estimate $\hat{c}_1$ is the argument (angle) of the complex-valued Fourier transform F_h calculated at the point where the maximum value of the Fourier spectrum is achieved. The formulas (4) show that the standard Fourier transform can be used for implementation of the proposed algorithm.

According to the model (2), the local polynomial model for the neighborhood of the pixel (x, y) is $\hat{\varphi}(x + x_s, y + y_s) \equiv p^T(x_s, y_s)\hat{\mathbf{c}}(x, y)$. In particular, we have

$$\hat{\varphi}(x, y) = \hat{\mathbf{c}}_1(x, y). \qquad (5)$$

In general, the estimate $\hat{\mathbf{c}}(x, y)$ depends of the coordinates (x, y) and the window size h. We wish to emphasize the nonparametric nature of the introduced estimator. Indeed, we start from the parametric linear on x and y model in (2) and could expect that the approximation (estimate) is also linear on x and y. However, the fit is used in the polynomial approximation (2) only for one "central" point $x_s = y_s = 0$. The result of this point-wise approximation is that the parametric estimate (2) becomes nonparametric, with $\hat{\varphi}(x, y)$ depending in a nonlinear way on x and y [5]. All ideas of the standard LPA concerning the window w (shape, anisotropy, directionality, etc.), the scaling h (scalar, multivariate), estimation of

the signal and derivatives are naturally valid in this nonparametric pointwise estimation.

As final remark on the LPA approach, we note that a zero-order approximation of the phase would lead to the estimate

$$\tilde{\varphi}(x,y) = \tilde{c}_1(x,y) = \text{angle}\left(F_h(0,0)\right). \quad (6)$$

3. ESTIMATE ACCURACY AND ADAPTATION

Given a two-dimensional window function w and a window size (*scale*) parameter $h > 0$, we define the scaled window $w_{h,s} \equiv w(x_s/h, y_s/h)$, centered at (x,y). In particular, if we take the square uniform window $w = 1$ for $|x| \leq 1$, $|y| \leq 1$ and $w = 0$ otherwise, we have $w_h = 1$ for $|x| \leq h$, $|y| \leq h$ and $w_h = 0$ otherwise. A smaller or larger h narrows or widens the window w_h, respectively.

It is shown in Proposition 2 of [9] that, asymptotically for small noise level and symmetrical windows, the phase estimate $\hat{\varphi}(x,y) = \hat{c}_1(x,y)$ (5) is unbiased with the variance

$$\sigma_h^2 = \frac{\sigma^2}{A^2} \frac{\sum_s w_{h,s}^2}{(\sum_s w_{h,s})^2}. \quad (7)$$

This result is used for the adaptive selection of the window size h. Let H be a set of the ordered window sizes $H = \{h_1 < h_2 < ... < h_J\}$ and $\hat{\varphi}_h(x,y)$, for $h \in H$, the respective phase estimates. A statistic known as the intersection of confidence interval (ICI) rule is exploited in order to select the best window size. Given the estimates $\hat{\varphi}_h(x,y)$ and the respective variance, for $h \in H$, the confidence intervals of these estimates are defined as

$$Q_h = \{\hat{\varphi}_h - \Gamma \cdot \sigma_h, \ \hat{\varphi}_h + \Gamma \cdot \sigma_h\}. \quad (8)$$

where $\Gamma > 0$ is a parameter of the algorithm and σ_h is calculated according to (7).

The ICI rule defines the adaptive window size, denoted by h^+, as the largest $h \in H$ for which the estimate $\hat{\varphi}_h$ does not differ "significantly" from the estimates corresponding to the smaller window sizes. In order to identify this adaptive h^+, the successive intersection of the confidence intervals Q_h is considered starting from Q_{h_1} and Q_{h_2}. Specifically, *the pairwise intersection of the intervals Q_{h_j}, $1 \leq h_j \leq h_i$, is considered with increasing h_i. Let h^+ be the largest of those h_i for which the intervals Q_{h_j}, $1 \leq h_j \leq h_i$, have a point in common. This h^+ defines the adaptive window size and the adaptive estimate as $\hat{\varphi}_{h^+}$.*

For the varying pointwise adaptive estimation, these calculations are produced for all points (pixels). In implementation, the ICI algorithm is used when the estimates for all points (x,y) are already calculated for all h. Then the algorithm works as a selector of the proper window size estimate for each point from a given set of the estimates for all window sizes ([5], Chapter 6).

Parameter Γ in (8) controls the bias-variance balance in the estimate. Decreasing Γ means a shift of this balance in favor of the bias, as smaller Γ results in smaller bias of the
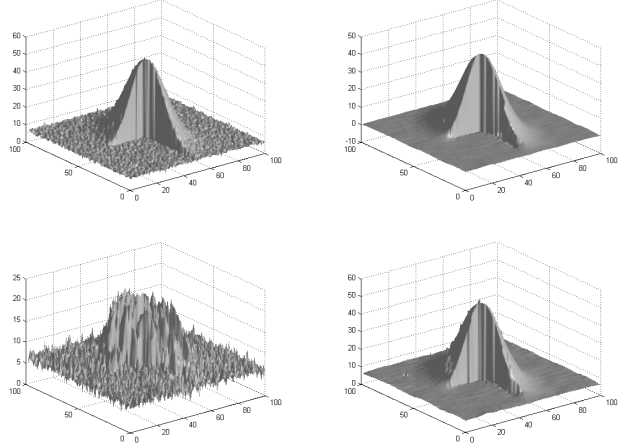

Fig. 1. From left to right: $PUMA$ reconstruction from the noisy data and from prefiltered $PEARLS$ data, first line $\sigma = 0.5$ and second line $\sigma = 0.75$.

estimate. On the contrary, increasing Γ means a shift in favor of the variance, as larger Γ results in smaller variance of the estimate but possible larger bias.

