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ADAPTIVE LQG CONTROLLER WITH LOOP TRANSFER RECOVERY

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SUMMARY

In this paper we propose for scalar plants an adaptive LQG controller with adaptive input sensitivity function/loop transfer recovery of an associated adaptive LQ design. The sensitivity recovery can be viewed as a frequency-shaped loop recovery where the weights involve a sensitivity function. The adaptive loop/sensitivity recovery is achieved by feeding back the estimation residuals to the control through a stable bounded input, bounded output (BIBO) adaptive filter Q_k. For simplicity we consider fixed but uncertain plants in the model set and identification schemes where there are consistent parameter estimates. For non-minimum phase plants an asymptotic partial recovery is achieved via a recursive least squares update of the BIBO filter Q_k. The degree of recovery can be prescribed *a priori* between zero and the maximum possible. For the case of minimum phase plant estimates, full loop recovery may be achieved asymptotically by prescribing a maximum degree of recovery.

The motivation for proposing the new adaptive control algorithm is to enhance robustness of adaptive LQG designs, taking advantage of the robustness enhancement properties of sensitivity/loop recovery for off-line designs. The robustness properties of the new algorithm are demonstrated by simulation results.

KEY WORDS Adaptive control Optimal control Loop transfer recovery Robust controller design

1. INTRODUCTION

The linear quadratic (LQ) controller design method for a nominal deterministic plant is a straightforward state feedback design approach that results in a simple controller which is optimal with respect to its quadratic performance index. Not only is the design optimal, it also has certain robustness properties. In classical terms a continuous time LQ design guarantees 60° phase margin and $[-6, \infty)$ dB gain margin (or multivariable equivalent). For discrete time LQ design the same attractive properties are not guaranteed. However, for most practical cases the attractive robustness property is still available.[1] In linear quadratic Gaussian (LQG) design, state estimates are used in lieu of states. The state estimator is designed based on a nominal white Gaussian noise environment to give optimal estimates of the nominal plant state. For the nominal noise environment the resulting LQG controller gives the optimal controller in that the expected value of the quadratic index is minimized. However, the result

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of incorporating a state estimator into the controller is that the resulting controller may no longer possess the desirable robustness present in the original LQ design. The LQG controller may have intolerable robustness.[2,3]

It is usually considered desirable in an LQG controller design to modify the design in such a way as to achieve full or partial loop transfer recovery (LTR) of the original state feedback.[4–6] This modification regains the robustness lost as a result of feeding back state estimates instead of the states. The technique of References 4–6 developed for minimum phase plants suffers a shortcoming in that there is no systematic way of coping with non-minimum phase plants. In an earlier paper[7] a new approach to achieving LTR is proposed. It is based on H^2- or H^∞-optimization over the class of all stabilizing controllers. Loop recovery (possibly only partial recovery) is achieved by feeding back output prediction errors (residuals) from the state estimator through a stable transfer function Q to the inputs. The approach constitutes a more systematic way of coping with non-minimum phase plants.

A fixed off-line designed controller, however robust, is limited in the class of plants it can stabilize and control adequately. It is thus necessary to turn to adaptive control techniques, involving on-line learning of the plant, to cope with a wider class of plants, plant variations or plant uncertainties. Of course, adaptive techniques cope with only parametrized structured uncertainties. To cope with unstructured uncertainties such as unmodelled dynamics, it is usual to implement adaptively robust controller design. Here we develop an adaptive control design based on the well understood off-line linear quadratic methods. The adaptive linear quadratic Gaussian (LQG) controller[8] calculates at each finite time step an estimate of the plant parameters from which the LQG controller parameters are updated. However, this adaptive controller is not an adaptive robust controller since it inherits the poor robustness properties of off-line LQG designs. Clearly there is motivation to seek an adaptive LQG/LTR scheme as a means to achieve an adaptive robust controller and thereby exploit the full power of linear quadratic design methodology. If a suitable recursive version of the LQG/LTR design approach could be devised, it would have a greater potential for robustness, inheriting the enhanced robustness associated with off-line LQG/LTR. The challenge in achieving a practical adaptive LQG/LTR algorithm is to minimize complexity increase over an adaptive LQG algorithm, to avoid high estimator loop gains and to cope with non-minimum phase plants.

In this paper we propose generalizations of the LQG/LTR off-line design approach in Reference 7 to the on-line adaptive LQG/LTR case. For minimum phase plants the modifications to the adaptive LQG schemes are straightforward applications of the off-line results of Reference 7 to the on-line case. For non-minimum phase plants the off-line optimization LTR procedure with a stability constraint as suggested in Reference 7 is too involved to be practically performed on-line. Here we propose a recursive optimization based on the LTR procedure of Reference 7 involving a standard recursive least squares algorithm. The computational effort involved in the recursive LTR is thus (loosely) of the same order as that of the identification and that of the LQG design. The algorithm, though developed with the non-minimum phase plants case in mind, is equally suited when applied to the minimum phase plants case.

The simulation studies of the paper aim to demonstrate that the proposed adaptive LQG/LTR scheme can be more robust than an adaptive LQG scheme.

In Section 2 we revise the LQG/LTR methodology of Reference 7 which optimizes over the class of all stabilizing controllers and present variations which are suitable for on-line implementation. In Section 3 we present the proposed adaptive LQG/LTR algorithm. Simulation results are presented in Section 4 and conclusions are drawn in Section 5.

2. OPTIMIZATION TASK

In this section we first revise the theory on the class of all stabilizing controllers and a method of off-line loop transfer recovery which can be viewed as an H^2- or H^∞-optimization over this class. Next, variations are studied which prove suitable for on-line implementation.

Deterministic plant model

Consider the following state space description of a linear time-invariant stabilizable and detectable plant G

$$G: \quad x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k + Du_k \tag{1a}$$

and transfer function matrix

$$G = C(zI - A)^{-1}B + D \equiv \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{\mathrm{T}} \in R_{\mathrm{p}} \tag{1b}$$

where R_{p} denotes the class of rational proper transfer function matrices. Throughout the paper, $[\ ]_{\mathrm{T}}$ denotes a transfer function matrix according to the convention of (1b).

