Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Adaptive Neural Control for Uncertain Nonlinear Systems in Pure-feedback Form with Hysteresis Input

Beibei Ren[1], Shuzhi Sam Ge[1*], Tong Heng Lee[1] and Chun-Yi Su[2]

***Abstract*— In this paper, adaptive neural control is investigated for a class of unknown nonlinear systems in pure-feedback form with the generalized Prandtl-Ishlinskii hysteresis input. The non-affine problem both in the pure-feedback form and in the generalized Prandtl-Ishlinskii hysteresis input function is solved by adopting the Mean Value Theorem. By utilizing Lyapunov synthesis, the closed-loop control system is proved to be semi-globally uniformly ultimately bounded (SGUUB), and the tracking error converges to a small neighborhood of zero. Simulation results are provided to illustrate the performance of the proposed approach.**

I. INTRODUCTION

Control of nonlinear systems with unknown hysteresis nonlinearities has been an active topic, since hysteresis nonlinearities are common in many industrial processes. It is challenging to control a system with hysteresis nonlinearities, because they severely limit system performance such as giving rise to undesirable inaccuracy or oscillations and even may lead to instability [1]. In addition, due to the nonsmooth characteristics of hysteresis nonlinearities, traditional control methods are insufficient in dealing with the effects of unknown hysteresis. Therefore, advanced control techniques are much needed to mitigate the effects of hysteresis.

One of the most common approaches is to construct an inverse operator to cancel the effects of the hysteresis as in [1] and [2]. However, it is a challenging task to construct the inverse operator for the hysteresis, due to its complexity and uncertainty. To circumvent these difficulties, alternative control approaches that do not need an inverse model have also been developed in [3]- [6]. In [3] and [4], robust adaptive control and adaptive backstepping control were, respectively, investigated for a class of nonlinear system with unknown backlash-like hysteresis. In [5] and [6], adaptive variable structure control and adaptive backstepping methods, respectively, were proposed for a class of continuous-time nonlinear dynamic systems preceded by a hysteresis nonlinearity with the conventional Prandtl-Ishlinskii model representation.

In this paper, we consider a class of unknown nonlinear systems in pure-feedback form which are preceded by a generalized Prandtl-Ishlinskii hysteresis input. Compared with the backlash-like hysteresis and the conventional Prandtl-Ishlinskii hysteresis model discussed in the above works [3]- [6], the generalized Prandtl-Ishlinskii hysteresis model proposed in [7], can capture the hysteresis phenomenon more accurately and accommodate more general classes of hysteresis shapes, by adjusting not only the density function, but also the input function. However, the difficulty in dealing with the generalized Prandtl-Ishlinskii hysteresis model lies in that the input function in the generalized Prandtl-Ishlinskii hysteresis model is unknown and non-affine. Motivated by [8], in this paper, we will adopt the Mean Value Theorem to transform the unknown non-affine input function to a partially affine form, which can be handled by extending some available techniques for affine nonlinear system control in the literature.

For pure-feedback systems, the cascade and non-affine properties make it quite difficult to find the explicit virtual controls and the actual control to stabilize the pure-feedback systems. In [9] and [10], much simpler pure-feedback systems where the last one or two equations were assumed to be affine, were discussed. In [11], an "ISS-modular" approach combined with small gain theorem was presented for adaptive neural control of the completely non-affine pure-feedback system. In this paper, we also consider a class of unknown nonlinear systems in pure-feedback form. The non-affine problem in the control variable and virtual ones is dealt with by adopting the Mean Value Theorem, motivated by the works [8], without the strong assumptions that the last one or two equations are affine as in [9] and [10]. The unknown virtual control directions are dealt with by using Nussbaum functions. To the best of our own knowledge, it is the first time, in the literature, to investigate the tracking control problem of unknown nonlinear systems in pure-feedback form with the generalized Prandtl-Ishlinskii hysteresis input.

II. PROBLEM FORMULATION AND PRELIMINARIES

Throughout this paper, $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$, $\|\cdot\|$ denotes the 2-norm, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalues of a square matrix $(\cdot)$, respectively.

A. Problem Formulation

Consider the following class of unknown nonlinear system in pure-feedback form whose input is preceded by the uncertain generalized Prandtl-Ishlinskii hysteresis:

$$\begin{aligned} \dot{x}_j &= f_j(\bar{x}_j, x_{j+1}), \quad 1 \le j \le n-1 \\ \dot{x}_n &= f_n(\bar{x}_n, u) + d(t) \\ y &= x_1 \end{aligned} \tag{1}$$

This work is partially supported by A*STAR SERC Singapore (Grant No. 052 101 0097).

[1]Beibei Ren, Shuzhi Sam Ge and Tong Heng Lee are with Social Robotics Lab, Interactive Digital Media Institute and Department of Electrical and Computer Engineering, National University of Singapore, Singapore 117576. (Email: renbeibei@nus.edu.sg, samge@nus.edu.sg, eleleeth@nus.edu.sg).

[2]Chun-Yi Su is with the Department of Mechanical & Industrial Engineering, Concordia University, Montreal, Quebec, Canada H3G 1M8. (Email: cysu@alcor.concordia.ca).

where $\bar{x}_j = [x_1, ..., x_j]^T \in R^j$ is the vector of states of the first j differential equations, and $\bar{x}_n = [x_1, ..., x_n]^T \in R^n$; $f_j(\cdot)$ and $f_n(\cdot)$ are unknown smooth functions; $d(t)$ is a bounded disturbance; $y \in R$ is the output of the system; and $u \in R$ is the input of the system and the output of the hysteresis nonlinearity, which is represented by the generalized Prandtl-Ishlinskii model in [7] as follows

$$\begin{aligned} u(t) &= h(v)(t) - \int_0^D p(r) F_r[v](t) dr \quad (2)\\ F_r[v](0) &= h_r(v(0), 0)\\ F_r[v](t) &= h_r(v(t), F_r[v](t_i)), \text{ for } t_i < t \le t_{i+1},\\ & \quad 0 \le i \le N-1\\ h_r(v, w) &= \max(v - r, \min(v + r, w)) \end{aligned}$$

where v is the input to the hysteresis model; $0 = t_0 < t_1 < ... < t_N = t_E$ is a partition of $[0, t_E]$ such that the function v is monotone on each of the subintervals $(t_i, t_{i+1}]$; $p(r)$ is a given density function, satisfying $p(r) \ge 0$ with $\int_0^\infty rp(r)dr < \infty$; D is a constant so that density function $p(r)$ vanishes for large values of D; $F_r[v](t)$ is known as the play operator; and $h(v)$ is the hysteresis input function that satisfies the following assumptions [7]:

Assumption 1: The function $h : R \to R$ is odd, non-decreasing, locally Lipschitz continuous, and satisfies $\lim_{v\to\infty} h(v) \to \infty$ and $\frac{dh(v)}{dv} > 0$ for almost every $v \in R$.

