Adaptive Nonlinear Design with Controller-Identifier Separation and Swapping

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***Abstract*—We present a new adaptive nonlinear control design which achieves a complete controller-identifier separation. This modularity is made possible by a strong input-to-state stability property of the new controller with respect to the parameter estimation error and its derivative as inputs. These inputs are independently guaranteed to be bounded by the identifier. The new design is more flexible than the Lyapunov-based design because the identifier can employ any standard update law gradient and least-squares, normalized and unnormalized. A key ingredient in the identifier design and convergence analysis is a nonlinear extension of the well-known linear swapping lemma.**

I. Introduction

The estimation-based approach to adaptive control has been extremely successful in linear systems. In contrast to the Lyapunov-based approach, which restricts the choice of parameter update laws and controller structures, the estimation-based designs are versatile. For linear systems, any common update law and any stabilizing controller can be employed as long as the boundedness properties of the identifier are sufficient to allow a "certainty-equivalence" design of the controller. This versatility is of conceptual and practical importance. It is due to a modularity feature: the identifier module achieves its boundedness properties independently of the controller module.

Thanks to its versatility, the estimation-based approach unifies many diverse adaptive schemes. For linear systems, this unification, initiated by Egardt [5], was extended by Goodwin and Mayne [6].

Attempts to apply estimation-based designs to nonlinear systems have had only limited success. The nonlinearities were either matched [24], [2], [3] or severely restricted [27], [35], [9], [10], [39]. Otherwise the results were local, i.e., valid in regions which were not a priori verifiable. A cause for this difficulty is a fundamental difference between the instability phenomena in linear and nonlinear systems. The states of an unstable linear system remain bounded over any finite interval, so that there is enough time for the identifier to "catch up." The situation is fundamentally different in a system with nonlinearities whose growth is faster than linear (x^2, x_1x_2, e^x, etc.). Even a small parameter estimation error may drive the state of such a nonlinear system to infinity in finite time. This explains why estimation-based designs have been mostly for systems with linearly bounded nonlinearities. Typically, linear growth constraints had to be imposed not only on the plant nonlinearities, but also on those derived during the design.

The only nonlinear estimation-based results which go beyond the linear growth constraints were obtained by Praly *et al.* [29]–[33]. In [32] a unified framework of control Lyapunov functions was used to characterize relationships between nonlinear growth constraints and controller stabilizing properties. In the absence of matching conditions, all the nonlinear estimation-based schemes presented in [32] involved some growth restrictions.

In contrast to the difficulties experienced by the estimation-based designs, the new recursive Lyapunov-based designs for systems in the parametric-strict-feedback form [12], [8], [19], [36] and the output-feedback form [21], [22], [14] were successful in achieving global boundedness and tracking without any restrictions on nonlinearities. However, these designs do not allow any flexibility in the choice of the parameter update law, excluding, for example, the least-squares update laws.

In spite of the previous difficulties with nonlinear estimation-based approaches, their flexibility and modularity motivate us to pursue their development. Since the independence of the identifier is not sufficient for modularity, we place the burden of the task of boundedness on the controller. For parametric-strict-feedback systems we seek (and find!) nonlinear controllers which guarantee boundedness in the presence of bounded parameter uncertainty. More precisely, we consider the parameter estimation error and its derivative as two independent disturbance inputs and design controllers which achieve input-to-state stability [37] (ISS) with respect to those inputs. In addition to such ISS-controllers, we also design weaker SG-controllers which only provide a small gain property and are presented for comparison with linear designs.

These new controllers create a possibility for a complete identifier-controller modularity. The remaining task is to design identifiers with guaranteed boundedness properties. A key ingredient in the identifier design and convergence analysis in this paper is our nonlinear extension of the well known linear swapping lemma [25]. Various forms of swapping were also used in most of the early nonlinear estimation-based results [27], [24], [29], [30], [35], [2], [9], [10], [39]. The identifiers in this paper are based on two different parametric models: the plant model and the error system. They allow a wide variety of update laws—gradient and least-squares, normalized and unnormalized.

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The paper is organized as follows. After the problem statement in Section II, in Section III we design the ISS controllers and prove that the input-to-state stability is achieved. Section IV presents the nonlinear swapping lemma. Parameter identifiers with gradient and least-squares update laws are developed in Section V, and the stability proofs for the resulting adaptive systems are given in Section VI. In Section VII we analyze performance of the new adaptive systems. To reveal the connection with linear estimation-based designs we present in Section VIII the design of a weaker SG-controller. The new controller designs and performance are illustrated by examples in Section IX.

II. Problem Statement

The problem is to adaptively control nonlinear systems transformable into the parametric-strict-feedback form

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T \varphi_i(x_1, \cdots, x_i), \quad 1 \le i \le n-1 \\ \dot{x}_n &= \beta_0(x)u + \theta^T \varphi_n(x) \\ y &= x_1 \end{aligned} \tag{2.1}$$

where $\theta \in \mathbb{R}^p$ is the vector of unknown constant parameters, β_0, and the components of $\phi = [\varphi_1, \cdots, \varphi_n]$ are smooth nonlinear functions in $\mathbb{R}^n$, and $\beta_0(x) \neq 0$, $\forall x \in \mathbb{R}^n$. Necessary and sufficient conditions for a nonlinear system to be transformable into the form (2.1) are given in [12]. It should be noted that (2.1) is feedback linearizable for any bounded $\theta \in \mathbb{R}^p$.

The control objective is to force the output y of the system (2.1) to asymptotically track the output y_r of a known linear reference model while keeping all the closed-loop signals bounded. The reference model has the form

$$\dot{x}_m = \begin{bmatrix} 0 & & \\ \vdots & & I_{n-1} \\ 0 & & \\ -m_0 & \cdots & -m_{n-1} \end{bmatrix} x_m + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_m \end{bmatrix} r$$

$$y_r = x_{m,1} \tag{2.2}$$

where $M(s) = s^n + m_{n-1}s^{n-1} + \cdots + m_1 s + m_0$ is Hurwitz, $k_m > 0$, and $r(t)$ is bounded and piecewise continuous. Another way of stating the same objective is to asymptotically track a given reference signal $y_r(t)$ with its first n derivatives known, bounded and piecewise continuous.

The above problem was first posed and solved in [12] using np estimates for p unknown parameters. This number of estimates was subsequently reduced in half in [8]. The over-parametrization was completely removed in [19] by the use of "tuning functions." In [40] the adaptive scheme of [12] was extended and recast in the observer-based setting. For the case when the nonlinearities in (2.1) are polynomial, a solution employing growth conditions was given in [33]. Possibilities to enlarge the class of systems that can be adaptively stabilized using the approach of [12] were explored in [1] and [36].

The Lyapunov-based results [12], [8], [19], and [36] employ only one type of parameter update laws. To increase the flexibility in the update law selection we now develop an estimation-based design which treats the controller and the identifier as separate modules.