4. *PEARLS* ALGORITHM

We name the proposed algorithm *PEARLS* for **P**hase **E**stimation using **A**daptive **R**egularization based on **L**ocal **S**moothing. The pseudo-code of this algorithm is as follows:

1: **For every pixel** (x,y) **and** $h \in H$:

   a: **calculate the zero-order phase estimate according to the formula (6) , *i.e.*,**

$$\tilde{\varphi}_h(x,y) = \text{angle}\left(\sum_s w_{h,s} z_\phi(x+x_s, y+y_s)\right);$$

   b: **apply the ICI rule to the estimates** $\tilde{\varphi}_h(x,y)$ **for the selection of the best window size** $h^+(x,y)$;

   c: **using the formulas (4) with** $h = h^+(x,y)$, **calculate** $\hat{\varphi}_{h^+}$, **the first-order phase estimates with adaptive window size** ;

  End for ;

2: **unwrap phase** $\hat{\varphi}_{h^+}$ **using one of the procedures developed for noise-less data, for example, the PUMA algorithm [1].**

We have used zero-order estimates in the ICI rule. There are at least two reasons for this: First, it makes calculations much faster. Second, the ICI rule works better with zero-order estimates than with first order ones as, the former are essentially less noisy.

5. SIMULATION EXPERIMENTS

In this section, we present two experiments illustrating the *PEARLS* competitiveness. For the phase unwrap of the filtered wrapped phase we use the $PUMA$ algorithm [1], which is able to work with discontinuities. In what follows, the LPA is exploited with the uniform square windows w_h defined on the integer symmetric grid $\{(x,y) : |x|, |y| \leq h\}$. The ICI parameter was set to $\Gamma = 2.0$ and the window sizes $H = \{1, 2, 3, 4\}$. The noisy observations were generated according to (1), with $A = 1$, on the square grid with integer arguments x, y, $-49 \leq x, y \leq 50$. The absolute phase is defined by the formula $\varphi(x,y) = A_\varphi \exp(-x^2/(2\sigma_x^2) - y^2/(2\sigma_y^2))$ with $\sigma_x = 10$, $\sigma_y = 15$, and $A_\varphi = 14\pi$. The the maximum value of φ is 14π and the maximum values of the first differences about 2.5 radians. Which such high phase differences, even a noise of small variance leads to a difficult unwrapping problem, due to many phase differences larger the π.

Table 1 shows the $RMSE$ values obtained with the *PEARLS* algorithm, as a function of the observation noise standard deviation σ. For comparison purposes, we have also computed the $RMSE$ obtained with the PhaseLa [10] and $Z\pi M$ [4] algorithms, which implement filtering plus unwrapping.

Table 1. $RMSE$ for $PEARLS$, $PhaseLa$ and $Z\pi M$ algorithms.

$Algorithm \setminus \sigma$	0.75	0.5	0.25	0.05	0.01
$PEARLS$	0.34	0.15	0.09	0.05	0.03
$PhaseLa$	$-$	0.18	0.11	0.04	0.03
$Z\pi M$	1.39	0.21	0.12	0.19	0.08

The unwrapping based on the *PEARLS* prefiltering yields consistently comparable or better performance than PhaseLa and $Z\pi M$ algorithms. The advantage increases for large values of σ, corresponding to the more challenging scenarios.

The $3D$ imaging in Figure 1 illustrates performance of the algorithm for a discontinuous phase. In the first line we can see the reconstructions obtained by the PUMA algorithm with no prefiltering and with prefiltering. The prefiltering obviously improves the reconstruction for the smooth areas where the larger values of the window sizes are used in *PEARLS* algorithm. A distribution of the adaptive window sizes for this case can be seen in Figure 2. These results are shown for $\sigma = 0.5$. For a larger noise level of $\sigma = 0.75$, the unwrapping result without prefiltering is catastrophic, whereas with prefiltering the reconstruction is quite acceptable, despite the high noise level present in the observed data. $PhaseLa$ and $Z\pi M$ were not compared in this experiment because they are conceived to blindly deal with discontinuities.

These results confirm conclusions obtained from multiple test experiments. The novel algorithm demonstrates very strong filtering properties enabling the posterior application of unwrapping.
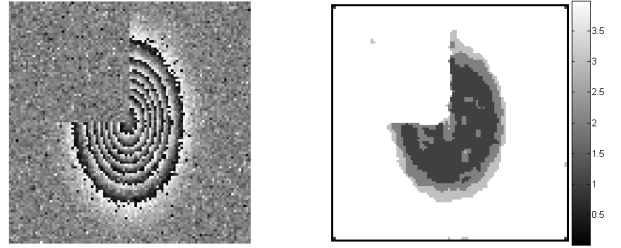

Fig. 2. Noise data and adaptive ICI window sizes, $\sigma = 0.5$.

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