INSTITUTE OF PHYSICS PUBLISHING PHYSIOLOGICAL MEASUREMENT

Physiol. Meas. **22** (2001) 91–96 www.iop.org/Journals/pm PII: S0967-3334(01)19904-1

Adaptive mesh refinement techniques for electrical impedance tomography

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Received 4 December 2000

Abstract
Adaptive mesh refinement techniques can be applied to increase the efficiency of electrical impedance tomography reconstruction algorithms by reducing computational and storage cost as well as providing problem-dependent solution structures. A self-adaptive refinement algorithm based on an *a posteriori* error estimate has been developed and its results are shown in comparison with uniform mesh refinement for a simple head model.

Keywords: nonlinear electrical impedance tomography, adaptive mesh refinement, h-refinement, p-refinement, efficiency, improved convergence

1. Introduction

Electrical impedance tomography (EIT) can provide images with well defined characteristics only when the full nonlinear reconstruction process is constrained by a property of the image such as its local smoothness, applied in parallel with the requirement to fit the data to within clearly defined statistical criteria (Blott *et al* 1998, 2000). The finite element forward solution is a significant part of the computational cost of such a reconstruction (Yorkey *et al* 1987, Johnson and MacLeod 1994). This cost grows quickly when the image is subdivided into smaller and smaller elements to obtain an image whose accuracy is governed by the quality of the input data alone and not by the choice of discretization.

To overcome the problems involved in handling high-density discretizations, sparse matrix techniques have been applied in the past (Pinheiro and Dickin 1997) without taking into account that a proportion of the cost could be avoided by using meshes adapted to the problem.

We have developed an algorithm which automatically adapts to the reconstructed image by producing finer meshes in areas where there are sharp gradients in the EIT image. Typically, refinement is required at interfaces between regions with differing conductivities. Although adaptive meshing has not yet been applied to resistance or impedance modelling in a biomedical context, there exists some work on applications of adaptive mesh refinement (AMR) in modelling heart current sources (Johnson and MacLeod 1994)—a topic related to EIT.

0967-3334/01/010091+06$30.00

We apply the auto-adaptive refinement method to the forward modelling problem and demonstrate that the method quickly reduces the number of nodes and elements so that the calculation converges much more rapidly to the true solution. In particular, to obtain the same accuracy as uniform refinement, the adaptive refinement reduces the number of elements by a factor of more than three, which yields up to an order of magnitude speed-up.

2. Methods

The basic equation of electrical impedance tomography in the case of imaging conductivity σ (or analogously for impedance imaging) is

$$\nabla \cdot \sigma \nabla \Phi = 0 \qquad \text{on } \Omega \tag{1}$$

subject to the boundary conditions:

$$\Phi = \Phi_0 \qquad \text{on } \Gamma_D \qquad \sigma \frac{\partial \Phi}{\partial n} = j_n \qquad \text{on } \Gamma_N \tag{2}$$

where Φ and Φ_0 represent electrostatic potentials resulting from the injected current density j_n in the direction of the surface normal n, and Γ_D and Γ_N are the boundaries with Dirichlet or von Neumann conditions, respectively. To solve equation (1) numerically, the problem domain Ω is subdivided into discrete finite elements. If insufficient elements are used, the choice of discretization will affect the accuracy of the potential distribution, and also the calculation of the Jacobian in the nonlinear reconstruction of the conductivities. It is therefore usual to refine the mesh globally to improve the accuracy of the solution across the whole domain.

However, it is in fact only necessary to refine the mesh where the error is large: the paradox is that the exact error is only known if the exact solution is available! We therefore use an *a posteriori* error estimate, which is used to determine whether refinement of the mesh is required. Starting with an initially rather coarse quality mesh (Shewchuk 1996), we refine according to this estimator and adapt the mesh to give an accurate solution.

