# Adaptive Moving Grid Solutions of a Shallow-water Transport Model with Steep Vertical Gradients

P. Zegeling\*, M. Borsboom\*\*, J. van Kester\*\*

*\*Dept. of Mathematics, Utrecht University, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands*

*\*\*Delft Hydraulics, P.O. Box 177, 2600 MH Delft, The Netherlands*

## Abstract

We investigate the use of a boundary-fitted moving adaptive grid technique in vertical direction for the numerical modeling of transport in multi-dimensional shallow-water applications. The difficulty with such modeling is that both the non-horizontal bottom and the free water surface, as well as nearly-horizontal regions of high gradient are to be represented accurately. The results that we present show that by using a moving grid that adapts automatically to the solution and is fitted to both the bottom and the free surface, it is possible in principle to model all relevant transport processes.

## Introduction

In stratified flows in estuaries and lakes, the exchange of matter in the vertical direction plays a crucial role in water quality studies. Steep vertical gradients of salinity or temperature may lead to locally very small turbulence levels that reduces significantly the vertical transport. In order to take proper account of this phenomenon, it is important to model steep vertical gradients accurately. Likewise, it is important to represent the free surface and the bathymetry adequately, for the modeling of the effect of, e.g., wind shear stress and bottom friction. Satisfying these demands with one of the standard numerical approaches is difficult Deleersnijder & Ruddick [5].

The use of a vertical coordinate system that is boundary-fitted to both

the bottom and the surface (the so-called sigma coordinate system) is well suited to represent a varying bottom topography and a moving free surface. On the other hand, the calculation of gradients in vertical direction depends critically on how the equations are transformed to sigma coordinates Huang & Spaulding [2], as well as on the size of discretization errors. The latter may lead to large modeling errors in the vicinity of the sharp interface between layers of different salinity or temperature Stelling & van Kester [1]. Since these layers usually extend (nearly) horizontally, their interface can be modeled adequately by using a Cartesian coordinate system with strictly horizontal grid surfaces Stelling & van Kester [1]. This, however, necessitates the use of a less accurate 'staircase' grid to represent the surface and bottom Deleersnijder & Ruddick [5].

To satisfy all modeling demands simultaneously, we propose an *adaptive* sigma coordinate system, where grid points are moved dynamically in the vertical to capture all important details. This includes the modeling of steep vertical gradients that do not extend in strictly horizontal direction, as well as steep gradients that move in time. Moving adaptive grid methods, also characterized by the term $r$-refinement, have been shown to be very useful for solving parabolic and hyperbolic partial differential equations (PDEs) involving small scale structures. For a recent overview of developments in adaptive grid methods in general, we refer to the proceedings of the Conference on Grid Adaptation in Computational PDEs, 1996, Edinburgh (Duncan [6]).

In one space dimension, moving adaptive grid methods have been applied successfully to a large class of PDE systems (see, e.g., Zegeling et al. [8, 7] and Borsboom & van der Marel [3]). In two space dimensions, application of moving adaptive grid methods is less trivial (Catherall [4]). Two, not just one, coordinates per grid point have to be determined, i.e., not only the grid stretching in two directions, but also the grid skewness has to be controlled.

In this paper we describe a two-dimensional finite-difference method with grid adaptation in one direction only. The latter reduces the complexity of the technique. Although keeping grid distortion sufficiently low, it is still an important issue. The moving adaptive grid technique that we employ is therfore based on an equidistribution principle with smoothing operators to ensure that the generated adaptive grids are smooth both in space and in time. As an illustration, we present results for a 2D transport problem with a steep vertical gradient that enters the domain from the inflow boundary. The problem is a model for the advection of a moving salt wedge that creates a sharp, moving interface between a layer of salt water near the bottom and

the fresh water area. Such a situation is typical of estuaria problems.