Assumption 2: The growth of the hysteresis function $h(v)$ is smooth, and there exist positive constants h_0 and h_1 such that $0 < h_0 \le \frac{dh(v)}{dv} \le h_1$.

The objective is to design adaptive neural control $v(t)$ for system (1) (2) such that all signals in the closed-loop system are bounded, while the tracking error converges to a small neighborhood of zero.

To facilitate the control design later in Section III, the following assumptions are needed.

Assumption 3: The desired trajectory y_d, and their time derivatives up to the nth order $y_d^{(n)}$, are continuous and bounded.

Based on Assumption 3, we define the trajectory vector $\bar{x}_{d(j+1)} = [y_d \; \dot{y}_d \; ... \; y_d^{(j)}]^T$, $j = 1, ..., n-1$, which is a vector from y_d to its j-th time derivative, $y_d^{(j)}$, which will be used in the subsequent control design.

Assumption 4: There exists an unknown constant d^* such that $|d(t)| \le d^*$.

Assumption 5: There exist a known constant p_{max}, such that $p(r) \le p_{max}$ for all $r \in [0, D]$.

According to the Mean Value Theorem [12], we can express $f_j(\cdot, \cdot)$ in (1) as follows:

$$\begin{aligned} f_j(\bar{x}_j, x_{j+1}) &= f_j(\bar{x}_j, x_{j+1}^0) + \frac{\partial f_j(\bar{x}_j, x_{j+1})}{\partial x_{j+1}}\Big|_{x_{j+1}=x_{j+1}^{\theta_j}}\\ & \quad \times (x_{j+1} - x_{j+1}^0), \quad 1 \le j \le n \quad (3) \end{aligned}$$

where $x_{n+1} = u$, and $x_{j+1}^{\theta_j} = \theta_j x_{j+1} + (1 - \theta_j) x_{j+1}^0$ with $0 < \theta_j < 1$. By choosing $x_{j+1}^0 = 0$, and define $g_j(\bar{x}_j, x_{j+1}^{\theta_j}) = [\partial f_j(\bar{x}_j, x_{j+1})/\partial x_{j+1}]\Big|_{x_{j+1}=x_{j+1}^{\theta_j}}$, (3) can be written as

$$f_j(\bar{x}_j, x_{j+1}) = f_j(\bar{x}_j, 0) + g_j(\bar{x}_j, x_{j+1}^{\theta_j}) x_{j+1} \quad (4)$$

Substituting (4) into (1), we have

$$\begin{aligned} \dot{x}_j &= f_j(\bar{x}_j, 0) + g_j(\bar{x}_j, x_{j+1}^{\theta_j}) x_{j+1}, \quad 1 \le j \le n-1\\ \dot{x}_n &= f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, u^{\theta_n}) u + d(t)\\ y &= x_1 \quad (5) \end{aligned}$$

In addition, according to the Mean Value Theorem [12], there also exists a constant θ_0 $(0 < \theta_0 < 1)$ such that the unknown input function $h(v)$ in (2) satisfies $h(v) = h(v^*) + \frac{\partial h(\cdot)}{\partial v}\Big|_{v=v^{\theta_0}} (v - v^*)$, where $v^{\theta_0} = \theta_0 v + (1 - \theta_0) v^*$. According to Assumptions 1 and 2, and the Implicit function Theorem [13], we can find v^* such that $h(v^*) = 0$. Defining $g_0(v^{\theta_0}) = \frac{\partial h(\cdot)}{\partial v}\Big|_{v=v^{\theta_0}}$, we have $h(v) = g_0(v^{\theta_0})(v - v^*)$. Therefore, we can rewrite (2) as

$$u(t) = g_0(v^{\theta_0}) v - g_0(v^{\theta_0}) v^* - \int_0^D p(r) F_r[v](t) dr \quad (6)$$

Substituting (6) into (5) leads to our unified system:

$$\begin{aligned} \dot{x}_j &= f_j(\bar{x}_j, 0) + g_j(\bar{x}_j, x_{j+1}^{\theta_j}) x_{j+1}, \quad 1 \le j \le n-1\\ \dot{x}_n &= f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, u^{\theta_n})[g_0(v^{\theta_0}) v - g_0(v^{\theta_0}) v^*\\ & \quad - \int_0^D p(r) F_r[v](t) dr] + d(t)\\ y &= x_1 \quad (7) \end{aligned}$$

Assumption 6: There exist constants $\underline{g}_j$ and $\bar{g}_j$ such that $0 < \underline{g}_j \le |g_j(\cdot)| \le \bar{g}_j < \infty$, for $j = 1, ..., n$.

The following lemma is useful for establishing the stability properties of the closed-loop system.

Lemma 1: [14] Let $V(\cdot)$, $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \ge 0$, $\forall t \in [0, t_f)$, and $N(\cdot)$ be an even smooth Nussbaum-type function. If the following inequality holds:

$$V(t) \le c_0 + e^{-c_1 t} \int_0^t [g(\cdot) N(\zeta) + 1] \dot{\zeta} e^{c_1 \tau} d\tau$$

where c_0 represents some suitable constant, c_1 is a positive constant, and $g(\cdot)$ is a time-varying parameter which takes values in the unknown closed intervals $I = [l^-, l^+]$, with $0 \notin I$, then $V(t)$, $\zeta(t)$, $\int_0^t g(\cdot) N(\zeta) \dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

III. Control Design and Stability Analysis

In this section, we will investigate adaptive neural control for the system (7) using the backstepping method [15] combined with neural networks approximation. The backstepping design procedure contains n steps and involves the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$, where α_i are virtual controls which shall be developed for the corresponding i-subsystem based on some appropriate Lyapunov functions V_i. The control law $v(t)$ is designed in the last step to stabilize the entire closed-loop system, and deal with the hysteresis term.