Error estimates for finite element analysis of elliptic problems have been extensively studied. We have chosen a residual-based energy norm estimator, which is robust (Salazar-Palma *et al* 1998) and physically sound. It is obtained by multiplying equation (1) by an arbitrary weighting function and integrating by parts. We then obtain a measure for the error $e = \Phi_{\text{exact}} - \Phi_{\text{approx}}$ of the numerically approximated potential distribution Φ_{approx}. The *a posteriori* energy norm error $||e||_E^2$ is estimated based on surface and lumped edge residuals $(r_{\text{surf}}, r_{\text{edge}})$ of the intra-element and inter-element current densities within a two-dimensional triangular mesh. The local error estimate, ε_l for a single element l with area Ω_l, longest side h_l and conductivity σ_l is computed as follows:

$$\varepsilon_l^2 = f_{\text{surf}} \frac{h_l^2}{\sigma_l p} \int_{\Omega_l} r_{\text{surf}}^2 \, \mathrm{d}\Omega + f_{\text{edge}} \frac{h_l}{\sigma_l p} \sum_{\substack{\Gamma_k \in \Gamma_{\Omega_l} \\ \Gamma_k \not\subset \Gamma_D}} \int_{\Gamma_k} r_{\text{edge}}^2 \, \mathrm{d}\Gamma \tag{3}$$

where f_{surf} and f_{edge} are numerically estimated factors (Salazar-Palma *et al* 1998) and the sum over edges Γ_k in the second term runs over all elements (Γ_{Ω_l}) which do not have Dirichlet boundary conditions (Γ_D). p represents the polynomial order of the basis function. The total error for the mesh can be approximated by summing the local estimates:

$$||e||_E^2 \approx \sum_{l=1}^{N_e} \varepsilon_l^2. \tag{4}$$

The error estimate can be used in three ways to refine the mesh:

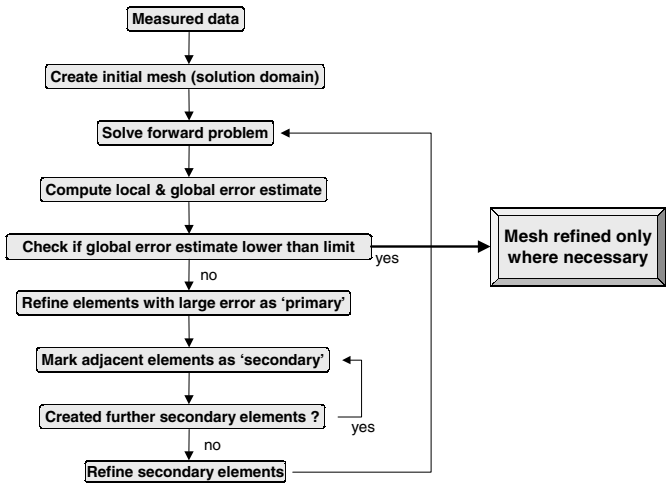

Figure 1. Steps in auto-adaptive mesh algorithm for solution of one instance of the forward problem.

(a) *h-refinement* consists of subdividing elements into two or more elements; h represents the element size (Burnett 1987).

(b) *p-refinement* increases the rate of convergence by using higher order interpolating basis functions on the elements (Zienkiewicz and Craig 1986).

(c) *r-refinement* relocates the existing nodes of a mesh in a more appropriate fashion without adding any new nodes (Shepard 1985).

Efficient hybrids of these methods also exist, but can be complicated to implement. In this article, we focus on h-refinement of linear elements, which is both fast and adds relatively few additional elements and nodes to the mesh. p-refinement is an already commonly used improvement but produces larger matrices with increasing polynomial order. r-refinement requires modification of the mesh at each refinement stage but does usually not significantly improve the solution; however, it might be useful in time-dependent problems such as monitoring breathing, where the nodes of the mesh can follow predefined trajectories.

3. Auto-adaptive mesh algorithm

Figure 1 shows the steps in the adaptive meshing algorithm. We initialize the procedure with a coarse mesh and with the configuration of electrodes used. For reasons of simplicity we assume point-sized electrodes; however, the method works equally well for the complete electrode model (Somersalo *et al* 1992). For each step in the forward solution of the EIT reconstruction, we estimate the local error using (3) and compute the global error using (4). If the global error estimate is larger than a pre-defined threshold, the refinement procedure is started. Otherwise, the refinement is complete. The refinement algorithm then starts by marking 'primary' elements, which are those with a local error estimate above a certain percentage of the maximum occurring local error (usually 40–50% for best efficiency).