# The Transport Model

We consider the time-dependent two-dimensional transport equation:

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(uc) + \frac{\partial}{\partial z}(wc) = \frac{\partial}{\partial x}\left(D_H \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial z}\left(D_V \frac{\partial c}{\partial z}\right) , \quad (1)$$

where $(u, w)^T$ is the velocity vector that satisfies the incompressibility constraint $\partial u/\partial x + \partial w/\partial z = 0$, and where $D_H$ and $D_V$ are respectively the coefficient of horizontal dispersion and of vertical turbulent diffusion. We seek for the solution $c(x, z, t)$ (salinity of the water) with $(x, z) \in \Omega$ and $t \in [0, T]$, in which $\Omega$ represents an estuary model defined by $x \in [0, L]$ and $z \in [z_b, \zeta]$, with $L$, $z_b$, and $\zeta$ respectively the length of the model, the bottom profile, and the position of the free surface.

The procedure (described below) for finding numerical solutions of (1) is based on the method of lines. This means that the discretization is carried out in two stages. In the first stage the space variables are discretized, giving a large system of ordinary differential equations (ODEs). The second stage deals with the numerical integration in time of this stiff ODE system to generate the desired numerical solution.

# The adaptive $\sigma$-transformation

The physical domain is transformed to the computational domain $[0, 1] \times [0, 1]$ by the mapping $(x, z, t) \rightarrow (\xi, \sigma, \tau)$:

$$\xi = \xi(x, z, t) ,$$

$$\sigma = \sigma(x, z, t) ,$$

$$\tau = t .$$

Using the inverse of this transformation, transport model (1) is written as:

$$\frac{\partial \mathcal{J} c}{\partial \tau} + \frac{\partial}{\partial \xi}[z_\sigma uc] + \frac{\partial}{\partial \sigma}\left[(x_\xi(w - z_\tau) - z_\xi u) c\right]$$

$$= \frac{\partial}{\partial \xi}\left[\frac{z_\sigma^2}{\mathcal{J}} D_H \frac{\partial c}{\partial \xi} - \frac{z_\sigma z_\xi}{\mathcal{J}} D_H \frac{\partial c}{\partial \sigma}\right] + \quad (2)$$

$$\frac{\partial}{\partial \sigma}\left[-\frac{z_\xi z_\sigma}{\mathcal{J}}D_H\frac{\partial c}{\partial \xi} + \left(\frac{z_\xi^2}{\mathcal{J}}D_H + \frac{x_\xi^2}{\mathcal{J}}D_V\right)\frac{\partial c}{\partial \sigma}\right] ,$$

where $\mathcal{J} = x_\xi z_\sigma$ is the Jacobian of the transformation. Note that we only consider grid point movement in vertical direction along strictly vertical grid lines, i.e., in (2) we have used $x_\tau = 0$ and $x_\sigma = 0$. We restrict ourselves to uniform grids in $x$-direction, so $x_\xi$ will be taken constant.

Equation (2) is space discretized on a uniform grid with grid cells of sizes $\Delta\xi$ and $\Delta\sigma$, using standard central differences. Dirichlet conditions are imposed at inflow boundaries and homogeneous Neumann conditions are applied elsewhere. Together, this leads to a system of ODEs for the vector of unknowns $C = \{c_{i,k}\}$, with $c_{i,k}$ the salinity per grid point. To complete this system, we need to define how the grid is moved with and adapted to the solution.

We have used a moving grid PDE that is based on an equidistribution principle enhanced with smoothing procedures in time and vertical space direction. The moving grid PDE is given by

$$\frac{\partial}{\partial \sigma}\left[\frac{\tilde{m} + \tau_s \dot{\tilde{m}}}{\mathcal{W}}\right] = 0 , \tag{3}$$

with $\tilde{m}$ the non-smoothed grid point concentration and weight function $\mathcal{W}$ taken as $\mathcal{W} = \sqrt{1 + \alpha(c_z)^2}$, where parameter $\alpha$ controls the level of adaptivity. A uniform grid in $z$-direction is obtained for $\alpha = 0$ when weight function $\mathcal{W}$ becomes equal to 1, whereas for $\alpha > 0$ the grid adapts to the solution gradient $\partial c/\partial z$.