Notation: For vectors we use $|x|_P \triangleq (x^T P x)^{1/2}$ to denote the weighted Euclidean norm of x. For matrices, $|X|_{\mathcal{F}} \triangleq (\mathrm{tr}\{X^T X\})^{1/2} = (\mathrm{tr}\{X X^T\})^{1/2}$ denote the Frobenius, and $|X|_2$ the induced 2-norm of X. The $\mathcal{L}_\infty$, $\mathcal{L}_2$ and $\mathcal{L}_1$ norms for signals are denoted by $\|\cdot\|_\infty$, $\|\cdot\|_2$ and $\|\cdot\|_1$ respectively. By referring to a matrix $A(t)$ as exponentially stable we mean that the corresponding LTV system $\dot{x} = A(t)x$ is exponentially stable. The spaces of all signals which are globally bounded, locally bounded and square-integrable on $[0, t_f)$, $t_f > 0$, are denoted by $\mathcal{L}_\infty[0, t_f)$, $\mathcal{L}_{\infty e}[0, t_f)$ and $\mathcal{L}_2[0, t_f)$, respectively. By saying that a signal belongs to $\mathcal{L}_\infty[0, t_f)$ or to $\mathcal{L}_2[0, t_f)$ we mean that the corresponding bound is independent of t_f.

III. ISS-Controller Design

Our modular estimation-based adaptive design for (2.1) places the burden of achieving boundedness on the controller module. We require that the controller guarantee input-to-state stability (ISS) with respect to the parameter error $\tilde{\theta} = \theta - \hat{\theta}$ and its derivative $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ as disturbance inputs.

Using the backstepping procedure, which is well known from [12], [19], and [18], the adaptive nonlinear controller is recursively designed as follows

$$z_i = x_i - x_{m,i} - \alpha_{i-1}$$

$$\begin{aligned} \alpha_i(\overline{x}_i, \hat{\theta}, \overline{x}_{m,i}) = &-z_{i-1} - c_i z_i - \hat{\theta}^T w_i \\ &+ \sum_{k=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial x_{m,k}} x_{m,k+1}\right) \\ &- s_i(\overline{x}_i, \hat{\theta}, \overline{x}_{m,i-1}) z_i \end{aligned}$$

$$w_i(\overline{x}_i, \hat{\theta}, \overline{x}_{m,i-1}) = \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k, \qquad i = 1, \cdots, n$$

$$\begin{aligned} u = \frac{1}{\beta_0(x)}[\alpha_n(x, \hat{\theta}, x_m) - m_0 x_{m,1} - \cdots \\ - m_{n-1} x_{m,n} + k_m r] \end{aligned} \tag{3.1}$$

where $\overline{x}_i = [x_1, \cdots, x_i]^T$, $\overline{x}_{m,i} = [x_{m,1}, \cdots, x_{m,i}]^T$, $c_i > 0$, $i = 1, \cdots, n$, and, for notational convenience $z_0 \triangleq 0$, $\alpha_0 \triangleq 0$. In these expressions the nonlinear damping functions $s_i(\overline{x}_i, \hat{\theta}, \overline{x}_{m,i-1})$ are yet to be designed. We will employ these functions to achieve the desired ISS property of the system obtained by the recursive design procedure (3.1). This nonlinear system, called the error system, is readily shown to be

$$\dot{z} = A_z(z, \hat{\theta}, t)z + W(z, \hat{\theta}, t)^T \tilde{\theta} + D(z, \hat{\theta}, t)^T \dot{\hat{\theta}}, \quad z \in \mathbb{R}^n \tag{3.2}$$

where $z_1 = x_1 - x_{m,1} = y - y_r$ represents the tracking error, and A_z, W, D are matrix-valued functions of z, $\hat{\theta}$ and t

$$A_z(z, \hat{\theta}, t) = \begin{bmatrix} -c_1 - s_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - s_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -c_n - s_n \end{bmatrix}$$

$$W(z, \hat{\theta}, t)^T = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix} \in \mathbb{R}^{n \times p},$$

$$D(z, \hat{\theta}, t)^T = \begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ -\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \end{bmatrix} \in \mathbb{R}^{n \times p}. \tag{3.3}$$

The explicit dependence of w_i and $\partial \alpha_{i-1}/\partial \hat{\theta}$ (and hence s_i) on t is due to the reference model; for example, $\varphi_1(x_1) = \varphi_1(z_1 + x_{m,1}(t))$.

Except for the term $D(z, \hat{\theta}, t)^T \dot{\hat{\theta}}$, the error system (3.2)–(3.3) is similar to the error system in [19] where the term $D(z, \hat{\theta}, t)^T \dot{\hat{\theta}}$ was accounted for by using tuning functions. Here we let both $\tilde{\theta}$ and $\dot{\hat{\theta}}$ appear as disturbance inputs. Their boundedness will later be guaranteed by parameter identifiers.

To design the nonlinear damping functions s_i we will employ the following lemma which evolved from [15] and [37].

Lemma 3.1 (Nonlinear Damping): Assume that for the system

$$\dot{x} = f(x, t) + g(x, t)[u + p(x, t)^T d(t)], \qquad x \in \mathbb{R}^n, u \in \mathbb{R} \tag{3.4}$$

a feedback control $u = \mu(x, t)$ guarantees

$$\frac{\partial V}{\partial x}[f(x, t) + g(x, t)\mu(x, t)] + \frac{\partial V}{\partial t} \le -U(x, t), \qquad \forall x \in \mathbb{R}^n, \forall t \ge 0 \tag{3.5}$$

where $V, U: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ are positive definite and radially unbounded and V is decrescent and continuously differentiable in x uniformly in t, $f: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $p: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^q$, $\mu: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ are continuously differentiable in x and piecewise continuous and bounded in t, and $d: \mathbb{R}_+ \to \mathbb{R}^q$ is piecewise continuous. Then the feedback control

$$u = \mu(x, t) - \lambda |p(x, t)|^2 \frac{\partial V}{\partial x}(x, t) g(x, t) \tag{3.6}$$

where $\lambda > 0$, guarantees that:

i) If $d \in \mathcal{L}_\infty$ then $x \in \mathcal{L}_\infty$.

ii) If $d \in \mathcal{L}_2$ and $U(x, t) \ge c|x|^2$, $\forall x \in \mathbb{R}^n$, $\forall t \ge 0$, $c > 0$, then $x \in \mathcal{L}_2$. If, in addition, $d \in \mathcal{L}_\infty$ then $x \in \mathcal{L}_\infty$ and $x(t) \to 0$ as $t \to \infty$.

Proof: i) Due to (3.5), the derivative of V along (3.4)–(3.6) is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\left[f + g\mu + g\left(-\lambda p^T p \frac{\partial V}{\partial x} g + p^T d\right)\right] + \frac{\partial V}{\partial t} \\ &\le -U - \lambda \left| p \frac{\partial V}{\partial x} g - \frac{1}{2\lambda} d \right|^2 + \frac{1}{4\lambda}|d|^2 \\ &\le -U + \frac{1}{4\lambda}|d|^2 \end{aligned} \tag{3.7}$$

and, hence $x \in \mathcal{L}_\infty$.

ii) Integrating (3.7) over $[0, \infty)$, we obtain

$$c\|x\|_2^2 \le \|U\|_1 \le \frac{1}{4\lambda}\|d\|_2^2 + V(0) \tag{3.8}$$

which implies that $x \in \mathcal{L}_2$. If, in addition, $d \in \mathcal{L}_\infty$ then by part i) of this lemma, $x \in \mathcal{L}_\infty$, and therefore, $u \in \mathcal{L}_\infty$. Hence $\dot{x} \in \mathcal{L}_\infty$. By Barbalat's lemma, $x(t) \to 0$ as $t \to \infty$. □

To apply this lemma to the error system (3.2)–(3.3) we first note that the coefficients multiplying $\tilde{\theta}$ and $\dot{\hat{\theta}}$ are w_i and $-\partial \alpha_{i-1}/\partial \hat{\theta}$, respectively. They play the role of the function p in the lemma, while the part of $(\partial V/\partial x)g$ in (3.6) is played by z_i. Therefore, our choice of nonlinear damping functions is

$$s_i = \kappa_i |w_i|^2 + g_i \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T \right|^2 \tag{3.9}$$

where κ_i, g_i, $i = 1, \cdots, n$ are positive scalar constants.[1] The usefulness of the first term for achieving boundedness was stressed by Kanellakopoulos [16].