Stabilizing controllers

A controller $K \in R_{\mathrm{p}}$ is said to stabilize $G \in R_{\mathrm{p}}$ if and only if

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \in RH^\infty \tag{2}$$

(Here we assume that the control loop is well posed or equivalently that the inverse in (2) exists.)

LQG design

Consider a standard LQG design for the plant G of (1) based on the separation principle,[2,3] as in Figure 1 for the case $Q(s) \equiv 0$. The steady state feedback gain vector F is obtained from an LQ design which involves the solution of the Riccati equation

$$\begin{aligned} P^{\mathrm{c}}_{k+1} &= A^{\mathrm{T}}[P^{\mathrm{c}}_k - P^{\mathrm{c}}_k B(B^{\mathrm{T}}P^{\mathrm{c}}_k B + R_{\mathrm{c}})^{-1}B^{\mathrm{T}}P^{\mathrm{c}}_k]A + Q_{\mathrm{c}} \\ \bar{P}^{\mathrm{c}} &= \lim_{k \to -\infty} P^{\mathrm{c}}_k, \qquad P^{\mathrm{c}}_0 = 0 \\ F &= -(B^{\mathrm{T}}\bar{P}^{\mathrm{c}}B + R_{\mathrm{c}})^{-1}B^{\mathrm{T}}\bar{P}^{\mathrm{c}}A \end{aligned} \tag{3}$$

parametrized by $R_{\mathrm{c}} = R_{\mathrm{c}}^{\mathrm{T}} > 0$ and $Q_{\mathrm{c}} = Q_{\mathrm{c}}^{\mathrm{T}} = C_{\mathrm{c}}^{\mathrm{T}}C_{\mathrm{c}} \geqslant 0$, $[A, C_{\mathrm{c}}]$ completely detectable, with interpretation as weighting coefficients in a quadratic index $\Sigma_0^\infty\,(x^{\mathrm{T}}Q_{\mathrm{c}}x + u^{\mathrm{T}}R_{\mathrm{c}}u)$. The closed loop arrangement arising from an LQ design is depicted in Figure 2.

Similarly, the steady state estimator gain is obtained from

$$\begin{aligned} P^{\mathrm{e}}_{k+1} &= A[P^{\mathrm{e}}_k - P^{\mathrm{e}}_k C^{\mathrm{T}}(CP^{\mathrm{e}}_k C^{\mathrm{T}} + R_{\mathrm{e}})^{-1}CP^{\mathrm{e}}_k]A^{\mathrm{T}} + Q_{\mathrm{e}} \\ \bar{P}^{\mathrm{e}} &= \lim_{t \to \infty} P^{\mathrm{e}}_k, \qquad P^{\mathrm{e}}_0 = 0 \\ H &= -A\bar{P}^{\mathrm{e}}C^{\mathrm{T}}(C\bar{P}^{\mathrm{e}}C^{\mathrm{T}} + R_{\mathrm{e}})^{-1} \end{aligned} \tag{4}$$

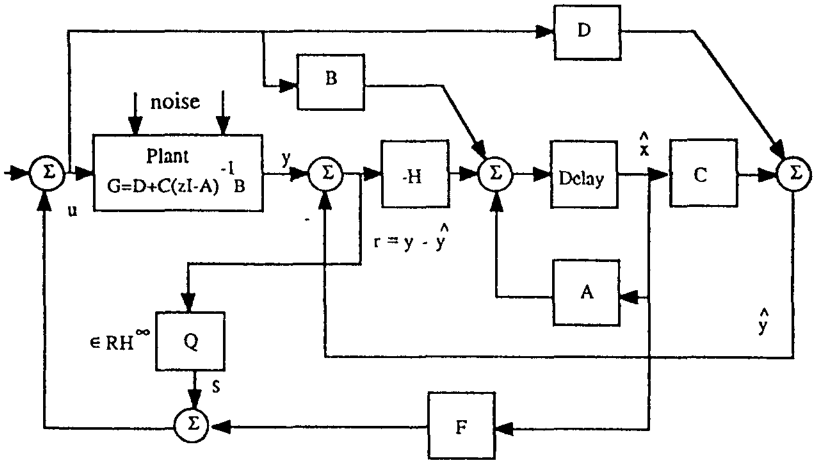

Figure 1. The class of all stabilizing controllers

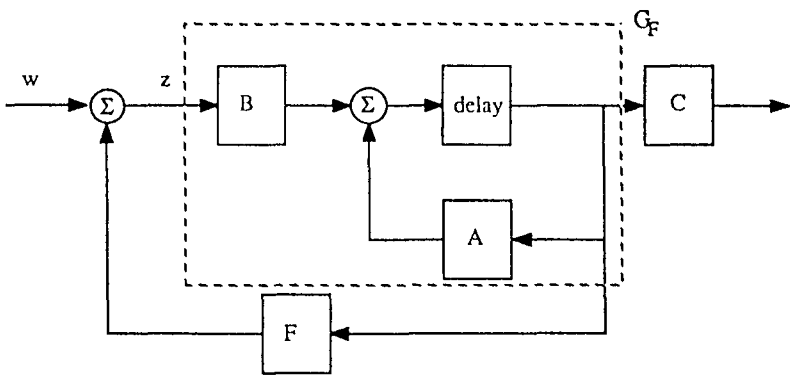

Figure 2. State feedback design

parametrized by $R_e = R_e^T > 0$, $Q_c = Q_e^T = B_e B_e^T \geqslant 0$, $[A, B_e]$ completely stabilizable, with interpretation as intensities of independent plant measurement and plant process noise disturbances in a stochastic model of the plant based on (1). The state estimator loop is depicted in Figure 3.