The grid point concentration $m = 1/z_\sigma$ that is used in (2) is obtained by smoothing $\tilde{m}$:

$$\left[I - \kappa(\kappa + 1)(\Delta\sigma)^2\frac{\partial^2}{\partial \sigma^2}\right]m = \tilde{m} , \tag{4}$$

with $\kappa$ the spatial smoothing parameter. Loosely speaking, the weight function $\mathcal{W}$ determines the shape of the grid distribution and $\kappa$ the level of clustering. With $\kappa = O(1)$, rather modestly stretched space grids are obtained. The parameter $\tau_s$ in (3) controls the temporal smoothness of the grid; it serves as a delay factor for the grid movement and provides a means for suppressing grid oscillations in time. See Zegeling [7] for more details on the use of $\kappa$, $\tau_s$ and equations like (3).

The discretization of (3) and (4) in space gives the system of ODEs for $Z = \{z_{i,k}\}$, with $z_{i,k}$ the vertical coordinate per grid point. When

combined with the space discretization of (2) and its boundary conditions, we obtain the complete system of ODEs:

$$\mathcal{M}(C, Z)(\dot{C}, \dot{Z})^T = \mathcal{H}(C, Z) \,, \quad t > 0 \ .$$

# Numerical Results

In this section we present some results obtained with the moving finite-difference method (MFD) described before. In the experiment we have used a time tolerance $tol = 10^{-3}$ for DASSL, a uniform starting grid, and a zero initial concentration field. A flat bottom profile $z_b = 0\ [m]$ and a uniform water level $\zeta = 10\ [m]$ have been specified, while the length $L$ of the model was taken equal to $100\ [m]$. The values of the parameters in the model equation (1) were: $D_H = 0.1\ [m^2/s]$, $D_V = 0.001\ [m^2/s]$, $u = 0.5\ [m/s]$, $w = 0\ [m/s]$. These values model typical shallow-water conditions. The value of $D_V$ has been taken small enough to investigate the numerical problems associated with the modeling of small turbulent diffusion in the vertical.

At the left boundary, the following inflow Dirichlet condition was imposed:

$$c(0, z, t) = 0 \,, \quad z > f(t) \,; \quad c(0, z, t) = 1 \,, \quad z \leq f(t) \,, \qquad (5)$$

with $f(t) = 3 - 3*(t/200 - 1)^2 + z_b$ to model an incoming salt wedge that slowly grows in time. To avoid spurious oscillations in space, this inflow condition was smoothed slightly at $z = f(t)$ by means of a steep cosine function.

The time interval that we have considered is $t \in [0, 150]\ [sec]$. Note that during this period, the salt wedge specified in (5) does not enter fully into the domain. Further, 'standard' values have been used for the moving adaptive-grid parameters. They have been scaled to account for the spatial and time scale of the model, i.e., we have used $\alpha = 10^4$, $\kappa = 1$ and $\tau_s = 10^{-3}$.

In figure 1 numerical results are shown for the adaptive moving-grid method with $30 \times 15$ grid cells at $t = 150\ [sec]$. On the left the solution is displayed for different values of $x$ ($x = 0,\ 10,\ 30,\ 50,\ 70\ [m]$).

On the right of figure 1 the adapted grid is shown. It is seen that the grid adapts nicely to the transition region. For lower values of $x$, the concentration of grid points follows the sharp interface between low and high concentration. One clearly recognizes the parabolic shape of the salt wedge

imposed at the inflow boundary (5). Horizontal dispersion causes the inter-
face to smear out, so further downstream of the inlet there is no need for a
high local grid point concentration. The result shows that the adaptive grid
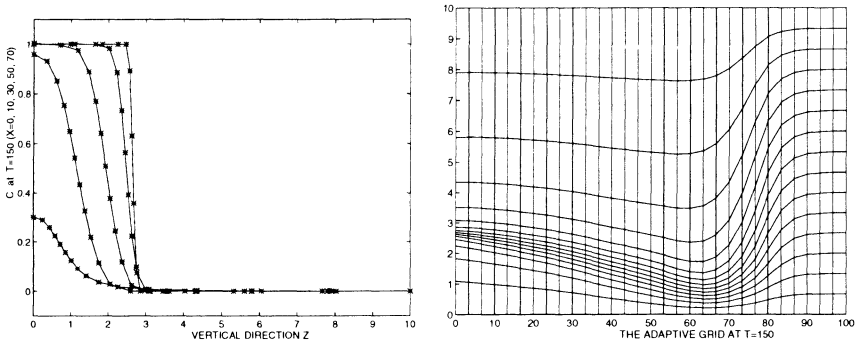algorithm takes care of this automatically.