As shown in figure 2, the primary elements are then subdivided into four smaller elements. This yields 'floating' nodes on the sides of adjacent secondary elements, which must be subsequently refined to maintain the required node connectivity within the mesh.

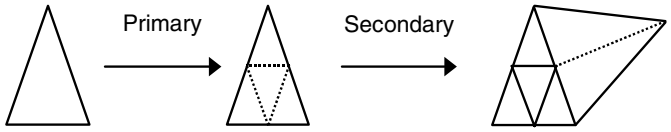

Figure 2. Refinement of triangular elements.

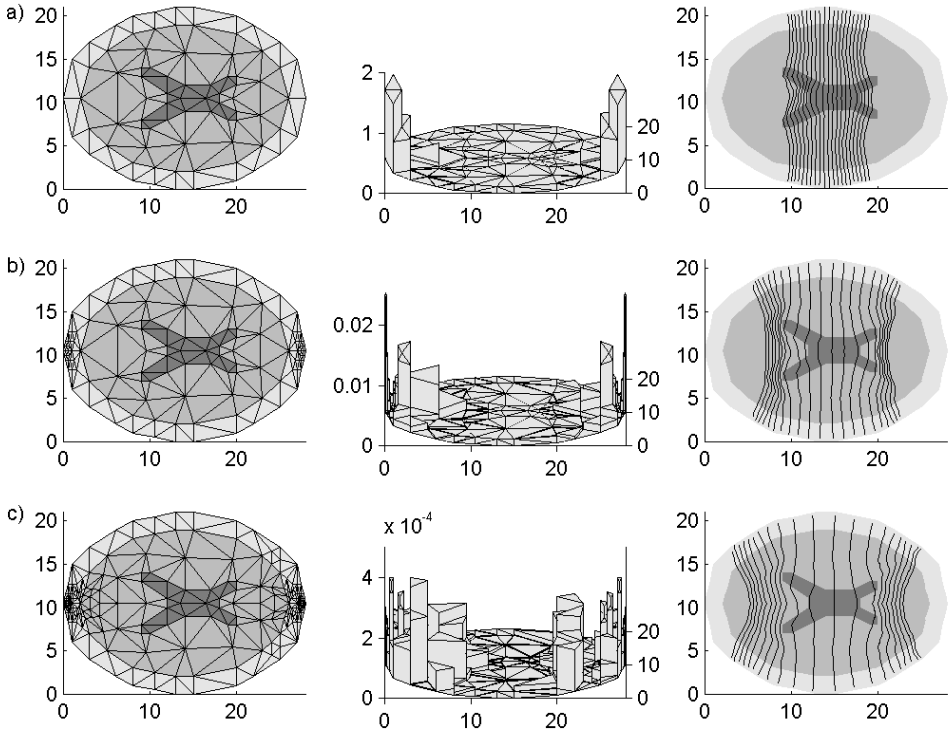

Figure 3. Example of auto-adaptive meshing for a head model showing bone, white matter and ventricles. Left, finite element mesh. Centre, local error estimates for each element in z-direction. Right, potential distribution in the region of interest for (a) the initial mesh, (b) after three refinement steps and (c) after seven refinement steps.

If we are using a fully nonlinear reconstruction algorithm (Blott *et al* 2000), then the mesh refinement steps form a natural part of the iterative solver.

4. Results

We have implemented the adaptive-meshing algorithm described in the previous section. Our results demonstrate the application of the method when solving the forward problem i.e. computing the potentials given the conductivities. Figure 3 shows a model of a transverse slice of the head with two electrodes, where we have used typical values for the conductivities of the tissue (0.25 S m^{-1}), bone (0.018 S m^{-1}) and cerebro-spinal fluid (1.79 S m^{-1}). This configuration could be used to monitor intra-ventricular haemorrhaging. As the mesh is refined, the equi-potential lines tend to the true solution, which we have verified using an independent FE solver.