The output feedback LQG (stabilizing controller) constructed from F and H based on the separation principle is then given as

$$K = \left[\begin{array}{c|c} A + BF + HC + HDF & -H \\ \hline F & 0 \end{array}\right]_T \tag{5}$$

See Figure 1 with $Q(s) \equiv 0$.

Class of all stabilizing controllers

Now define coprime factorizations for G and K as

$$\begin{aligned} G &= NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad N, M, \tilde{N}, \tilde{M} \in RH^\infty \\ K &= UV^{-1} = \tilde{V}^{-1}\tilde{U}, \quad U, V, \tilde{U}, \tilde{V} \in RH^\infty \end{aligned} \tag{6}$$

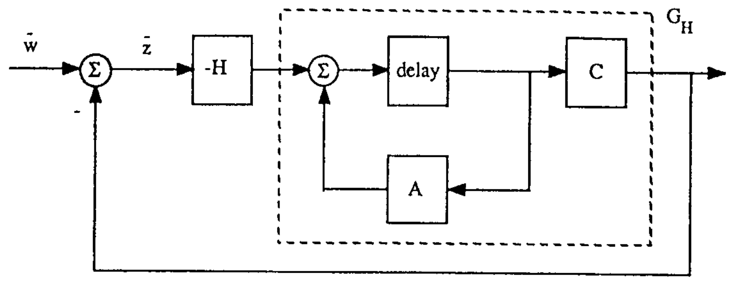

Figure 3. Estimator feedback loop design

such that the following double Bezout equation is satisfied.

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} M & U \\ N & V \end{bmatrix}\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{7}$$

The set of factorizations we work with here are given in terms of F and H and are as follows (for a more detailed discussion of coprime factorization see Reference 9).

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -(B+HD) & H \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]_T \tag{8}$$

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -H \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right]_T \tag{9}$$

It is shown in Reference 9 that the class of all stabilizing controllers $K(Q)$ for G can be uniquely parametrized in terms of arbitrary $Q \in RH^\infty$ as

$$K(Q) = (U+MQ)(V+NQ)^{-1} = K + \tilde{V}^{-1}Q(I+V^{-1}NQ)^{-1}V^{-1} \tag{10}$$

Duals can also be defined as

$$K(Q) = (\tilde{V}+Q\tilde{N})^{-1}(\tilde{U}+Q\tilde{M}) = K + \tilde{V}^{-1}(I+Q\tilde{N}\tilde{V}^{-1})^{-1}QV^{-1} \tag{11}$$

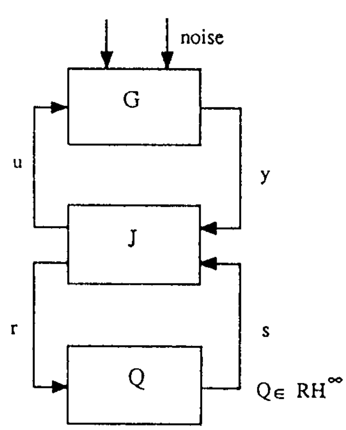

Figure 4. Class of all stabilizing controllers

It can be shown from (6)–(9) that the class of all stabilizing controllers can be organized as in Figure 1[10] or in Figure 4 where

$$J = \begin{bmatrix} K & \tilde{V}^{-1} \\ V^{-1} & -V^{-1}N \end{bmatrix} \tag{12}$$

Here $Q \in RH^\infty$ is interpreted as an augmentation to the nominal LQG controller K feeding back the estimation residuals $r = (y - \hat{y})$ to the plant controls.

Loop/sensitivity recovery

To regain robustness at the plant input, it is proposed in Reference 6 that $Q \in RH^\infty$ be selected to achieve partial (or perhaps full) loop/sensitivity recovery of the original LQ controller. Loop recovery is said to take place if $F(zI - A)^{-1}B - K(Q)G \to 0$ and input sensitivity recovery takes place if $[I - F(zI - A)^{-1}B]^{-1} - [I - K(Q)G]^{-1} \to 0$. It is pointed out in Reference 7 that the sensitivity difference terms above can be viewed as a frequency-shaped loop difference where the frequency weightings are the sensitivities, which weight most heavily the crossover frequencies of the loop gain. Thus simple manipulations give, as in Reference 7,

$$[I - F(zI - A)^{-1}B]^{-1} - [I - K(Q)G]^{-1} = M[F(zI - A)^{-1}B - K(Q)G]\,[I - K(Q)G]^{-1}$$

Now define the sensitivity difference as

$$\varepsilon_Q^i = [I - F(zI - A)^{-1}B]^{-1} - [I - K(Q)G]^{-1} = M(I - \tilde{V}) - MQ\tilde{N} \tag{13}$$

Full input sensitivity function recovery is achieved if ε_Q^i is zero subject to closed loop stability, i.e. $Q \in RH^\infty$. Correspondingly, partial input sensitivity function recovery is achieved if ε_Q^i is made small in some sense. Typical criteria are the two norm or infinity norm.

Towards an adaptive loop/sensitivity recovery

Let us first define a frequency-shaped sensitivity error so as to achieve calculational simplicity, particularly for on-line versions. The error we consider is

$$\begin{aligned} G\varepsilon_Q^i &= G[I - F(zI - A)^{-1}B]^{-1} - G[I - K(Q)G]^{-1} \\ &= N[F(zI - A)^{-1}B - K(Q)G]\,[I - K(Q)G]^{-1} = N(I - \tilde{V}) - NQ\tilde{N} \end{aligned} \tag{14}$$

Clearly, from (14), achieving a small $G\varepsilon_Q^i$ can be viewed as frequency-shaped loop recovery of the original LQ design where the frequency weights are N and the sensitivity function is $[I - K(Q)G]^{-1}$. The appropriateness of working with a minimization of $G\varepsilon_Q^i$ is virtually identical to that of working with ε_Q^i, as studied in Reference 6, and so is not repeated here.