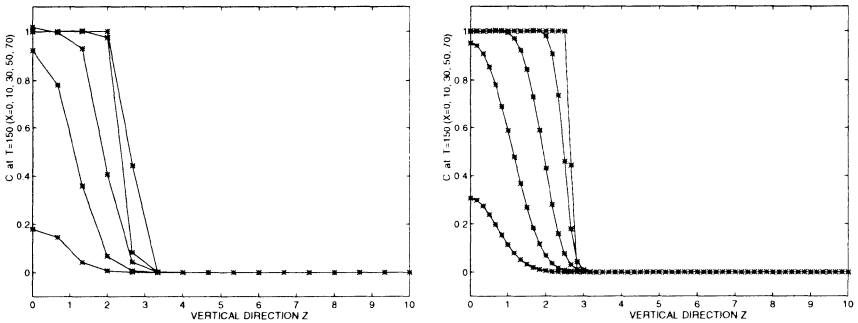Figure 1: Moving adaptive grid results (left: solution, right: grid).



Figure 2: Uniform grid solutions (left: $31 \times 16$ grid, right: $31 \times 61$ grid).

In figure 2 results are given for the fixed uniform grid case ($\alpha = 0$). On
the left the rather inaccurate solution obtained with a $31 \times 16$ grid is shown.
This uniform grid solution is improved by adding more grid points in the
vertical direction: the right plot in figure 2 shows the results for $31 \times 61$
uniformly distributed grid points.

A comparison between the figures 1 and 2 shows that when a uniform
grid is used, about 4 times more grid points are required to get a solution
of the same quality as the one obtained with the proposed grid adaptation
algorithm. Note however the slight 'lift-up' around $z = 2.5 \ [m]$ in the
adaptive grid result at the position $x = 70 \ [m]$ that is not present in the

uniform grid solution. Detailed inspection of the results at earlier time levels has revealed that this is caused by the excessive grid distortion of the adapted grid that occurs when the salt wedge starts entering the domain. This sudden change of the solution at $(x, z) = (0, 0)$ leads to a strong local clustering of grid points in the beginning of the calculation, and hence to rather large discretization errors because of the excessive skewness of the grid. The negative effect of this on the solution is transported downstream, and reaches position $x = 70$ $[m]$ at about $t = 150$ $[sec]$. It is this mechanism that causes the small error observed in figure 1.

Not shown in a figure, but important to mention is that results obtained with different values of $\kappa$ and $\tau_s$ have shown the importance of smoothing the grid both in time and in space (see eq. (3)). Grid smoothing is essential for letting the grid move with and adapt to the solution in a satisfactory way. This phenomenon has already been observed by others, see, e.g., Duncan [6].

# Conclusions

The performance of the adaptive moving-grid method presented in this paper is promising. Satisfactory results were obtained when the method was used to solve a time-dependent 2D transport model of an incoming salt wedge with steep salt-fresh water interface. When a uniform grid is used, 4 times more grid points have to be used in the vertical direction to produce comparable results.

It was observed that the smoothing of the grid both in time and in space is necessary to prevent the creation of irregular adaptive grids, although it did not fully suppress the effect of grid distortion on the solution. This is actually not surprising, since the weight function $\mathcal{W}$ that we used in (3) does not monitor the gradients in $\xi$-direction. The grid distortion in $x$-direction is therefore not properly detected by the algorithm.

The design of a weight function that monitors the behavior of the solution in both coordinate directions is presently under study. Another topic of research is the development and use of a discretization technique that is less sensitive to grid distortion. In the near future we will also investigate the performance of the method for other transport problems, in particular problems defined in geometries with varying bottom slope and non-uniform, moving free surface.

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