With this choice of s_i we now prove input-to-state stability of the error system (3.2), (3.3), (3.9), making use of the following constants: $c_0 = \min_{1 \le i \le n} c_i$, $1/\kappa_0 = \sum_{i=1}^n (1/\kappa_i)$ and $1/g_0 = \sum_{i=1}^n (1/g_i)$.

Lemma 3.2 (ISS): In the error system (3.2), (3.3), (3.9), if $\tilde{\theta}, \dot{\hat{\theta}} \in \mathcal{L}_\infty[0, t_f)$ then $z, x \in \mathcal{L}_\infty[0, t_f)$, and

$$|z(t)| \le \frac{1}{2\sqrt{c_0}}\left(\frac{1}{\kappa_0}\|\tilde{\theta}\|_\infty^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}\|_\infty^2\right)^{1/2} + |z(0)|e^{-c_0 t}. \tag{3.10}$$

Proof: Differentiating $\frac{1}{2}|z|^2$ along the solutions of (3.2) we compute

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}|z|^2\right) &= -\sum_{i=1}^n c_i z_i^2 - \sum_{i=1}^n \left(\kappa_i |w_i|^2 + g_i \left|\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T\right|^2\right) z_i^2 \\ &\quad + \sum_{i=1}^n z_i \left(w_i^T \tilde{\theta} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}\right) \\ &\le -c_0|z|^2 - \sum_{i=1}^n \kappa_i \left| w_i z_i - \frac{1}{2\kappa_i}\tilde{\theta}\right|^2 \\ &\quad - \sum_{I=1}^n g_i \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T z_i + \frac{1}{2g_i}\dot{\hat{\theta}} \right|^2 \\ &\quad + \left(\sum_{i=1}^n \frac{1}{4\kappa_i}\right)|\tilde{\theta}|^2 + \left(\sum_{i=1}^n \frac{1}{4g_i}\right)|\dot{\hat{\theta}}|^2 \end{aligned} \tag{3.11}$$

[1] The constant coefficients g_i are not components of the vector field $g(x)$.

and arrive at

$$\frac{d}{dt}\left(\frac{1}{2}|z|^2\right) \le -c_0|z|^2 + \frac{1}{4}\left(\frac{1}{\kappa_0}|\tilde{\theta}|^2 + \frac{1}{g_0}|\dot{\hat{\theta}}|^2\right). \quad (3.12)$$

From Lemma A.1(i), it follows that

$$\begin{aligned}|z(t)|^2 &\le |z(0)|^2 e^{-2c_0 t} \\ &\quad + \frac{1}{2}\int_0^t e^{-2c_0(t-\tau)}\left(\frac{1}{\kappa_0}|\tilde{\theta}(\tau)|^2 + \frac{1}{g_0}|\dot{\hat{\theta}}(\tau)|^2\right) d\tau \\ &\le |z(0)|^2 e^{-2c_0 t} + \frac{1}{4c_0}\left(\frac{1}{\kappa_0}\|\tilde{\theta}\|_\infty^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}\|_\infty^2\right)\end{aligned} \quad (3.13)$$

which proves $z \in \mathcal{L}_\infty$ and (3.10), and by (3.1), $x \in \mathcal{L}_\infty$. □

The quadratic form of the nonlinear damping functions is only one out of many possible forms. Any power greater than one would yield an ISS property, but the proof with quadratic nonlinear damping is by far the simplest.

A consequence of Lemma 3.2 is that, even when the adaptation is switched off, that is, when the parameter estimate $\hat{\theta}$ is constant ($\dot{\hat{\theta}} = 0$) and the only disturbance input is $\tilde{\theta}$, the state z of the error system (3.2), (3.3), (3.9) remain bounded and converges exponentially to a positively invariant compact set. (Note that since $\dot{\hat{\theta}} = 0$, the terms $-g_i|(\partial\alpha_{i-1}/\partial\hat{\theta})^T|^2 z_i$ are not needed.) Moreover, when the adaptation is switched off, this boundedness result holds even when the unknown parameter is time varying.

Corollary 3.1 (Boundedness Without Adaptation): If $\theta: \mathbb{R}_+ \to \mathbb{R}^p$ is piecewise continuous and bounded, and $\hat{\theta}$ is constant, then $z, x \in \mathcal{L}_\infty$, and

$$|z(t)| \le \frac{1}{2\sqrt{c_0\kappa_0}} \sup_{\tau\ge 0}|\theta(\tau) - \hat{\theta}| + |z(0)|\mathrm{e}^{-c_0 t}. \quad (3.14)$$

Proof: Since $\dot{\hat{\theta}}(t) \equiv 0$, (3.12) holds with $\tilde{\theta}(t) = \theta(t) - \hat{\theta}$. □

Thus, the controller module alone guarantees boundedness, and the task of the adaptation is to achieve tracking.

IV. Nonlinear Swapping

The desired boundedness property having been achieved by the controller module, we can now proceed to the identifier module design. To make this design as close to linear designs as possible, we derive a nonlinear counterpart of the ubiquitous Swapping Lemma [25]. This lemma is an analytical device which uses regressor filtering to account for the time-varying nature of the parameter estimates. It was used in the early nonlinear estimation-based results [27], [24], [29], [30], [35], [2], [9], [10], [39]. For a class of nonlinear systems, including our error system (3.2), we provide the following two nonlinear swapping lemmas.

Lemma 4.1 (Nonlinear Swapping): Consider the nonlinear time-varying system

$$\Sigma_1: \quad \begin{aligned} \dot{z} &= A(z, t)z + g(z, t)W(z, t)^T\tilde{\theta} - D(z, t)^T\dot{\hat{\theta}} \\ y_1 &= h(z, t)z + l(z, t)W(z, t)^T\tilde{\theta} \end{aligned} \quad (4.1)$$

where $\tilde{\theta}: \mathbb{R}_+ \to \mathbb{R}^p$ is differentiable, $A: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}, g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, $W: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{p\times m}$, $D: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{p\times n}$, $l: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{r\times m}$ are locally Lipschitz in z and continuous and bounded in t, and $h: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{r\times n}$ is bounded in z and t. Along with (4.1) consider the linear time-varying systems

$$\Sigma_2: \quad \begin{aligned} \dot{\chi}^T &= A(z, t)\chi^T + g(z, t)W(z, t)^T \\ y_2 &= h(z, t)\chi^T + l(z, t)W(z, t)^T \end{aligned} \quad (4.2)$$

$$\Sigma_3: \quad \begin{aligned} \dot{\psi} &= A(z,t)\psi + \chi^T\dot{\tilde{\theta}} + D(z, t)^T\dot{\hat{\theta}} \\ y_3 &= -h(z, t)\psi. \end{aligned} \quad (4.3)$$

Assume that $z(t)$ is continuous on $[0, \infty)$ and there exists a continuously differentiable function $V: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\alpha_1|\zeta|^2 \le V(\zeta, t) \le \alpha_2|\zeta|^2 \quad (4.4)$$

and for each $z \in C^0$

$$\frac{\partial V}{\partial \zeta}A(z, t)\zeta + \frac{\partial V}{\partial t} \le -\alpha_3|\zeta|^2 \quad (4.5)$$

$\forall t \ge 0, \forall\zeta \in \mathbb{R}^n, \alpha_1, \alpha_2, \alpha_3 > 0$. Then for $\forall z(0), \psi(0) \in \mathbb{R}^n, \forall\chi(0) \in \mathbb{R}^{p\times n}, \forall t \ge 0$ the outputs of systems (4.1)–(4.3) are related by

$$y_1 = y_2\tilde{\theta} + y_3 + y_\epsilon \quad (4.6)$$

where y_ϵ is bounded and exponentially decaying.