Let us define, in obvious operator notation,

$$e_{1k} = (G\varepsilon_Q^i)w_k \tag{15}$$

where w_k is a sample function of a zero-mean, unit-covariance white noise and e_{1k} is the filtered response to w_k. Clearly, the H^2-minimization of $G\varepsilon_Q^i$ is equivalent to the minimization task

$$\min_{Q \in RH^\infty} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_{1i}^T e_{1i} \tag{16}$$

This minimization task can be performed off-line. Here we wish to apply standard least squares techniques so as to achieve a practical on-line optimization. For minimum phase plants the task (16) can be achieved using least squares and the least squares Q turns out to be stable anyway. For non-minimum phase plants the minimization of (16) cannot be performed using a standard recursive least squares minimization algorithm since such an algorithm will yield a Q that is unstable. To see this, note that with $N^{-\mathrm{L}}$ denoting the left inverse, $Q = (I - \tilde{V})\tilde{N}^{-\mathrm{L}} \notin RH^{\infty}$ is the unconstrained least squares solution giving a zero index.

Least squares minimization task

A variation of the minimization task (16) which can be made arbitrarily close to the task (16) is the more standard least squares minimization task over all Q of appropriate dimension

$$\min_{Q} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_i^{\mathrm{T}} e_i \tag{17}$$

where with $0 < \gamma < \infty$ and applying (14)

$$e_k = \begin{bmatrix} e_{1k} \\ e_{2k} \end{bmatrix} = \begin{bmatrix} G\varepsilon_Q^i \\ \gamma Q\tilde{M} \end{bmatrix} w_k = \begin{bmatrix} N(I - \tilde{V}) - NQ\tilde{N} \\ \gamma Q\tilde{M} \end{bmatrix} w_k \tag{18}$$

We propose to implement a recursive minimization

$$\min_{Q} \frac{1}{k} \sum_{i=1}^{k} \hat{e}_i^{\mathrm{T}} \hat{e}_i \tag{19}$$

where $\hat{e}_i$ denotes an estimate of e_i.

Stability of least squares Q

We will now show the key theoretical result of the paper, crucial for achieving the proposed practical adaptive LQG/LTR controller. For a controllable and observable scalar plant G the minimization of (17) for $\gamma > 0$ ensures the stability of Q and allows a rationalization for specific disturbance response construction of (18) and least squares Q selection.

Lemma 1

Consider the controllable and observable scalar plant G of (1) and a controller $K(Q)$ with (6), (7) and (10), (11) holding. Consider also a zero-mean, unit-variance white noise w_k and that the noise response e_k is given by (18). Assume a minimal realization of Q. Consider the least squares minimization task of (17) with $\gamma > 0$ giving a least squares Q denoted Q^{LS}. Then the optimal e_k has a bounded variance, as does $Q^{\mathrm{LS}} w_k$. Moreover, Q^{LS} is stable and is given by

$$Q^{\mathrm{LS}} = Z^{-1}\left[\left(\frac{N\tilde{N}}{Z}\right)^{*} N(I - \tilde{V})\right]_{\mathrm{S}} \in RH^{\infty} \tag{20}$$

where $Z^{*}Z = (N\tilde{N})^{*}(N\tilde{N}) + \gamma\tilde{M}^{*}\tilde{M}$ with $Z, Z^{-1} \in RH^{\infty}$ and $[X]_{\mathrm{S}}$ denotes the stable terms in a partial fraction expansion of X.

Proof: Part(i). From (18) we see that with $Q = 0$, then e_k is the output of a linear stable time-invariant system driven by w_k and so has a bounded variance, being

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_i^{\mathrm{T}} e_i.$$

It is clear that for the optimal selection of Q the variance of e_k is less than this variance and is bounded.

We have that $(\tilde{N}, \tilde{M})$ are coprime. Also, from (7), $V\tilde{M} - N\tilde{U} = I$ and in the scalar plant case equivalently $V\tilde{M} - \tilde{U}N = I$ so that $(N, \tilde{M})$ are coprime. These two coprime conditions imply that the pair $(-N\tilde{N}, \gamma\tilde{M})$ is also coprime for $\gamma > 0$. Thus there exist $X_1, X_2 \in RH^{\infty}$ such that

$$(\gamma X_2\tilde{M} - X_1 N\tilde{N}) = I \tag{21}$$

For the scalar plant case, when $Q\tilde{N} = \tilde{N}Q$ and $Q\tilde{M} = \tilde{M}Q$, premultiplying (18) by $[X_1 \;\; X_2]$ yields under (21)

$$[X_1 \;\; X_2]\, e_k - X_1 N(I - \tilde{V})w_k = Qw_k$$

Since the operators on e_k and w_k on the left-hand side are time-invariant and stable, and e_k (and of course w_k) has a bounded variance, under the minimization (17) yielding an optimum Q as Q^{LS}, then $Q^{\mathrm{LS}}w_k$ has a bounded variance. Consequently, under the assumption on w_k, it is clear that for Q^{LS} time-invariant, Q^{LS} must be stable.