Step 1: Since $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, the derivative of z_1 is

$$\dot{z}_1 = g_1(\bar{x}_1, x_2^{\theta_1})(z_2 + \alpha_1) + Q_1(Z_1) \quad (8)$$

where $Q_1(Z_1) = f_1(\bar{x}_1, 0) - \dot{y}_d$ with $Z_1 = [\bar{x}_1, \dot{y}_d] \in \Omega_{Z_1} \subset R^2$. To compensate for the unknown function $Q_1(Z_1)$, we can use radial basis function neural network (RBFNN), $\hat{W}_1^T S(Z_1)$, with $\hat{W}_1 \in R^{l\times 1}$, $S(Z_1) \in R^{l\times 1}$, and the NN node number $l > 1$, to approximate the function $Q_1(Z_1)$ on the compact set Ω_{Z_1} as follows

$$Q_1(Z_1) = \hat{W}_1^T S(Z_1) - \tilde{W}_1^T S(Z_1) + \varepsilon_1(Z_1) \quad (9)$$

where the approximation error $\varepsilon_1(Z_1)$ satisfies $|\varepsilon_1(Z_1)| \le \varepsilon_1^*$ with a positive constant ε_1^*. Substituting (9) into (8), we obtain

$$\dot{z}_1 = g_1(\bar{x}_1, x_2^{\theta_1})(z_2 + \alpha_1) + \hat{W}_1^T S(Z_1) - \tilde{W}_1^T S(Z_1) + \varepsilon_1(Z_1) \quad (10)$$

Choose the following virtual control and adaptation laws:

$$\alpha_1 = N(\zeta_1)[k_1 z_1 + \hat{W}_1^T S(Z_1)] \quad (11)$$

$$\dot{\zeta}_1 = k_1 z_1^2 + z_1 \hat{W}_1^T S(Z_1) \quad (12)$$

$$\dot{\hat{W}}_1 = \Gamma_1[z_1 S(Z_1) - \sigma_1 \hat{W}_1] \quad (13)$$

where $\Gamma_1 = \Gamma_1^T \in R^{l\times l} > 0$, $k_1 > 0$ and $\sigma_1 > 0$.

Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{W}_1^T \Gamma_1^{-1} \tilde{W}_1 \quad (14)$$

The time derivative of (14) along with (10)-(13) is

$$\dot{V}_1 \le -k_1 z_1^2 + [g_1(\bar{x}_1, x_2^{\theta_1})N_1(\zeta_1) + 1]\dot{\zeta}_1 + g_1(\bar{x}_1, x_2^{\theta_1}) z_1 z_2 - \sigma_1 \tilde{W}_1^T \hat{W}_1 + |z_1|\varepsilon_1^* \quad (15)$$

Using the Young's inequality, the following inequalities hold:

$$-\sigma_1 \tilde{W}_1^T \hat{W}_1 \le -\frac{\sigma_1 \|\tilde{W}_1\|^2}{2} + \frac{\sigma_1 \|W_1^*\|^2}{2} \quad (16)$$

$$|z_1|\varepsilon_1^* \le \frac{z_1^2}{4c_{11}} + c_{11}\varepsilon_1^{*2} \quad (17)$$

$$g_1(\bar{x}_1, x_2^{\theta_1}) z_1 z_2 \le \frac{z_1^2}{4c_{12}} + c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 \quad (18)$$

Substituting (16)-(18) into (15) results in

$$\dot{V}_1 \le -\gamma_1 V_1 + [g_1(\bar{x}_1, x_2^{\theta_1})N_1(\zeta_1) + 1]\dot{\zeta}_1 + \rho_1 + c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 \quad (19)$$

where γ_1 and ρ_1 are positive constants, which are defined as

$$\gamma_1 = \min\{2(k_1 - \frac{1}{4c_{11}} - \frac{1}{4c_{12}}), \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}\}$$

$$\rho_1 = \frac{\sigma_1 \|W_1^*\|^2}{2} + c_{11}\varepsilon_1^{*2}$$

Multiplying both sides of (19) by $e^{\gamma_1 t}$ and integrating it over $[0, t]$, we have

$$V_1 \le \frac{\rho_1}{\gamma_1} + V_1(0) + e^{-\gamma_1 t}\int_0^t [g_1(\bar{x}_1, x_2^{\theta_1})N_1(\zeta_1) + 1]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau + e^{-\gamma_1 t}\int_0^t c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 e^{\gamma_1 \tau} d\tau \quad (20)$$

Noting Assumption 6, the last term of (20) satisfies

$$e^{-\gamma_1 t}\int_0^t c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 e^{\gamma_1 \tau} d\tau \le \frac{c_{12}}{\gamma_1}\bar{g}_1^2 \sup_{\tau\in[0,t]}[z_2^2(\tau)]$$

where $\bar{g}_1$ is the upper bound for $|g_1(\cdot)|$ as defined in Assumption 6. Therefore, if z_2 can be kept bounded over a finite time interval $[0, t_f)$, we can obtain the boundedness of the term $e^{-\gamma_1 t}\int_0^t c_{12} g_1^2(\bar{x}_1, x_2^{\theta_1}) z_2^2 e^{\gamma_1 \tau} d\tau$. Furthermore, (20) can be written as

$$V_1 \le c_1 + e^{-\gamma_1 t}\int_0^t [g_1(\bar{x}_1, x_2^{\theta_1})N_1(\zeta_1) + 1]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau \quad (21)$$

where $c_1 = \frac{\rho_1}{\gamma_1} + V_1(0) + \frac{c_{12}}{\gamma_1}\bar{g}_1^2 \sup_{\tau\in[0,t_f]}[z_2^2(\tau)]$. According to Lemma 1, we can conclude that V_1, ζ_1, $\hat{W}_1$, $\int_0^t [g_1(\bar{x}_1, x_2^{\theta_1})N_1(\zeta_1) + 1]\dot{\zeta}_1 e^{\gamma_1 \tau} d\tau$ are all bounded on $[0, t_f)$. According to Proposition 2 [16], $t_f = \infty$ and we know that z_1 and $\hat{W}_1$ are SGUUB. The boundedness of z_2 will be dealt with in the following steps.