Proof: Due to the continuity of $z(t)$, we see that $g(z(t), t)$, $W(z(t), t)$ and $D(z(t), t)$ are continuous in t. Since $gW \in \mathcal{L}_{\infty e}$ and Σ_2 is a linear time-varying system, then $\chi \in \mathcal{L}_{\infty e}$. Therefore $(\chi + D)^T\dot{\hat{\theta}} \in \mathcal{L}_{\infty e}$, which implies $\psi \in \mathcal{L}_{\infty e}$ because Σ_3 is a linear time-varying system. Differentiating $\tilde{\epsilon} = z + \psi - \chi^T\tilde{\theta}$, we obtain

$$\dot{\tilde{\epsilon}} = \dot{z} + \dot{\psi} - \dot{\chi}^T\tilde{\theta} - \chi^T\dot{\tilde{\theta}} = A(z, t)\tilde{\epsilon} \quad (4.7)$$

which together with (4.4)–(4.5) yields

$$\dot{V}(\tilde{\epsilon}, t) = \frac{\partial V}{\partial\tilde{\epsilon}}A(z, t)\tilde{\epsilon} + \frac{\partial V}{\partial t} \le -\alpha_3|\tilde{\epsilon}|^2 \le -\frac{\alpha_3}{\alpha_2}V. \quad (4.8)$$

Therefore $V(t) \le V(0)e^{-(\alpha_3/\alpha_2)t}$, and, hence

$$|\tilde{\epsilon}(t)| \le \sqrt{\frac{\alpha_2}{\alpha_1}}|\tilde{\epsilon}(0)|e^{-(\alpha_3/2\alpha_2)t}. \quad (4.9)$$

Now, (4.1)–(4.3) imply that $y_\epsilon = y_1 - y_2\tilde{\theta} - y_3 = h(z, t)\tilde{\epsilon}$. Since $h(z, t)$ is bounded then y_ϵ is bounded and decays to zero exponentially.

Remark 4.1: When $D(z, t) \equiv 0$, the result of Lemma 4.1 is reminiscent of Morse's linear Swapping Lemma [25]. To see this we rewrite (4.6) as

$$T_z[W^T\tilde{\theta}] = T[W^T]\tilde{\theta} + T_h[T_g[W^T]\dot{\tilde{\theta}}] + y_\epsilon. \quad (4.10)$$

In this notation $T_z: W^T\tilde{\theta} \mapsto y_1$ is the nonlinear operator defined by (4.1) with $D(z, t) \equiv 0$, while the system

$$\dot{\xi} = A(z(t), t)\xi + g(z(t), t)u$$
$$y = h(z(t), t)\xi + l(z(t), t)u \tag{4.11}$$

is used to define the linear time-varying operators: $T: u \mapsto y$, $T_g: u \mapsto y$ for $h = I$ and $l = 0$, $T_h: u \mapsto y$ for $g = I$ and $l = 0$. When A, g, h and l are constant, then the operator $T_z(s) = T(s) = h(sI - A)^{-1}g + l$ is a proper stable rational transfer function, $T_g(s) = (sI - A)^{-1}g$, $T_h(s) = -h(sI - A)^{-1}$, and Lemma 4.1 reduces to Lemma 3.6.5 from [34]. □

In some texts on adaptive linear control, an extended result which guarantees that $\tilde{\theta} \in \mathcal{L}_2 \Rightarrow T_z[W^T\tilde{\theta}] - T[W^T]\tilde{\theta} \in \mathcal{L}_2$ is also referred to as Swapping Lemma. Our next lemma is a nonlinear time-varying generalization of this result.

Lemma 4.2: Consider systems (4.1)–(4.3) with the same set of assumptions as in Lemma 4.1. Further, assume that $z \in \mathcal{L}_\infty$. If $\dot{\tilde{\theta}} \in \mathcal{L}_2$, then

$$y_1 - y_2\tilde{\theta} \in \mathcal{L}_2. \tag{4.12}$$

If $\dot{\tilde{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, then

$$\lim_{t\to\infty} [y_1(t) - y_2(t)\tilde{\theta}(t)] = 0. \tag{4.13}$$

Proof: Since $z \in \mathcal{L}_\infty$, then gW^T, $D \in \mathcal{L}_\infty$. Due to the exponential stability of $A(z, t)$, it follows that $\chi \in \mathcal{L}_\infty$. By Lemma 4.1, $y_\epsilon \in \mathcal{L}_2$. We need to prove that $y_3 \in \mathcal{L}_2$. The solution of (4.3) is

$$\psi(t) = \Phi_z(t, 0)\psi(0) + \int_0^t \Phi_z(t, \tau)[\chi(\tau) + D(z(\tau), \tau)]^T\dot{\tilde{\theta}}(\tau)\, d\tau \tag{4.14}$$

where (4.4)–(4.5) guarantee that the state transition matrix $\Phi_z: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is such that $|\Phi_z(t, \tau)|_2 \leq ke^{-\alpha(t-\tau)}$, k, $\alpha > 0$. Since χ and D are bounded then

$$\begin{aligned}|\psi(t)| &\leq ke^{-\alpha t}|\psi(0)| + k\|\chi + D\|_\infty \int_0^t e^{-\alpha(t-\tau)}|\dot{\tilde{\theta}}(\tau)|\, d\tau \\ &\leq ke^{-\alpha t}|\psi(0)| + k\|\chi + D\|_\infty \left(\int_0^t e^{-\alpha(t-\tau)}\, d\tau\right)^{1/2} \cdot \left(\int_0^t e^{-\alpha(t-\tau)}|\dot{\tilde{\theta}}(\tau)|^2\, d\tau\right)^{1/2} \\ &\leq ke^{-\alpha t}|\psi(0)| + k\|\chi + D\|_\infty \frac{1}{\sqrt{\alpha}} \cdot \left(\int_0^t e^{-\alpha(t-\tau)}|\dot{\tilde{\theta}}(\tau)|^2\, d\tau\right)^{1/2}\end{aligned} \tag{4.15}$$

where the second inequality is obtained using the Schwartz inequality. By squaring (4.15) and integrating over $[0, t]$ we obtain

$$\int_0^t |\psi(\tau)|^2\, d\tau \leq \frac{k^2}{2\alpha}|\psi(0)|^2 + \frac{k^2}{\alpha}\|\chi + D\|_\infty^2 \cdot \int_0^t \left[\int_0^\tau e^{-\alpha(\tau-s)}|\dot{\tilde{\theta}}(s)|^2\, ds\right] d\tau. \tag{4.16}$$