Part (ii). For the minimization of (17) consider a Q of appropriate order. Then from the results of the first part of this lemma we have by the ergodic theorems

$$\min_{Q} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_i^{\mathrm{T}} e_i = \min_{Q \in RH^{\infty}} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_i^{\mathrm{T}} e_i = \min_{Q \in RH^{\infty}} E\| e_k \|_2^2$$

Now for w_k a zero-mean, unit-variance white noise sequence we have

$$\min_{Q \in RH\infty} E\| e_k \|_2^2 = \min_{Q \in RH\infty} \left\| \begin{bmatrix} N(I - \tilde{V}) - N\tilde{N}Q \\ \tilde{M}Q \end{bmatrix} \right\|_2^2 \tag{22}$$

Define $Z^*Z = (N\tilde{N}) + \gamma\tilde{M}^*\tilde{M}$ with $Z, Z^{-1} \in RH^{\infty}$. It is immediate from spectral factorization theory that a stable and minimum phase spectral factor Z exists. That $(Z^*Z)^{-1} \in R_{\mathrm{p}}$ and thereby $Z^{-1} \in R_{\mathrm{p}}$ follows since $\tilde{M}^{-1} \in R_{\mathrm{p}}$ and $\gamma > 0$. Decomposing $[-N\tilde{N} \;\; \gamma\tilde{M}]^{\mathrm{T}}$ into an inner–outer factor pair, from (22) we then have

$$\begin{aligned}
&\min_{Q \in RH^{\infty}} \left\| \begin{bmatrix} N(I - \tilde{V}) \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{-N\tilde{N}}{Z} \\ \dfrac{\gamma\tilde{M}}{Z} \end{bmatrix} ZQ \right\|_2^2 \\
&= \min_{Q \in RH\angle} \left\| \begin{bmatrix} \left(\dfrac{-N\tilde{N}}{Z}\right)^* & \left(\dfrac{\gamma\tilde{M}}{Z}\right)^* \end{bmatrix} \begin{bmatrix} N(I - \tilde{V}) \\ 0 \end{bmatrix} + ZQ \right\|_2^2 \\
&= \min_{Q \in RH\angle} \left\| \left(\dfrac{-N\tilde{N}}{Z}\right)^* N(I - \tilde{V}) + ZQ \right\|_2^2
\end{aligned} \tag{23}$$

Then, from Reference 8, with $Q = Q^{\mathrm{LS}}$ of (20), (23) is minimized as claimed. □

Rationale for definition of e_k

In defining a disturbance response e_k as in (18) for minimization in the task (17), the inclusion of the term e_{1k} is justified in terms of sensitivity recovery. The inclusion of the term $e_{2k} = \gamma Q\tilde{M}w_k$, $\gamma > 0$, in the minimization task (17) achieves the requirement that $Q \in RH^{\infty}$.

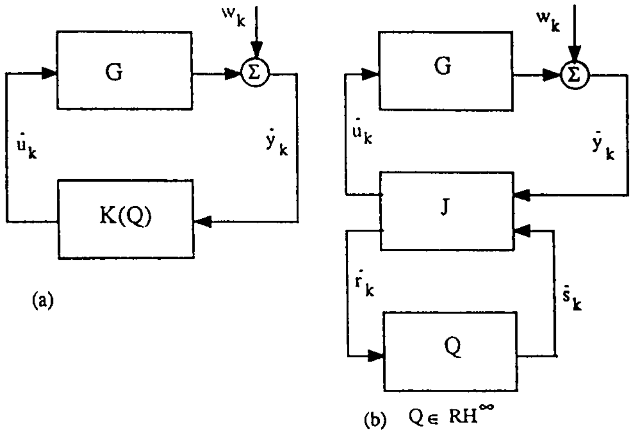

Figure 5. Stability of closed loop system

Clearly, performing the minimization task of (16) without any constraint on Q and taking $\gamma = 0$ for the scalar plant case will yield $Q = (I - \tilde{V})\tilde{N}^{-1}$. It is shown in Reference 6 that consequently the zeros of the plant G are contained in the modes of Q. If G is non-minimum phase, then Q is unstable. From (10) it is straightforward to show that this is equivalent to the plant–controller pair containing 'unstable zero–pole cancellations'. Therefore the closed loop system is not stable in the definition of (2). It is well known that such a situation will not occur if in Figure 5(a) $\bar{u}_k$ is bounded for bounded w_k. This can be translated[9] to requiring $\bar{s}_k$ of Figure 5(b) to be bounded. Now the transfer function from w_k to $\bar{s}_k$ is given by $\bar{s}_k = Q\tilde{M}w_k$. Thus selecting $e_{2k} = \gamma\bar{s}_k$, $\gamma > 0$, in the minimization of (17) avoids the 'unstable zero–pole cancellations' mentioned above.

Degree of sensitivity/loop recovery, γ

Loop recovery is achieved in Figure 1 by adding the signal s derived from passing the estimation residue r through a stable transfer function Q to the control. Thus $s = 0$ implies no loop recovery. On the other hand, zero constraint on s allows for the possibility of full loop recovery. In our case, selecting the penalizing constant γ to be either infinity or zero in the minimization task (19) caters for the two extreme cases. Thus choosing a finite γ between zero and infinity can be interpreted as determining the degree of loop recovery desired.

3. ADAPTIVE LQG/LTR ALGORITHM

In this section we present an adaptive LQG controller for scalar plants with sensitivity recovery (frequency-shaped loop transfer recovery) motivated by the off-line theory of Reference 7 reviewed in the previous section. The rationale for the algorithm is implicit in the development of the previous section.

The algorithm proposed consists of three parts. In the first part a standard identification algorithm such as the recursive least squares or the extended least squares algorithm is used to identify a model of the plant. In the second part a standard LQG controller using the parameter estimates and based on a single- (or multiple-) step update Riccati equation[8] is used to achieve a state feedback control signal u_k^{LQG}. In the third part there is adaptive sensitivity recovery by means of an adaptive Q_k feeding back estimation residuals r_k, giving $s_k = Q_k r_k$, which adds to the state feedback control u_k^{LQG}. Details are now developed.