Step j ($2 \le j < n$)**:** The derivative of $z_j = x_j - \alpha_{j-1}$ is

$$\dot{z}_j = f_j(\bar{x}_j, 0) + g_j(\bar{x}_j, x_{j+1}^{\theta_j}) x_{j+1} - \dot{\alpha}_{j-1} \quad (22)$$

Since α_{j-1} is a function of $\bar{x}_{j-1}, \bar{x}_{dj}, \zeta_{j-1}, \hat{W}_1, ..., \hat{W}_{j-1}$, its derivative, $\dot{\alpha}_{j-1}$, can be expressed as

$$\dot{\alpha}_{j-1} = \sum_{k=1}^{j-1}\frac{\partial \alpha_{j-1}}{\partial x_k} f_k(\bar{x}_k, x_{k+1}) + \phi_{j-1} \quad (23)$$

where

$$\phi_{j-1} = \frac{\partial \alpha_{j-1}}{\partial \zeta_{j-1}}\dot{\zeta}_{j-1} + \frac{\partial \alpha_{j-1}}{\partial \bar{x}_{dj}}\dot{\bar{x}}_{dj} + \sum_{k=1}^{j-1}\frac{\partial \alpha_{j-1}}{\partial \hat{W}_k}\dot{\hat{W}}_k \quad (24)$$

which is computable. As such, $\dot{\alpha}_{j-1}$ can be seen as a function of $\bar{x}_j, \sum_{k=1}^{j-1}\frac{\partial \alpha_{j-1}}{\partial x_k}, \phi_{j-1}$. Further, we can rewrite (22) as

$$\dot{z}_j \le g_j(\bar{x}_j, x_{j+1}^{\theta_j})(z_{j+1} + \alpha_j) + (\hat{W}_j^T - \tilde{W}_j^T)S(Z_j) + \varepsilon_j^* \quad (25)$$

where $\hat{W}_j^T S(Z_j)$ is used to approximate the unknown function $Q_j(Z_j) = f_j(\bar{x}_j, 0) - \dot{\alpha}_{j-1}$ on the compact set Ω_{Z_j} with $Z_j = [\bar{x}_j, \sum_{k=1}^{j-1}\frac{\partial \alpha_{j-1}}{\partial x_k}, \phi_{j-1}] \in \Omega_{Z_j} \subset R^{2j}$, and the approximation error $\varepsilon_j(Z_j)$ satisfies $|\varepsilon_j(Z_j)| \le \varepsilon_j^*$ with positive constants ε_j^*.

Similar to the discussion in Step 1, we consider the following Lyapunov function candidates, virtual controls and adaptation laws:

$$V_j = \frac{1}{2}z_j^2 + \frac{1}{2}\tilde{W}_j^T \Gamma_j^{-1} \tilde{W}_j \quad (26)$$

$$\alpha_j = N(\zeta_j)[k_j z_j + \hat{W}_j^T S(Z_j)] \quad (27)$$

$$\dot{\zeta}_j = k_j z_j^2 + z_j \hat{W}_j^T S(Z_j) \quad (28)$$

$$\dot{\hat{W}}_j = \Gamma_j[z_j S(Z_j) - \sigma_j \hat{W}_j] \quad (29)$$

where $\Gamma_j = \Gamma_j^T > 0$, k_j and σ_j are positive constants.

Following the procedures outlined in Step 1, we have

$$V_j \le c_j + e^{-\gamma_j t}\int_0^t [g_j(\bar{x}_j, x_{j+1}^{\theta_j})N_j(\zeta_j) + 1]\dot{\zeta}_j e^{\gamma_j \tau} d\tau \quad (30)$$

where $c_j = \frac{\rho_j}{\gamma_j} + V_j(0) + \frac{c_{j2}}{\gamma_j}\bar{g}_j^2 \sup_{\tau\in[0,t_f]}[z_{j+1}^2(\tau)]$, $\gamma_j = \min\{2(k_j - \frac{1}{4c_{j1}} - \frac{1}{4c_{j2}}), \frac{\sigma_j}{\lambda_{\max}(\Gamma_j^{-1})}\}$, and $\rho_j = \frac{\sigma_j\|W_j^*\|^2}{2} + c_{j1}\varepsilon_j^{*2}$. Then, applying Lemma 1, the boundedness of V_j, ζ_j, $\hat{W}_j$, $\int_0^t [g_j(\bar{x}_j, x_{j+1}^{\theta_j})N_j(\zeta_j) + 1]\dot{\zeta}_j e^{\gamma_j\tau}d\tau$ can be readily obtained. The boundedness of z_{j+1} will be dealt with in the Step $(j+1)$.

Step n: In this final step, we will design the control input $v(t)$. Since $z_n = x_n - \alpha_{n-1}$, its derivative is given by

$$\begin{aligned}\dot{z}_n &= g_n(\bar{x}_n, u^{\theta_n})[g_0(v^{\theta_0})v - g_0(v^{\theta_0})v^* \\ &\quad - \int_0^D p(r)F_r[v](t)dr] + \hat{W}_n^T S(Z_n) - \tilde{W}_n^T S(Z_n) \\ &\quad + \varepsilon_n(Z_n) + d(t) \end{aligned} \quad (31)$$

where $\hat{W}_n^T S(Z_n)$ is used to approximate the unknown function $Q_n(Z_n) = f_n(x, 0) - \dot{\alpha}_{n-1}$ on the compact set $\Omega_{Z_n} \subset R^n$ with $Z_n = [\bar{x}_n, \frac{\partial\alpha_{n-1}}{\partial x_1}, ..., \frac{\partial\alpha_{n-1}}{\partial x_{n-1}}, \phi_{n-1}] \in \Omega_{Z_n} \subset R^{2n}$, and the approximation error $\varepsilon_n(Z_n)$ satisfies $|\varepsilon_n(Z_n)| \le \varepsilon_n^*$ with a positive constant ε_n^*.