Changing the sequence of integration, (4.16) becomes

$$\begin{aligned}\int_0^t |\psi(\tau)|^2\, d\tau &\leq \frac{k^2}{2\alpha}|\psi(0)|^2 + \frac{k^2}{\alpha}\|\chi + D\|_\infty^2 \cdot \int_0^t e^{\alpha s}|\dot{\tilde{\theta}}(s)|^2 \left(\int_s^t e^{-\alpha\tau}\, d\tau\right) ds \\ &\leq \frac{k^2}{2\alpha}|\psi(0)|^2 + \frac{k^2}{\alpha}\|\chi + D\|_\infty^2 \cdot \int_0^t e^{\alpha s}|\dot{\tilde{\theta}}(s)|^2 \frac{1}{\alpha}e^{-\alpha s}\, ds\end{aligned} \tag{4.17}$$

because $\int_s^t e^{-\alpha\tau}\, d\tau = 1/\alpha(e^{-\alpha s} - e^{-\alpha t}) \leq (1/\alpha)e^{-\alpha s}$. Now, the cancellation $e^{\alpha s}e^{-\alpha s} = 1$ in (4.17) yields

$$\|\psi\|_2 \leq \frac{k}{\sqrt{2\alpha}}|\psi(0)| + \frac{k}{\alpha}\|\chi + D\|_\infty\|\dot{\tilde{\theta}}\|_2 < \infty \tag{4.18}$$

which proves $\psi \in \mathcal{L}_2$. Due to the uniform boundedness of h, it follows that $y_3 \in \mathcal{L}_2$. This proves (4.12). When $\dot{\tilde{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ then $\psi \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{\psi} \in \mathcal{L}_\infty$. Thus, by Barbalat's lemma, $\psi(t) \to 0$ and hence $y_3(t) \to 0$ as $t \to \infty$. This proves (4.13) because $y_\epsilon(t) \to 0$ as $t \to \infty$.

Remark 4.2: When $D(z, t) \equiv 0$, we rewrite (4.12) as

$$T_z[W^T\tilde{\theta}] - T[W^T]\tilde{\theta} \in \mathcal{L}_2 \tag{4.19}$$

and (4.13) as

$$\lim_{t\to\infty} \{T_z[W^T\tilde{\theta}](t) - (T[W^T]\tilde{\theta})(t)\} = 0 \tag{4.20}$$

with T_z and T as in Remark 4.1. For constant A, g, h and l, the operator $T_z = T$ is a proper stable rational transfer function, and Lemma 4.2 reduces to Lemma 2.11 from [28]. □

V. Parameter Identifiers

We are now in the position to design the identifier module by applying the Nonlinear Swapping Lemma 4.1 to either z- or x-system. Each of the two types of identifiers, with z-swapping and with x-swapping, can be implemented with either gradient or least-squares update laws. These parameter identifiers are variants of the regressor filtering identifiers in [32].

A. z-Swapping

For the error system (3.2) we introduce the filters

$$\dot{\chi}_0 = A_z(z, \hat{\theta}, t)\chi_0 + W(z, \hat{\theta}, t)^T\hat{\theta} - D(z, \hat{\theta}, t)^T\dot{\hat{\theta}}, \quad \chi_0 \in \mathbb{R}^n \tag{5.1}$$

$$\dot{\chi}^T = A_z(z, \hat{\theta}, t)\chi^T + W(z, \hat{\theta}, t)^T, \quad \chi \in \mathbb{R}^{p\times n} \tag{5.2}$$

and define the estimation error as

$$\epsilon = z + \chi_0 - \chi^T\hat{\theta}, \quad \epsilon \in \mathbb{R}^n. \tag{5.3}$$

Along with ϵ we define

$$\tilde{\epsilon} = z + \chi_0 - \chi^T\theta, \quad \tilde{\epsilon} \in \mathbb{R}^n. \tag{5.4}$$

Then we obtain

$$\epsilon = \chi^T\tilde{\theta} + \tilde{\epsilon} \tag{5.5}$$

and, by differentiating (5.4) and substituting (3.2), (5.1) and (5.2), recognize that $\tilde{\epsilon}$ is governed by

$$\dot{\tilde{\epsilon}} = A_z(z, \hat{\theta}, t)\tilde{\epsilon}. \tag{5.6}$$

The update laws for $\hat{\theta}$ employ the estimation error ϵ and the filtered regressor χ. The gradient update law is

$$\dot{\hat{\theta}} = \Gamma\frac{\chi\epsilon}{1+\nu|\chi|_{\mathcal{F}}^2}, \qquad \Gamma = \Gamma^T > 0, \nu \geq 0 \tag{5.7}$$

and the least-squares law is

$$\dot{\hat{\theta}} = \Gamma\frac{\chi\epsilon}{1+\nu|\chi|_{\mathcal{F}}^2}$$

$$\dot{\Gamma} = -\Gamma\frac{\chi\chi^T}{1+\nu|\chi|_{\mathcal{F}}^2}\Gamma \qquad \Gamma(0) = \Gamma^T(0) > 0,\ \nu \geq 0. \tag{5.8}$$

By allowing $\nu = 0$ we encompass unnormalized update laws.

Since the regressor χ is a matrix, we use the Frobenius norm $|\chi|_{\mathcal{F}}$ to avoid the need for on-line matrix inversion, as well as unnecessary algebraic complications in the stability arguments that would arise from applying update laws $\dot{\hat{\theta}} = \Gamma\chi(I_p + \nu\chi^T\Gamma\chi)^{-1}\epsilon$ with Γ fixed or updated with $\dot{\Gamma} = -\Gamma\chi(I_p + \nu\chi^T\Gamma\chi)^{-1}\chi^T\Gamma$.

The boundedness properties of the z-swapping identifiers are as follows.

Lemma 5.1: Suppose the solution $x(t)$ is defined on $[0, t_f)$. The update laws (5.7) and (5.8) guarantee that

1) if $\nu = 0$ then $\tilde{\theta} \in \mathcal{L}_\infty[0, t_f)$ and $\epsilon \in \mathcal{L}_2[0, t_f)$,
2) if $\nu > 0$ then $\tilde{\theta} \in \mathcal{L}_\infty[0, t_f)$ and

$$\dot{\tilde{\theta}},\ \frac{\epsilon}{\sqrt{1+\nu|\chi|_{\mathcal{F}}^2}} \in \mathcal{L}_2[0, t_f) \cap \mathcal{L}_\infty[0, t_f).$$

Proof: (Sketch) Noting from (5.6) and (3.3) that $d/dt\left(\frac{1}{2}|\tilde{\epsilon}|^2\right) = -\sum_{i=1}^n c_i\tilde{\epsilon}_i^2 \leq -c_0|\tilde{\epsilon}|^2$ it is clear that the positive definite function $V = \frac{1}{2}|\tilde{\theta}|_{\Gamma^{-1}}^2 + 1/2c_0|\tilde{\epsilon}|^2$ can be used as in [6], [34], [7] to prove the lemma. □

As explained in [6], various modifications of the least-squares algorithm (covariance resetting, exponential data weighting, etc.,) do not affect the properties established by Lemma 5.1. *A priori* knowledge of parameter bounds can also be incorporated via projection.