Signal model

We shall assume the scalar variable ARMAX model

$$\begin{aligned}&\bar{A}y_k = \bar{B}u_k + \bar{C}w_k\\ &\bar{A} = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}, \qquad \bar{B} = b_1 z^{-1} + \cdots + b_n z^{-n},\\ &\qquad\qquad \bar{C} = 1 + c_1 z^{-1} + \cdots + c_n z^{-n}\end{aligned} \tag{24}$$

where w_k is a random white noise sequence. This can be restructured as

$$\begin{aligned}y_k &= \theta^{\mathrm{T}}\varphi_k + w_k\\ \theta^{\mathrm{T}} &= [a_1 \ldots a_n \;\; b_1 \ldots b_n \;\; c_1 \ldots c_n],\\ \varphi_k^{\mathrm{T}} &= [-y_{k-1} \ldots -y_{k-n} \;\; u_{k-1} \ldots u_{k-n} \;\; w_{k-1} \ldots w_{k-n}]\end{aligned} \tag{25}$$

Identification: extended least squares

The following standard algorithm is used to obtain plant parameter estimates $\hat{\theta}_k$ from input/output measurements:

$$\begin{aligned}\hat{\theta}_k &= \hat{\theta}_{k-1} + \hat{P}_k\hat{\varphi}_k(y_k - \hat{\varphi}_k^{\mathrm{T}}\hat{\theta}_{k-1})\\ \hat{P}_k &= \left(\sum_{i=1}^{k} \hat{\varphi}_i\hat{\varphi}_i^{\mathrm{T}}\right)^{-1} = \hat{P}_{k-1} - \hat{P}_{k-1}\hat{\varphi}_k(I + \hat{\varphi}_k^{\mathrm{T}}\hat{P}_{k-1}\hat{\varphi}_k)^{-1}\hat{\varphi}_k^{\mathrm{T}}\hat{P}_{k-1}, \text{ suitably initialized}\\ \hat{\varphi}_k^{\mathrm{T}} &= [-y_{k-1} \ldots -y_{k-n} \;\; u_{k-1} \ldots u_{k-n}\hat{w}_{k-1} \ldots \hat{w}_{k-n}], \;\; \hat{w}_k = y_k - \hat{\varphi}_k^{\mathrm{T}}\hat{\theta}_k\\ \hat{\theta}_k &= [\hat{a}_{1k} \ldots \hat{a}_{nk} \;\; \hat{b}_{1k} \ldots \hat{b}_{nk} \;\; \hat{c}_{1k} \ldots \hat{c}_{nk}]\end{aligned} \tag{26}$$

Standard conditions such as persistence of excitation, etc. are necessary to ensure consistency of the parameter estimates. These conditions will not be discussed here. See Reference 11 for more details.

LQG adaptive controller

The parameter estimates $\hat{\theta}_k$ of (26) allow estimates of the deterministic plant state space matrix written in companion matrix form as (1) with

$$A_k = \begin{bmatrix} -\hat{a}_{1k} & 1 & 0 & \ldots \\ -\hat{a}_{2k} & 0 & 1 & \ldots \\ \vdots & & & 1 \\ -\hat{a}_{nk} & 0 & & 0 \end{bmatrix}, \qquad B_k = \begin{bmatrix} \hat{b}_{1k} \\ \vdots \\ \hat{b}_{nk} \end{bmatrix}, \qquad C_k = [1 \;\; 0 \;\; \ldots \;\; 0], \qquad D_k = 0 \tag{27}$$

For the performance index $\Sigma_{i=1}^{\infty}(y_i^2 + R_c u_i^2)$, where R_c is a scalar constant, the following are standard equations defining an adaptive LQG control u_k^{LQG}:[12]

$$
\begin{aligned}
u_k^{\mathrm{LQG}} &= F_k \hat{x}_k \\
Q_c &= C_k^{\mathrm{T}} C_k, \qquad \hat{P}_0^c = 0 \\
F_k &= -(B_k^{\mathrm{T}} \hat{P}_k^c B_k + R_c)^{-1} B_k^{\mathrm{T}} \hat{P}_k^c A_k \\
\hat{P}_{k+1}^c &= A_k^{\mathrm{T}} [\hat{P}_k^c - \hat{P}_k^c B_k (B_k^{\mathrm{T}} \hat{P}_k^c B_k + R_c)^{-1} B_k^{\mathrm{T}} \hat{P}_k^c] A_k + Q_c \\
\hat{x}_{k+1} &= A_k \hat{x}_k + B_k u_k - H_k r_k, \qquad r_k = y_k - C_k \hat{x}_k \\
H_k &= [(\hat{a}_{1k} - \hat{c}_{1k}) \ldots (\hat{a}_{nk} - \hat{c}_{nk})]^{\mathrm{T}}
\end{aligned}
\tag{28}
$$

Variations on these recursions allowing multistep iterations or central tendency implementation[10] can also be accommodated.

Adaptive Q_k

Consider the adaptive LQG controller with a stable adaptive filter Q_k tuned on-line to achieve sensitivity recovery of the original LQ design. Let us first parametrize Q_k (here a scalar) in terms of parameters Θ for the time-invariant case as

$$
Q_\Theta = \frac{\beta_1 z^{-1} + \cdots + \beta_p z^{-p}}{1 + \alpha_1 z^{-1} + \cdots + \alpha_m z^{-m}}, \qquad \Theta = [\alpha_1 \ldots \alpha_m \ \beta_1 \ldots \beta_p] \tag{29}
$$

For the time-varying Q_k case there is an obvious generalization with Θ now time-varying as

$$
\Theta_k = [\alpha_{1k} \ldots \alpha_{mk} \ \beta_{1k} \ldots \beta_{pk}]
$$

We proceed to define an algorithm for Θ_k selections based on Lemma 1 but for the time-varying case as here where G is estimated on-line. Using (27) and (28), define from the time-invariant filters of (8) the time-varying version

$$
\begin{aligned}
[\tilde{N}_k \ \tilde{M}_k] &= \left[\begin{array}{c|cc} A_k + H_k C_k & B_k & H_k \\ \hline C_k & 0 & I \end{array}\right]_{\mathrm{T}} \\
I - \tilde{V}_k &= \left[\begin{array}{c|c} A_k + H_k C_k & B_k \\ \hline F_k & 0 \end{array}\right]_{\mathrm{T}}, \qquad N_k = \left[\begin{array}{c|c} A_k + B_k F_k & B_k \\ \hline C_k & 0 \end{array}\right]_{\mathrm{T}}
\end{aligned}
\tag{30}
$$