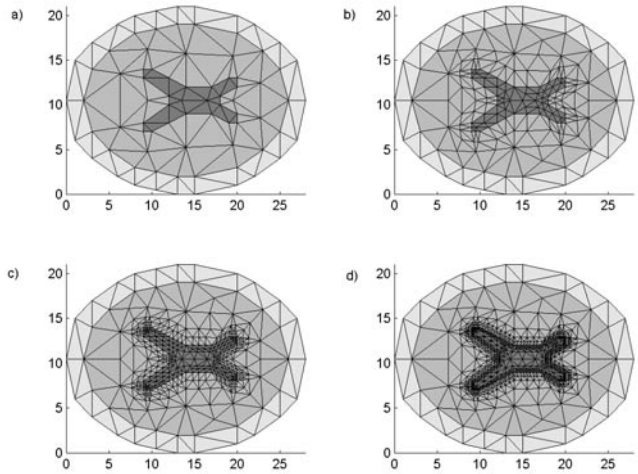

Figure 4. Material gradient-dependent mesh refinement provides higher resolution at material boundaries (for example for smoothing constraints).

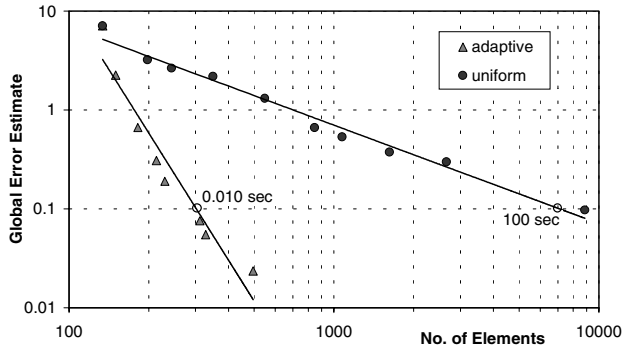

Figure 5. Comparison of error estimates for adaptive and uniform refinement for the head example.

Figure 4 shows the method being applied where we refine the mesh at boundaries between materials with significantly differing conductivities. In a practical application of our technique, it would be desirable to combine both strategies to yield an accurate solution, which can give good resolution of material boundaries.

If the image contains materials with differing conductivities, we can replace the error estimate, which determines when elements are refined, with one which refines elements based on the gradient of the reconstructed conductivity. This allows such boundaries to be more sharply resolved.

In figure 5 we show the performance benefits of our approach, by comparing the convergence of our method with a uniform refinement strategy. The adaptive algorithm requires only a small fraction of the number of elements/nodes in the uniformly refined mesh to achieve a given global error estimate. In particular, to attain a global error of 0.1 requires 300 elements and 0.01 s using the adaptive technique compared to 7000 elements and 100 s of computer time for the global refinement strategy. Since the solution of the forward problem scales with order between $N^{1.46}$ and N^2, where N is the number of nodes, reducing N saves both computation time and storage requirements for the system matrix.

Whilst there are many benefits of the adaptive refinement procedure, a number of numerical issues can arise. For example, the method for subdividing the elements has to be chosen in a way to avoid degenerate elements of high aspect ratios (small angles) and subsequent incorrect solutions (Salazar-Palma *et al* 1998). In addition, non-smooth transitions between regions of low and high mesh densities are likely to produce less accurate results (Burnett 1987).

Our results were obtained on a 500 MHz AMD Athlon PC running Windows NT 4.0; the code is written in C++ and compiled using MS Visual C++ 6.0.

5. Conclusions and further work

We have developed an efficient adaptive mesh refinement algorithm and applied it to improve the performance of EIT reconstruction algorithms by reducing both computational and storage requirements. We demonstrate its application to imaging of a section through the head and show that (i) the accuracy of the forward solution is improved using considerably fewer elements than a global refinement strategy and (ii) the resolution of interfaces between materials with differing conductivities is improved.

Our results indicate that it is possible to reduce the number of nodes required by at least a factor of three to obtain an accurate image reconstruction, over a uniform refinement strategy. This results in at least an order of magnitude improvement in the speed of the forward problem and increases the feasibility of performing fully nonlinear reconstructions for complex large-scale biomedical problems in real-time using standard PC technology. Future work will integrate our method into a full nonlinear solver for 3D reconstruction.

Acknowledgments

We thank J Shewchuk for the meshing program TRIANGLE. MM is grateful to EPSRC for funding.

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