B. x-Swapping

A different identifier results if instead of the error system (3.2) we consider the plant (2.1) rewritten in the form

$$\dot{x} = Ex + e_n\beta_0(x)u + \phi(x)^T\theta \tag{5.9}$$

where

$$E = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \cdots & 0 \end{bmatrix}.$$

We employ the following filters

$$\dot{\Omega}_0 = \overline{A}(t)(\Omega_0 - x) + Ex + e_n\beta_0(x)u, \qquad \Omega_0 \in \mathbb{R}^n \tag{5.10}$$

$$\dot{\Omega}^T = \overline{A}(t)\Omega^T + \phi(x)^T, \qquad \Omega \in \mathbb{R}^{p\times n} \tag{5.11}$$

where $\overline{A}(t)$ is an exponentially stable matrix. We define the estimation error vector

$$\epsilon = x - \Omega_0 - \Omega^T\hat{\theta}, \qquad \epsilon \in \mathbb{R}^n. \tag{5.12}$$

and along with it

$$\tilde{\epsilon} = x - \Omega_0 - \Omega^T\theta, \qquad \tilde{\epsilon} \in \mathbb{R}^n. \tag{5.13}$$

Then we obtain

$$\epsilon = \Omega^T\tilde{\theta} + \tilde{\epsilon} \tag{5.14}$$

and, by differentiating (5.13) and substituting (5.9), (5.10) and (5.11), recognize that $\tilde{\epsilon}$ is governed by

$$\dot{\tilde{\epsilon}} = \overline{A}(t)\tilde{\epsilon}. \tag{5.15}$$

The update laws for $\hat{\theta}$ employ the estimation error ϵ and the filtered regressor Ω. The gradient update law is

$$\dot{\hat{\theta}} = \Gamma\frac{\Omega\epsilon}{1+\nu|\Omega|_{\mathcal{F}}^2}, \qquad \Gamma = \Gamma^T > 0, \nu \geq 0 \tag{5.16}$$

and the least-squares law is

$$\dot{\hat{\theta}} = \Gamma\frac{\Omega\epsilon}{1+\nu|\Omega|_{\mathcal{F}}^2}$$

$$\dot{\Gamma} = -\Gamma\frac{\Omega\Omega^T}{1+\nu|\Omega|_{\mathcal{F}}^2}\Gamma \qquad \Gamma(0) = \Gamma^T(0) > 0, \nu \geq 0. \tag{5.17}$$

Again, by allowing $\nu = 0$ we encompass unnormalized gradient and least-squares. Concerning the update law modifications, the same comments from the preceding subsection are also in order here.

Lemma 5.2: Suppose $x(t)$ is defined on $[0, t_f)$, and $\overline{A}(t)$ is continuous and bounded on $[0, t_f)$ and exponentially stable. The update laws (5.16) and (5.17) guarantee that

1) if $\nu = 0$ then $\tilde{\theta} \in \mathcal{L}_\infty[0, t_f)$ and $\epsilon \in \mathcal{L}_2[0, t_f)$,
2) if $\nu > 0$ then $\tilde{\theta} \in \mathcal{L}_\infty[0, t_f)$ and

$$\dot{\tilde{\theta}},\ \frac{\epsilon}{\sqrt{1+\nu|\Omega|_{\mathcal{F}}^2}} \in \mathcal{L}_2[0, t_f) \cap \mathcal{L}_\infty[0, t_f).$$

Proof: (Sketch) There exists a continuously differentiable, bounded, positive definite, symmetric $P: \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ such that $\dot{P} + P\overline{A} + \overline{A}^T P = -I, \forall t \in [0, t_f)$, and the positive definite function $V = \frac{1}{2}\|\tilde{\theta}\|_{\Gamma^{-1}}^2 + |\tilde{\epsilon}|_P^2$ can be used as in [6], [34], [7] to prove the lemma. □

VI. Stability and Tracking

Either of the identifiers from the preceding sections can now be connected with the ISS-controller (3.1), (3.9). We give stability proofs for the resulting adaptive systems. These proofs encompass both normalized and unnormalized update laws.

Theorem 6.1 (z-Swapping Scheme): All the signals in the adaptive system consisting of the plant (2.1), controller (3.1), (3.9), filters (5.1), (5.2), and either the gradient (5.7) or the least-squares (5.8) update law, are globally uniformly bounded for all $t \geq 0$, and $\lim_{t\to\infty} z(t) = 0$. This means, in particular, that global asymptotic tracking is achieved: $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$. Furthermore, if $\lim_{t\to\infty} r(t) = 0$ and $\phi(0) = 0$ then $\lim_{t\to\infty} x(t) = 0$.

Proof: Due to the continuity of x_m and the smoothness of the nonlinear terms appearing in (2.1), (3.1), (3.9), (5.1), (5.2), (5.7), (5.8), the solution of the closed-loop adaptive system exists and is unique. Let its maximum interval of existence be $[0, t_f)$.

For the normalized update laws, from Lemma 5.1 we obtain

$$\tilde{\theta},\, \dot{\hat{\theta}},\, \frac{\epsilon}{\sqrt{1+v|\chi|_{\mathcal{F}}^2}} \in \mathcal{L}_\infty[0, t_f).$$

When the update laws are unnormalized, Lemma 5.1 gives only $\tilde{\theta} \in \mathcal{L}_\infty[0, t_f)$ and we have to establish boundedness of $\dot{\hat{\theta}}$. To this end, we treat (5.2) in a fashion similar to (3.11)

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{2}|\chi|_{\mathcal{F}}^2\right)\\ &= \mathrm{tr}\frac{d}{dt}\left(\frac{1}{2}\chi\chi^T\right)\\ &\le -c_0\,\mathrm{tr}\{\chi\chi^T\} - \mathrm{tr}\,\{\chi\,\mathrm{diag}\,(\kappa_1|w_1|^2,\cdots,\kappa_n|w_n|^2)\chi^T\}\\ &\quad + \mathrm{tr}\,\{W^T\chi\}\\ &= -c_0|\chi|_{\mathcal{F}}^2 + \sum_{i=1}^n(-\kappa_i|w_i|^2|\chi_i|^2 + w_i^T\chi_i)\\ &\le -c_0|\chi|_{\mathcal{F}}^2 + \frac{1}{4\kappa_0}. \end{aligned} \tag{6.1}$$

This proves that $\chi \in \mathcal{L}_\infty[0, t_f)$. Therefore, by (5.5) and because of the boundedness of $\tilde{\epsilon}$ we conclude that $\epsilon \in \mathcal{L}_\infty[0, t_f)$. Now by (5.7) or (5.8), $\dot{\hat{\theta}} \in \mathcal{L}_\infty[0, t_f)$. Therefore, by Lemma 3.2, $z, x \in \mathcal{L}_\infty[0, t_f)$. Finally, by (5.3), $\chi_0 \in \mathcal{L}_\infty[0, t_f)$.

We have thus shown that all of the signals of the closed-loop adaptive system are bounded on $[0, t_f)$ by constants depending only on the initial conditions, design gains, the external signals x_m and r, and not depending on t_f. The independence of the bounds of t_f proves that $t_f = \infty$. Hence, all signals are globally uniformly bounded on $[0, \infty)$.