Filtered regressors

Now, with $\hat{w}_k$ derived from (26), define filtered versions of $\hat{w}_k$ as

$$
\zeta_k = \begin{bmatrix} N_k (I - \tilde{V}_k) \\ 0 \end{bmatrix} \hat{w}_k, \qquad \xi_k = \begin{bmatrix} N_k \tilde{N}_k \\ -\gamma \tilde{M}_k \end{bmatrix} \hat{w}_k, \quad 0 < \gamma < \infty
$$

and regressors

$$
\phi_k = [(e_{k-1} - \zeta_{k-1}) \ldots (e_{k-m} - \zeta_{k-m}) \ \xi_{k-1} \ldots \xi_{k-p}]^{\mathrm{T}}, \qquad e_k = \zeta_k - \Theta_k^{\mathrm{T}} \phi_k \tag{31}
$$

Least squares Θ_k selection

Consider the on-line least squares index

$$
\min_\Theta \frac{1}{k} \sum_{i=1}^{k} e_{i/\Theta}^{\mathrm{T}} e_{i/\Theta}, \qquad e_{k/\Theta} = \zeta_k - \Theta^{\mathrm{T}} \phi_k \tag{32}
$$

This optimization leads to parameters Θ_k with the associated least squares algorithm given as follows:

$$\Theta_k = \Theta_{k-1} + \hat{\Gamma}_k \hat{\phi}_k (\zeta_k - \Theta_{k-1}^{\mathrm{T}} \hat{\phi}_k)$$

$$\hat{\Gamma}_k = \left(\sum_{i=1}^{k} \hat{\phi}_k \hat{\phi}_k^{\mathrm{T}} \right)^{-1} = \hat{\Gamma}_{k-1} - \hat{\Gamma}_{k-1} \hat{\phi}_k (I + \hat{\phi}_k^{\mathrm{T}} \hat{\Gamma}_{k-1} \hat{\phi}_k)^{-1} \hat{\phi}_k^{\mathrm{T}} \hat{\Gamma}_{k-1} \tag{33}$$

$$\hat{\phi}_k = [(\hat{e}_{k-1} - \zeta_{k-1}) \ldots (\hat{e}_{k-m} - \zeta_{k-m}) \ \xi_{k-1} \ldots \xi_{k-p}]^{\mathrm{T}}, \qquad \hat{e}_k = \zeta_k - \Theta_k^{\mathrm{T}} \hat{\phi}_k$$

Note here that $\hat{e}_k$, ζ_k and ξ_k are vectors.

Adaptive loop

Define a signal $s_k = Q_k r_k$ as follows, recalling that $r_k = y_k - C_k \hat{x}_k$. Thus

$$s_k = \Theta_{k-1}^{\mathrm{T}} \psi_k, \qquad \psi_k^{\mathrm{T}} = [-s_{k-1} \ldots -s_{k-m} \ r_{k-1} \ldots r_{k-p}] \tag{34}$$

Then the plant input is given by

$$u_k = u_k^{\mathrm{LQG}} + s_k \tag{35}$$

where u_k^{LQG} is given from (26).

Stability of adaptive controller

The first two parts of the adaptive scheme consisting of the parameter estimator and the adaptive LQG controller are standard and analysis with regards to consistency of parameter estimates and stability of the LQG controller can be found for example in Reference 12. For our scheme, if asymptotically the plant estimate $\hat{G} \to G$ and the resultant adaptive LQG controller is stabilizing, then from Lemma 1 the optimization task will yield a stable Q_k. From the theory on the class of all stabilizing controllers we then conclude that asymptotically the resultant adaptive controller consisting of the adaptive LQG controller and the stable operator Q_k derived from the LTR optimization algorithm will yield a stable closed loop.

Closed loop stability and good performance, however, cannot be guaranteed during the transient stage. To achieve closed loop stability and a reasonable performance during the transient stage, switching algorithms such as those presented in References 11 and 13 may have to be used, in which case stability analysis in our case will be no different from that presented in those references. These issues will not be pursued further here.

4. SIMULATION RESULTS

Example

A third-order actual plant described by an ARMAX model given as follows is used in this example to demonstrate the robustness properties of the adaptive LQG/LTR controller.

$$(1 - 1{\cdot}55z^{-1} + 0{\cdot}695z^{-2} + 0{\cdot}085z^{-3})y_k = (z^{-1} - 3{\cdot}5z^{-2} + 3{\cdot}0z^{-3})u_k + w_k \tag{36}$$

The plant has stable poles at $z = 0{\cdot}2$, $0{\cdot}5$ and $0{\cdot}85$ and non-minimum phase zeros at $z = 1{\cdot}5$ and 2. To demonstrate the robustness of the adaptive LQG/LTR controller, we undermodel the plant order in the recursive least squares algorithm. A second-order model is assumed for the identification algorithm instead of a third-order model.

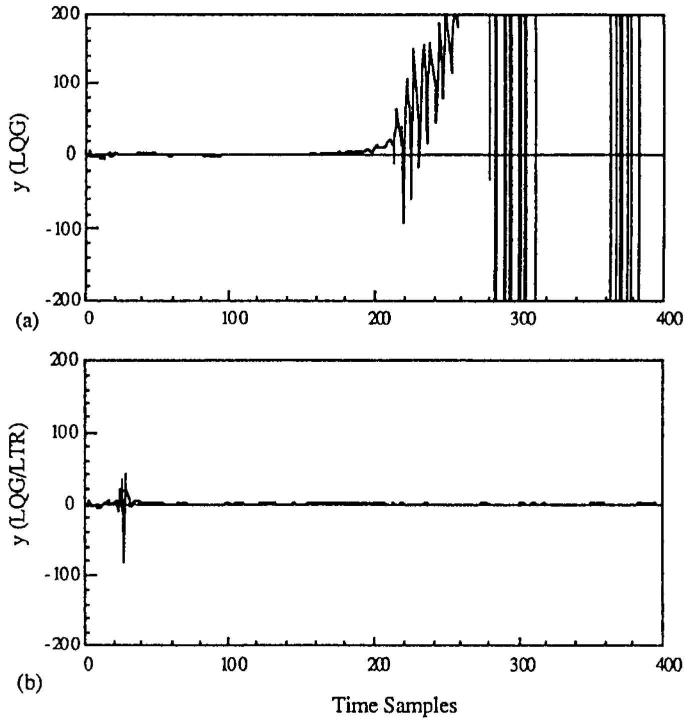

Figure 6. Plots of plant output. Top: adaptive LQG controller. Bottom: adaptive LQG/LTR controller