Choose the following Lyapunov function candidate

$$\begin{aligned}V_n &= \frac{1}{2}z_n^2 + \frac{1}{2}\tilde{W}_n^T\Gamma_n^{-1}\tilde{W}_n + \frac{1}{2\gamma_d}\tilde{d}^2 \\ &\quad + \frac{\bar{g}_n}{2\gamma_p}\int_0^D \tilde{p}^2(t,r)dr \end{aligned} \quad (32)$$

where $\tilde{d} = \hat{d} - d^*$, $\tilde{p}(t,r) = \hat{p}(t,r) - p_{\max}$, $\hat{d}$ and $\hat{p}(t,r)$ are the estimates of the disturbance bound d^* and the density function of $p(r)$ respectively, $\Gamma_n = \Gamma_n^T > 0$, and γ_d, γ_p are positive constants.

The derivative of V_n defined in (32) along (31) is

$$\begin{aligned}\dot{V}_n &= z_n g_n(\bar{x}_n, u^{\theta_n})[g_0(v^{\theta_0})v - \int_0^D p(r)F_r[v](t)dr] - \\ &\quad z_n g_n(\bar{x}_n, u^{\theta_n})g_0(v^{\theta_0})v^* + z_n\hat{W}_n^T S(Z_n) - \\ &\quad z_n\tilde{W}_n^T S(Z_n) + z_n\varepsilon_n(Z_n) + z_n d(t) + \tilde{W}_n^T\Gamma_n^{-1}\dot{\hat{W}}_n \\ &\quad + \frac{1}{\gamma_d}\tilde{d}\dot{\hat{d}} + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \end{aligned} \quad (33)$$

From Assumptions 2 and 6, we know that $|g_n(x, u^{\theta_n})g_0 v^*| \le C$, where C is a positive constant. Due to $|\varepsilon_n(Z_n)| \le \varepsilon_n^*$ and Assumption 4, (33) becomes

$$\begin{aligned}\dot{V}_n &\le z_n g_n(\bar{x}_n, u^{\theta_n})[g_0(v^{\theta_0})v - \int_0^D p(r)F_r[v](t)dr] \\ &\quad + z_n\hat{W}_n^T S(Z_n) - z_n\tilde{W}_n^T S(Z_n) + |z_n|(C + \varepsilon_n^*) \\ &\quad + |z_n|d^* + \tilde{W}_n^T\Gamma_n^{-1}\dot{\hat{W}}_n + \frac{1}{\gamma_d}\tilde{d}\dot{\hat{d}} \\ &\quad + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\tilde{p}(t,r)dr \end{aligned} \quad (34)$$

The following control and adaptation laws are proposed:

$$\begin{aligned} v &= N(\zeta_n)\Big[k_n z_n + \hat{W}_n^T S(Z_n) + \hat{d}\tanh(\frac{z_n}{\omega})\Big] \\ &\quad + v_h \end{aligned} \quad (35)$$

$$v_h = -\mathrm{sign}(z_n)\int_0^D \frac{\hat{p}(t,r)}{h_0}|F_r[v](t)|dr \quad (36)$$

$$\dot{\zeta}_n = k_n z_n^2 + z_n\hat{W}_n^T S(Z_n) + z_n\hat{d}\tanh(\frac{z_n}{\omega}) \quad (37)$$

$$\dot{\hat{W}}_n = \Gamma_n[z_n S(Z_n) - \sigma_n\hat{W}_n] \quad (38)$$

$$\dot{\hat{d}} = \gamma_d[z_n\tanh(\frac{z_n}{\omega}) - \sigma_d\hat{d}] \quad (39)$$

$$\frac{\partial}{\partial t}\hat{p}(t,r) = \begin{cases} -\gamma_p\sigma_p\hat{p}(t,r), & \hat{p}(t,r) \ge p_{max} \\ \gamma_p[|z_n||F_r[v](t)| - \sigma_p\hat{p}(t,r)], & 0 \le \hat{p}(t,r) < p_{max} \end{cases} \quad (40)$$

where σ_p and ω are positive constants.

Substituting (35)-(39) into (34), and using Young's Inequality and the property of the hyperbolic tangent function $0 \le |z_n| - z_n\tanh(\frac{z_n}{\omega}) \le 0.2785\omega$, we obtain that

$$\begin{aligned}\dot{V}_n &\le -(k_n - \frac{1}{4c_{n1}})z_n^2 + [g_n(x, u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n) \\ &\quad + 1]\dot{\zeta}_n - \frac{\sigma_n\|\tilde{W}_n\|^2}{2} - \frac{\sigma_d\tilde{d}^2}{2} + \frac{\sigma_n\|W_j^*\|^2}{2} \\ &\quad + \frac{\sigma_d d^{*2}}{2} + 0.2785\omega d^* + c_{n1}(\varepsilon_n^* + C)^2 + \Delta \end{aligned} \quad (41)$$

where c_{n1} is a positive constant and

$$\begin{aligned}\Delta &= g_n(x, u^{\theta_n})\Big[-g_0(v^{\theta_0})|z_n|\int_0^D \frac{\hat{p}(t,r)}{h_0}|F_r[v](t)|dr \\ &\quad - z_n\int_0^D p(r)F_r[v](t)dr\Big] \\ &\quad + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\hat{p}(t,r)dr \\ &\le -g_n(x, u^{\theta_n})|z_n|\int_0^D \tilde{p}(t,r)|F_r[v](t)|dr \\ &\quad + \frac{\bar{g}_n}{\gamma_p}\int_0^D \tilde{p}(t,r)\frac{\partial}{\partial t}\hat{p}(t,r)dr \end{aligned} \quad (42)$$