Now we set out to prove that $z \in \mathcal{L}_2$, and eventually that $z(t) \to 0$ as $t \to \infty$. For the normalized update laws, from Lemma 5.1 we obtain $\dot{\hat{\theta}}, \in /\sqrt{1+\nu|\chi|_{\mathcal{F}}^2} \in \mathcal{L}_2$. Since $\chi \in \mathcal{L}_\infty$ then $\epsilon \in \mathcal{L}_2$. When the update laws are unnormalized Lemma 5.1 gives $\epsilon \in \mathcal{L}_2$, and since $\chi \in \mathcal{L}_\infty$ then by (5.7) or (5.8), $\dot{\hat{\theta}} \in \mathcal{L}_2$. Consequently in both the normalized and the unnormalized cases $\chi^T\tilde{\theta} \in \mathcal{L}_2$ because $\tilde{\epsilon} \in \mathcal{L}_2$. With $V = \frac{1}{2}|\zeta|^2$, all the conditions of Lemmas 4.1 and 4.2 are satisfied. Thus, by Lemma 4.2, $z - \chi^T\tilde{\theta} \in \mathcal{L}_2$. Hence $z \in \mathcal{L}_2$. To prove the convergence of z to zero, we note that (3.2), (3.3) implies that $\dot{z} \in \mathcal{L}_\infty$. Therefore, by Barbalat's lemma $z(t) \to 0$ as $t \to \infty$. When $r(t) \to 0$ then $x_m(t) \to 0$ as $t \to \infty$, and from the definitions in (3.1) we conclude that, if $\phi(0) = 0$, then $x(t) \to 0$ as $t \to \infty$. □

Now we proceed to prove stability of the x-swapping scheme. With normalized update laws, the proof is similar to the proof of Theorem 6.1. With the unnormalized update laws, it is not clear how to prove boundedness of all signals for an arbitrary exponentially stable $\overline{A}(t)$. We avoid this difficulty by designing

$$\overline{A}(t) = A_0 - \lambda\phi^T(x)\phi(x)P \tag{6.2}$$

where $\lambda > 0$ and A_0 is an arbitrary constant matrix that satisfies $PA_0 + A_0^TP = -I, P = P^T > 0$. With this design the matrix $\overline{A}(t)$ is exponentially stable because

$$P\overline{A}(t) + \overline{A}^T(t)P = -I - 2\lambda P\phi^T\phi P \le -I. \tag{6.3}$$

Theorem 6.2 (x-Swapping Scheme): All the signals in the adaptive system consisting of the plant (2.1), controller (3.1), (3.9), filters (5.10), (5.11), and either the gradient (5.16) or the least-squares (5.17) update law are globally uniformly bounded for all $t \ge 0$, and $\lim_{t\to\infty} z(t) = 0$. This means, in particular, that global asymptotic tracking is achieved: $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$. Furthermore, if $\lim_{t\to\infty} r(t) = 0$ and $\phi(0) = 0$ then $\lim_{t\to\infty} x(t) = 0$.

Proof: We first consider the normalized update laws. As in the proof of Theorem 6.1, we show that $\hat{\theta}, \dot{\hat{\theta}}, z, x \in \mathcal{L}_\infty[0, t_f)$ and hence $u \in \mathcal{L}_\infty[0, t_f)$. From (5.10) and (5.11) it follows that Ω_0, Ω, and therefore ϵ are in $\mathcal{L}_\infty[0, t_f)$. Now, by the same argument as in the proof of Theorem 6.1 we conclude that $t_f = \infty$.

Second, we consider the unnormalized update laws (5.16) and (5.17) with $\overline{A}(t)$ given by (6.2). Along the solutions of (5.11) we have

$$\begin{aligned}\frac{d}{dt}(\Omega P\Omega^T) &= -\Omega\Omega^T - 2\lambda\Omega P\phi^T\phi P\Omega^T + \Omega P\phi^T + \phi P\Omega^T\\ &= -\Omega\Omega^T - 2\lambda\left(\phi P\Omega^T - \frac{1}{2\lambda}I_p\right)^T\\ &\quad \cdot\left(\phi P\Omega^T - \frac{1}{2\lambda}I_p\right) + \frac{1}{2\lambda}I_p \end{aligned} \tag{6.4}$$

which implies

$$\frac{d}{dt}(\mathrm{tr}\,\{\Omega P\Omega^T\}) \le -\mathrm{tr}\,\{\Omega\Omega^T\} + \frac{p}{2\lambda}. \tag{6.5}$$

Hence $\Omega \in \mathcal{L}_\infty[0, t_f)$. Lemma 5.2 gives[2] $\tilde{\theta} \in \mathcal{L}_\infty[0, t_f)$, and from (5.14) and (5.15) we conclude that $\epsilon \in \mathcal{L}_\infty[0, t_f)$. Now by (5.16) or (5.17), $\dot{\hat{\theta}} \in \mathcal{L}_\infty[0, t_f)$. Therefore, by Lemma 3.2, $z, x \in \mathcal{L}_\infty[0, t_f)$. Finally, by (5.12), $\Omega_0 \in \mathcal{L}_\infty[0, t_f)$. As before, $t_f = \infty$.

Now we set out to prove that $z \in \mathcal{L}_2$. For normalized update laws, from Lemma 5.2, we have that $\dot{\hat{\theta}}, \epsilon/\sqrt{1+\nu|\Omega|_{\mathcal{F}}^2} \in \mathcal{L}_2$. Since $\Omega \in \mathcal{L}_\infty$ then $\epsilon \in \mathcal{L}_2$. When the update laws are unnormalized, Lemma 5.2 gives $\epsilon \in \mathcal{L}_2$, and since $\Omega \in \mathcal{L}_\infty$ then by (5.16) or (5.17), $\dot{\hat{\theta}} \in \mathcal{L}_2$. Consequently for both the normalized and the unnormalized cases, $\Omega^T\tilde{\theta} \in \mathcal{L}_2$ because $\tilde{\epsilon} \in \mathcal{L}_2$. Now, as in Theorem 6.1, we invoke Lemma 4.2 to deduce that $z - \chi^T\tilde{\theta} \in \mathcal{L}_2$. To show that $z \in \mathcal{L}_2$, we need to prove that $\Omega^T\tilde{\theta} \in \mathcal{L}_2$ implies $\chi^T\tilde{\theta} \in \mathcal{L}_2$, or, in the notation of Lemma A.2 from the Appendix, that $T_{\overline{A}}[\phi^T]\tilde{\theta} \in \mathcal{L}_2$ implies

[2] Since $\overline{A}(t)$ depends on $x(t)$ whose boundedness is yet to be proven, in invoking Lemma 5.2 we violate the boundedness condition for $\overline{A}(t)$. This, however, causes no difficulty because the boundedness condition is required only in order to establish the existence of P, and we know P in (6.2) *a priori*.

$T_{A_s}[W^T]\tilde{\theta} \in \mathcal{L}_2$. To apply this lemma to our adaptive system we note from (3.3) and (3.1) that

$$W^T(z, \hat{\theta}, t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial \alpha_1}{\partial x_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial \alpha_{n-1}}{\partial x_1} & -\frac{\partial \alpha_{n-1}}{\partial x_2} & \cdots & 1 \end{bmatrix} \phi^T(x) \triangleq M(z, \hat{\theta}, t)\phi^T(x). \tag{6.6}$$

Since $M(z(t), \hat{\theta}(t), t)$ satisfies the conditions of Lemma A.2 then $\chi^T\tilde{\theta} \in \mathcal{L}_2$ and hence $z \in \mathcal{L}_2$. The rest of the proof is the same as for Theorem 6.1. □

Remark 6.1: All the above results are presented for the parametric-strict-feedback form (2.1) without zero dynamics. As in [12], they can be readily modified for the strict-feedback systems with zero-dynamics

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T\varphi_i(x_1, \cdots, x_i, x^r), \qquad 1 \le i \le n-1 \\ \dot{x}_n &= \beta_0(x)u + \theta^T\varphi_n(x) \\ \dot{x}^r &= \Phi_0(y, x^r) + \Phi(y, x^r)\theta \\ y &= x_1 \end{aligned} \tag{6.7}$$

where the x^r-subsystem has a bounded-input bounded-state (BIBS) property with respect to y as its input. The procedure can also be modified, as in [12], to obtain a local result for the parametric-pure-feedback systems, i.e., the systems in which φ_i also depends on x_{i+1}. As in [40], the subset of pure-feedback systems that can be controlled globally can be enlarged using an appropriate filter and parameter estimate initialization. □

VII. $\mathcal{L}_\infty$, Mean-Square and $\mathcal{L}_2$ Performance

For linear systems the issue of transient performance has recently received considerable attention (see [4], [20] and references therein). For the adaptive schemes presented in the preceding sections we now derive $\mathcal{L}_\infty$, mean-square, and $\mathcal{L}_2$ bounds for the error state z, which incorporate the bounds for the tracking error $y - y_r$.