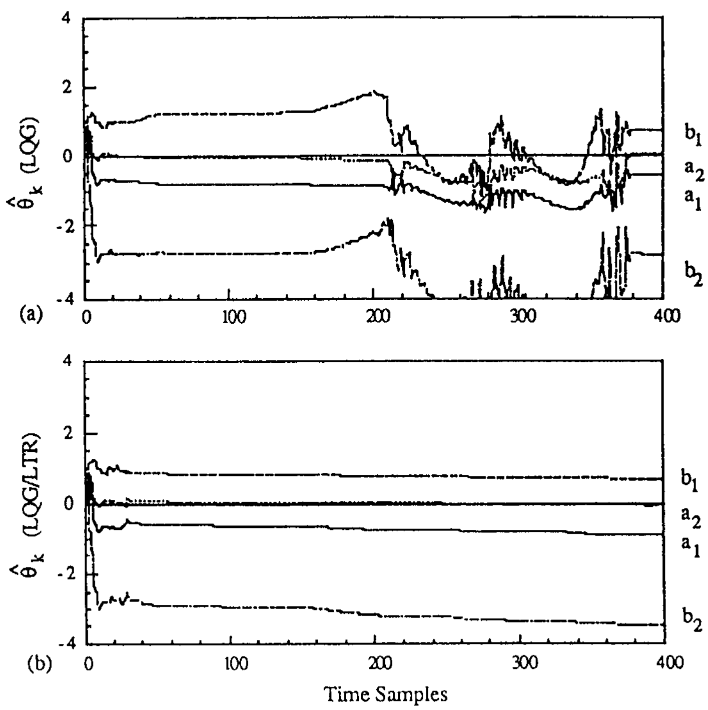

Figure 7. Plots of parameter estimates of plant model: (a) adaptive LQG controller; (b) adaptive LQG/LTR controller

Consider first the implementation of a standard adaptive LQG controller, i.e. without on-line loop recovery. The results of a simulation are summarized by Figures 6(a) and 7(a). The estimated plant parameters of Figure 7(a) drift and eventually the closed loop system exhibits unstable behaviour as shown by the plant output in Figure 6(a). The adaptive LQG controller, without modification, is clearly not robust enough to handle this undermodelling situation in the identification algorithm.

The adaptive LQG/LTR controller as described in Section 3 is next implemented for the same plant with undermodelling in the identification algorithm as above. The penalty on s_k, γ of (18), is chosen as $\gamma = 0{\cdot}01$ and a third-order Q_k is used. Figure 6(b) shows the plant output for this simulation run. The plant output remains bounded. The plant parameter estimates are shown in Figure 7(b). The parameters drift as in the adaptive LQG case; however, as a result of the more robust controller implemented, the plant input and output signals remain bounded and therefore there is no wild swing in the estimates. Clearly the adaptive LQG/LTR controller is more robust in this situation.

Discussions

1. For a given fixed γ and a Q of appropriate dimension, asymptotically the operator Q_k approaches the transfer function given by (20). As $\gamma \to 0$, it is easy to deduce that the poles of the time-invariant optimal Q approach the minimum phase zeros of the plant and the reflection of any non-minimum phase zeros of the plant into the unit circle. Clearly, if the plant possesses zeros on or close to the unit circle, asymptotically Q_k approaches a marginally stable transfer function. In simulation runs (not reported here) where the plant contains zeros close to the unit circle a selection of a small γ gives rise to a marginally stable Q, which results in a marginally stable closed loop. Thus γ has to be chosen large to avoid such a situation. However, with a large γ, $Q_k \to 0$ and little loop recovery is achieved.

2. As mentioned before, our proposed scheme is an adaptive robust controller as opposed to a robust adaptive controller. Thus for a situation as depicted in the simulation runs above where the parameter estimates drift slowly, the additional LTR component serves to make the 'almost time-invariant' controller more robust. This more robust controller can then cope with more diverse variations than the less robust LQG (i.e. without LTR) controller. However, if the plant is subjected to rapid variations, causing the parameter estimates to change rapidly, the adaptive LQG/LTR controller is not going to perform any better than the adaptive LQG controller. To cope with more rapid variations in plant uncertainties, it is necessary to look at robust adaptive schemes, e.g. the switching schemes of References 11 and 13.

6. CONCLUSIONS

In this paper an adaptive LQG controller with adaptive input sensitivity function/loop transfer recovery of an associated adaptive LQ design for a scalar plant is described. Loop recovery is achieved by feeding back estimation residuals to the control through a stable operator Q_k and is motivated by robustness properties of off-line LQG/LTR techniques. The algorithm adapts on-line a robust LQG/LTR controller. On-line LTR is achieved by optimizing over Q_k using straightforward least squares techniques. The least squares index is constructed both to achieve loop recovery and to maintain closed loop stability of the system.

The proposed adaptive algorithm appears attractive from both a computational complexity and performance/robustness viewpoint. As expected, the adaptive LQG/LTR algorithm appears to have at least as good a robustness as the corresponding adaptive LQG scheme and is more robust than the adaptive LQG scheme in some situations. It appears worthwhile to apply analytical and simulation studies to reflect a comparison between the proposed algorithm and adaptive predictive schemes and others in common use.

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