According to (40), the adaptation law for the estimate of density function $\hat{p}(t,r)$ comprises two cases, due to the different regions which $\hat{p}(t,r)$ belong to. Therefore, we also need to consider two cases for the analysis of (42):

Case(a): When $r \in D_{max} = \{r : \hat{p}(t,r) \ge p_{max}\} \subset [0, D]$, according to (40), we have

$$\tilde{p}(t,r) \ge 0, \qquad \frac{\partial}{\partial t}\hat{p}(t,r) = -\gamma_p\sigma_p\hat{p}(t,r) \quad (43)$$

Substituting (43) into (42), we have

$$\Delta \le -\sigma_p\bar{g}_n\int_{r\in D_{max}} \tilde{p}(t,r)\hat{p}(t,r)dr \quad (44)$$

Case (b): When $r \in D_{max}^c$, which is the complement set of D_{max} in $[0, D]$, i.e., $0 \le \hat{p}(t,r) < p_{max}$. In this case, from(40), we have

$$\tilde{p}(t,r) < 0 \quad (45)$$

$$\frac{\partial}{\partial t}\hat{p}(t,r) = \gamma_p[|z_n||F_r[v](t)| - \sigma_p\hat{p}(t,r)] \quad (46)$$

Substituting (45) and (46) into (42), we have

$$\begin{aligned}\Delta &\le -g_n(x,u^{\theta_n})|z_n|\int_{r\in D^c_{max}}\tilde{p}(t,r)|F_r[v](t)|dr \\ &\quad +\bar{g}_n|z_n|\int_{r\in D^c_{max}}\tilde{p}(t,r)|F_r[v](t)|dr \\ &\quad -\sigma_p\bar{g}_n\int_{r\in D^c_{max}}\tilde{p}(t,r)\hat{p}(t,r) \\ &\le -\sigma_p\bar{g}_n\int_{r\in D^c_{max}}\tilde{p}(t,r)\hat{p}(t,r)dr \quad (47)\end{aligned}$$

Combining Case (a) with Case (b), (42) can be written as

$$\Delta \le -\sigma_p\bar{g}_n\int_0^D\tilde{p}(t,r)\hat{p}(t,r)dr \quad (48)$$

By Young's Inequality, we can rewrite (48) further as

$$\Delta \le -\frac{\sigma_p\bar{g}_n}{2}\int_0^D\tilde{p}^2(t,r)dr + \frac{\sigma_p\bar{g}_nD}{2}p^2_{\max} \quad (49)$$

Substituting (49) into (41), we have

$$\dot{V}_n \le -\gamma_nV_n + [g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n)+1]\dot{\zeta}_n + \rho_n \quad (50)$$

where γ_n and ρ_n are positive constants defined as

$$\begin{aligned}\gamma_n &= \min\{2(k_n - \frac{1}{4c_{n1}}), \frac{\sigma_n}{\lambda_{\max}(\Gamma_n^{-1})}, \sigma_d\gamma_d, \sigma_p\gamma_p\} \\ \rho_n &= \frac{\sigma_n\|W_n^*\|^2}{2} + \frac{\sigma_d d^{*2}}{2} + 0.2785\omega d^* + c_{n1}(\varepsilon_n^* + C)^2 \\ &\quad + \frac{\sigma_p\bar{g}_nD}{2}p^2_{\max} \quad (51)\end{aligned}$$

Multiplying both sides of (50) and integrating over $[0,t]$, we have

$$\begin{aligned}V_n &\le \frac{\rho_n}{\gamma_n} + [V_n(0) - \frac{\rho_n}{\gamma_n}]e^{-\gamma_nt} + e^{-\gamma_nt}\int_0^t[g_n(x,u^{\theta_n}) \\ &\quad g_0(v^{\theta_0})N_n(\zeta_n)+1]\dot{\zeta}_ne^{\gamma_n\tau}d\tau \quad (52) \\ &\le c_n + e^{-\gamma_nt}\int_0^t[g_n(x,u^{\theta_n})g_0(v^{\theta_0})N_n(\zeta_n) \\ &\quad +1]\dot{\zeta}_ne^{\gamma_n\tau}d\tau \quad (53)\end{aligned}$$

where $c_n = \frac{\rho_n}{\gamma_n} + V_n(0)$. According to Assumptions 1, 2, and 6, we can regard $g_n(x,u)g_0(v)$ in (53) as $g(\cdot)$, which is a time-varying parameter and takes values in the known closed intervals $I = [h_0\underline{g}_n, h_1\bar{g}_n]$, with $0 \notin I$. Using Lemma 1, we can conclude that $V_n(t), \zeta_n(t)$ and hence $z_n(t), \hat{W}_n, \hat{\tilde{d}}_n$ are SGUUB. From the boundedness of $z_n(t)$, the boundedness of the extra term $e^{-\gamma_{n-1}t}\int_0^t c_{(n-1)2}g^2_{n-1}(\bar{x}_{n-1}, x_n^{\theta_{n-1}})z_n^2e^{\gamma_{n-1}\tau}d\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 1 for $(n-1)$ times backward, it can be seen from the above iterative design procedure that $V_j, z_j, \hat{W}_j, \hat{\tilde{d}}_j$, and hence, x_j, are SGUUB on $[0, t_f)$.

The following theorem shows the stability and control performance of the closed-loop adaptive system.

Theorem 1: Consider the closed-loop system consisting of the plant (1) with the unknown hysteresis nonlinearities (2), and the control and adaptation laws (35)-(40). Under Assumptions 1-6, given some initial conditions $z_i(0), \hat{W}_i(0), \hat{d}(0)$ $(i = 1,2,...,n)$, belong in Ω_0, we can conclude that the overall closed-loop neural control system is semi-globally uniformly ultimately bounded (SGUUB) in the sense that all of the signals in the closed-loop system are bounded i.e., the states and weights in the closed-loop system will remain in the compact set defined by

$$\begin{aligned}\Omega = &\Big\{z_j, \tilde{W}_j, \tilde{d}\Big||z_j| \le \sqrt{2\mu_j}, \|\tilde{W}_j\| \le \sqrt{\frac{2\mu_j}{\lambda_{\min}(\Gamma_j^{-1})}}, \\ &|\tilde{d}| \le \sqrt{2\gamma_d\mu_n}, \quad j = 1,...,n.\Big\} \quad (54)\end{aligned}$$

where $\mu_j = c_j + c_{j0}$ with c_{j0} being the upper bound of $e^{-\gamma_jt}\int_0^t[g_j(\bar{x}_j, x_{j+1}^{\theta_j})N_j(\zeta_j)+1]\dot{\zeta}_je^{\gamma_j\tau}d\tau$, $j = 1,...,n$; and $c_j = \frac{\rho_j}{\gamma_j} + V_j(0) + \frac{c_{j2}}{\gamma_j}\bar{g}_j^2\sup_{\tau\in[0,t]}[z^2_{j+1}(\tau)]$, $c_n = \frac{\rho_n}{\gamma_n} + V_n(0)$, $V_j(0) = \frac{1}{2}z_j^2(0) + \frac{1}{2}\tilde{W}_j^T(0)\Gamma_j^{-1}\tilde{W}_j(0)$, $V_n(0) = \frac{1}{2}z_n^2(0) + \frac{1}{2}\tilde{W}_n^T(0)\Gamma_n^{-1}\tilde{W}_n(0) + \frac{1}{2\gamma_d}\tilde{d}_n^2(0) + \frac{\bar{g}_n}{2\gamma_p}\int_0^D\tilde{p}^2(0,r)dr$, $j = 1,...,n-1$. Furthermore, the states and weights in the closed-loop system will eventually converge to the compact set defined by

$$\begin{aligned}\Omega_s = &\Big\{z_j, \tilde{W}_j, \tilde{d}\Big||z_j| \le \sqrt{2\mu_j^*}, \|\tilde{W}_j\| \le \sqrt{\frac{2\mu_j^*}{\lambda_{\min}(\Gamma_j^{-1})}}, \\ &|\tilde{d}| \le \sqrt{2\gamma_d\mu_n^*}, \quad j = 1,...,n.\Big\} \quad (55)\end{aligned}$$

where $\mu_j^* = c_j' + c_{j0}$, $j = 1,...,n$, and $c_j' = \frac{\rho_j}{\gamma_j} + \frac{c_{12}}{\gamma_j}\bar{g}_j^2\sup_{\tau\in[0,t]}[z^2_{j+1}(\tau)]$, $c_n' = \frac{\rho_n}{\gamma_n}$, $j = 1,...,n-1$.

Proof: Based on the previous iterative derivation procedures from Step 1 to Step n of backstepping, from (21) (30) to (53), and according to Lemma 1, we can conclude that $V_j, z_j, \hat{W}_j, \hat{\tilde{d}}$ and hence x_j are SGUUB, $i = 1,2,...,n$, i.e., all the signals in the closed-loop system are bounded.

From (53), letting c_{n0} be the upper bound of the term $e^{-\gamma_nt}\int_0^t[g_n(x,u^{\theta_n})g_0N_n(\zeta_n)+1]\dot{\zeta}_ne^{\gamma_n\tau}d\tau$, $\mu_n = c_n + c_{n0}$, and noting the definition of V_n in (32), we have

$$|z_n| \le \sqrt{2\mu_n}, \quad \|\tilde{W}_n\| \le \sqrt{\frac{2\mu_n}{\lambda_{\min}(\Gamma_n^{-1})}}, \quad |\tilde{\tilde{d}}| \le \sqrt{2\gamma_d\mu_n}$$

Similarly, in the rest of steps from $n-1$ to 1, letting c_{j0} be the upper bound of $e^{-\gamma_jt}\int_0^t[g_j(\bar{x}_j, x_{j+1}^{\theta_j})N_j(\zeta_j)+1]\dot{\zeta}_je^{\gamma_j\tau}d\tau$ and $\mu_j = c_j + c_{j0}$ in (30), we can obtain

$$|z_j| \le \sqrt{2\mu_j}, \quad \|\tilde{W}_i\| \le \sqrt{\frac{2\mu_j}{\lambda_{\min}(\Gamma_j^{-1})}}, \quad j = 1,2,...,n-1.$$

Furthermore, we can rewrite (52) as

$$V_n \le \frac{\rho_n}{\gamma_n} + [V_n(0) - \frac{\rho_n}{\gamma_n}]e^{-\gamma_nt} + c_{n0}$$

As $t \to \infty$, we have

$$V_n \quad \leq \quad c_n' + c_{n0}$$

where $c_n' = \frac{\rho_n}{\gamma_n}$. Therefore, define $\mu_n^* = c_n' + c_{n0}$, we can conclude that when $t \to \infty$,

$$|z_n| \leq \sqrt{2\mu_n^*}, \quad \|\tilde{W}_n\| \leq \sqrt{\frac{2\mu_n^*}{\lambda_{\min}(\Gamma_n^{-1})}}, \quad |\tilde{\hat{d}}| \leq \sqrt{2\gamma_d \mu_n^*}$$

Similar conclusions can be made about z_j, $\hat{W}_j$ as follows

$$|z_j| \leq \sqrt{2\mu_j^*}, \quad \|\tilde{W}_j\| \leq \sqrt{\frac{2\mu_j^*}{\lambda_{\min}(\Gamma_j^{-1})}}$$

with $\mu_j^* = c_j' + c_{j0}$ and $c_j' = \frac{\rho_j}{\gamma_j} + \frac{c_{j2}}{\gamma_j}\bar{g}_j^2 \sup_{\tau\in[0,t]}[z_{j+1}^2(\tau)]$ as $t \to \infty$. ■

IV. Simulation Studies

Consider a second-order nonlinear system with the generalized Prandtl-Ishlinskii hysteresis in (1), where $f_1 = x_2 + 0.05\sin(x_2)$, $f_2 = \frac{1-e^{-x_2}}{1+e^{-x_2}} + u + 0.1\sin(u)$, $d(t) = 0.1\sin(6t)$, the density function $p(r) = 0.08e^{-0.0024(r-1)^2}$, $r \in [0, 100]$, and $h(v)(t) = 0.4(|v|\arctan(v) + v)$. Our objective is to make the output, y, to track the desired trajectory, $y_d = 0.8\sin(0.5t) + 0.1\cos(t)$.

The simulation results are shown in Figures 1 and 2. From Figure 1, we can observe that the good tracking performance has been achieved and the tracking error converge to a small neighborhood of zero after a while. At the same time, the boundedness of the control signal v and the hysteresis output is shown in Figures 2.

V. Conclusion

Adaptive neural control has been proposed for a class of unknown nonlinear systems in pure-feedback form preceded by the uncertain generalized Prandtl-Ishlinskii hysteresis. We adopted the Mean Value Theorem to solve the non-affine problem both in the unknown nonlinear functions of the system dynamics and in the unknown input function of the generalized Prandtl-Ishlinskii hysteresis model. The closed-loop control system has been theoretically shown to be SGUUB using Lyapunov synthesis method.

References

[1] G. Tao and P. V. Kokotovic, "Adaptive control of plants with unknown hysteresis," *IEEE Transactions on Automatic Control*, vol. 40, pp. 200–212, 1995.

[2] X. Tan and J. S. Baras, "Modeling and control of hysteresis in magnestrictive actuators," *Automatica*, vol. 40, no. 9, pp. 1469–1480, 2004.

[3] C. Y. Su, Y. Stepanenko, J. Svoboda, and T. P. Leung, "Robust adaptive control of a class of nonlinear systems with unknown backlash-like hysteresis," *IEEE Transactions on Automatic Control*, vol. 45, no. 12, pp. 2427–2432, 2000.

[4] J. Zhou, C. Y. Wen, and Y. Zhang, "Adaptive backstepping control design of a class of uncertain nonlinear systems with unknown backlash-like hysteresis," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1751–1757, 2004.

[5] C. Y. Su, Q. Wang, X. Chen, and S. Rakheja, "Adaptive variable structure control of a class of nonlinear systems with unknown prandtl-ishlinskii hysteresis," *IEEE Transactions on Automatic Control*, vol. 50, no. 12, pp. 2069–2074, 2005.

[6] Q. Wang and C. Y. Su, "Robust adaptive control of a class of nonlinear systems including actuator hysteresis with prandtl-ishlinskii presentations," *Automatica*, vol. 42, no. 5, pp. 859–867, 2006.

[7] O. Klein and P. Krejci, "Outwards pointing hysteresis operators and asymptotic behaviour of evolution equations," *Nonlinear analysis: real world applications*, vol. 4, no. 5, pp. 755–785, 2003.

[8] S. S. Ge and J. Zhang, "Neural-network control of nonaffine nonlinear system with zero dynamics by state and output feedback," *IEEE Transactions on Neural Networks*, vol. 14, no. 4, pp. 900–918, 2003.

[9] D. Wang and J. Huang, "Adaptive neural network control for a class of uncertain nonlinear systems in pure-feedback form," *Automatica*, vol. 38, pp. 1365–1372, 2002.

[10] S. S. Ge and C. Wang, "Adaptive nn control of uncertain nonlinear pure-feedback systems," *Automatica*, vol. 38, no. 4, pp. 671–682, 2002.

[11] C. Wang, D. J. Hill, S. S. Ge, and G. Chen, "An ISS-modular approach for adaptive neural control of pure-feedback systems," *Automatica*, vol. 42, pp. 723–731, 2006.

[12] T. M. Apostol, *Mathematical Analysis*, 2nd ed. Reading, MA: Addison-Wesley, 1974.

[13] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall, 1996.

[14] S. S. Ge, F. Hong, and T. H. Lee, "Adaptive neural control of nonlinear time-delay systems with unknown virtual control coefficients," *IEEE Transactions on Systems Man and Cybernetics Part B-Cybernetics*, vol. 34, no. 1, pp. 499–516, 2004.

[15] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[16] E. P. Ryan, "A universal adaptive stabilizer for a class of nonlinear systems," *Systems & Control Letters*, vol. 16, pp. 209–218, 1991.

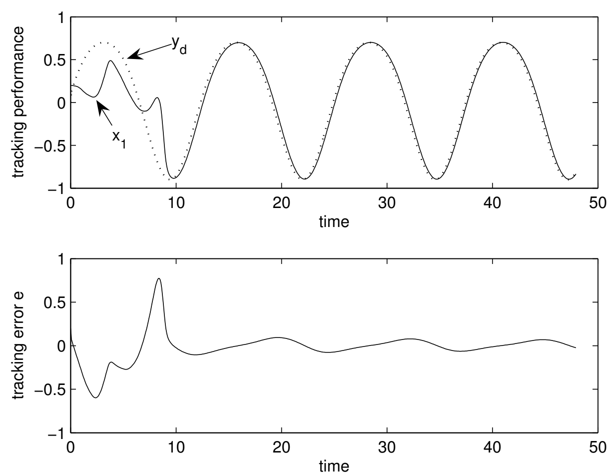

Fig. 1. Tracking performance

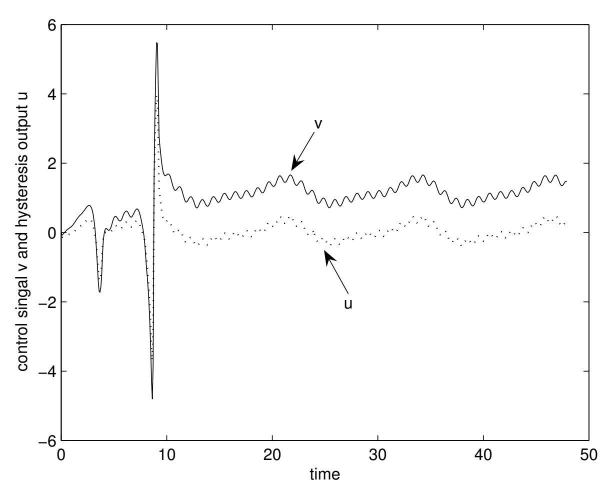

Fig. 2. Control signal and hysteresis output