First we give performance bounds for parameter identifiers and use them to establish $\mathcal{L}_\infty$ and mean-square bounds for z that are valid for both the z-swapping and the x-swapping schemes. Then we derive an $\mathcal{L}_2$ norm bound on z for the z-swapping scheme. For the x-swapping scheme a similar $\mathcal{L}_2$ bound is not yet available.

We analyze in detail the scheme with the normalized gradient update laws and suggest in Remarks 7.1 and 7.4 how to modify the derivations for other update laws.

Without loss of generality we assume in our analysis, and recommend for implementation, that $\tilde{\epsilon}(0)$, $\chi(0)$ (in the z-swapping scheme), and $\Omega(0)$ (in the x-swapping scheme), be set to zero. This can be achieved by initializing $\chi_0(0) = -z(0)$, $\chi(0) = 0$, in the z-swapping scheme, and $\Omega_0(0) = x(0)$, $\Omega(0) = 0$, in the x-swapping scheme. For simplicity, we also let $\Gamma = \gamma I$. We explain in Remark 7.3 how the performance bounds differ in the absence of initialization.

Lemma 7.1: For both the z-swapping (5.7) and the x-swapping (5.16) normalized ($\nu > 0$) gradient update laws, the following bounds hold

$$i) \quad \|\tilde{\theta}\|_\infty = |\tilde{\theta}(0)| \tag{7.1}$$

$$ii) \quad \|\dot{\hat{\theta}}\|_\infty \le \frac{\gamma}{\nu}|\tilde{\theta}(0)| \tag{7.2}$$

$$iii) \quad \|\dot{\hat{\theta}}\|_2 \le \sqrt{\frac{\gamma}{2\nu}}|\tilde{\theta}(0)|. \tag{7.3}$$

Proof: The proof is given for the z-swapping identifier (5.1), (5.2), (5.7). The proof for the x-swapping identifier (5.10), (5.11), (5.16) is identical.

Consider the positive definite function $V_{\tilde{\theta}} = 1/2\gamma|\tilde{\theta}|^2$.Its derivative along the solutions of (5.5), (5.7) is

$$\dot{V}_{\tilde{\theta}} = -\frac{|\epsilon|^2}{1+\nu|\chi|_{\mathcal{F}}^2} \le 0. \tag{7.4}$$

i) Due to the nonpositivity of $\dot{V}_{\tilde{\theta}}$ we have $V_{\tilde{\theta}}(t) \le V_{\tilde{\theta}}(0)$ which implies (7.1).

ii) From (5.7) we can write

$$\begin{aligned} |\dot{\hat{\theta}}|^2 &\le \gamma^2\frac{\epsilon^T\chi^T\chi^\epsilon}{(1+\nu|\chi|_{\mathcal{F}}^2)^2} \le \gamma^2\frac{|\epsilon|^2|\chi|_{\mathcal{F}}^2}{(1+\nu|\chi|_{\mathcal{F}}^2)^2} \\ &\le \frac{\gamma^2}{\nu}\frac{|\epsilon|^2}{1+\nu|\chi|_{\mathcal{F}}^2}. \end{aligned} \tag{7.5}$$

By using (5.5) we get

$$|\dot{\hat{\theta}}|^2 \le \frac{\gamma^2}{\nu}\frac{\tilde{\theta}^T\chi\chi^T\tilde{\theta}}{1+\nu|\chi|_{\mathcal{F}}^2} \le \frac{\gamma^2}{\nu}\frac{|\tilde{\theta}|^2|\chi|_{\mathcal{F}}^2}{1+\nu|\chi|_{\mathcal{F}}^2} \le \left(\frac{\gamma}{\nu}\right)^2|\tilde{\theta}|^2 \tag{7.6}$$

which, in view of (7.1), proves (7.2).

iii) By integrating (7.4) over $[0, \infty)$ we obtain

$$\left\|\frac{\epsilon}{\sqrt{1+\nu|\chi|_{\mathcal{F}}^2}}\right\|_2 \le \sqrt{V_{\tilde{\theta}}(0)} = \frac{1}{\sqrt{2\gamma}}|\tilde{\theta}(0)|. \tag{7.7}$$

Integration of (7.5) over $[0, \infty)$ and substitution of (7.7) yields (7.3). □

Remark 7.1: The only difference in the case of the normalized least-squares is that (7.3) becomes $\|\dot{\hat{\theta}}\|_2 \le \sqrt{\gamma/\nu}|\tilde{\theta}(0)|$. □

Theorem 7.1: In the adaptive system (2.1), (3.1) using either the identifier (5.1), (5.2), (5.7) or (5.10), (5.11), (5.16) with normalized update laws, the following inequalities hold

$$i) \quad |z(t)| \le \frac{|\tilde{\theta}(0)|}{2\sqrt{c_0}}\left(\frac{1}{\kappa_0} + \frac{\gamma^2}{g_0\nu^2}\right)^{1/2} + |z(0)|e^{-c_0t}, \tag{7.8}$$

$$ii) \quad \left(\frac{1}{t}\int_0^t |z(\tau)|^2\,d\tau\right)^{1/2} \le \frac{|\tilde{\theta}(0)|}{2\sqrt{c_0}}\left(\frac{1}{\kappa_0} + \frac{1}{t}\frac{\gamma^2}{2g_0\nu^2}\right)^{1/2} + \frac{1}{\sqrt{2c_0}}|z(0)|. \tag{7.9}$$

Proof: i) This bound follows by substituting (7.1) and (7.2) into (3.10).

ii) By integrating the first line of (3.13) we get

$$\int_0^t |z(\tau)|^2\,d\tau \le \frac{1}{2c_0}|z(0)|^2 + \frac{1}{4c_0\kappa_0}\|\tilde{\theta}\|_\infty^2 t + \frac{1}{2g_0}\int_0^t \left(\int_0^\tau e^{-2c_0(\tau-s)}|\dot{\hat{\theta}}(s)|^2\,ds\right)d\tau. \quad (7.10)$$

Now, to arrive at (7.9), the sequence of integration in (7.10) is interchanged as in the proof of Lemma A.1.(ii).

Remark 7.2: Although the initial states $z_2(0), \cdots, z_\rho(0)$ may depend on c_i, κ_i, g_i, this dependence can be removed by setting $z(0) = 0$ with the following initialization of the reference model

$$x_{m,i}(0) = x_i(0) - \alpha_{i-1}(\overline{x}_{i-1}(0), \hat{\theta}(0), \overline{x}_{m,i-1}(0)). \quad (7.11)$$

It can also be proven that in this initialization $x_m(0)$ does not depend on c_i, κ_i, g_i. Therefore, the bounds (7.8), (7.9) can be made as small as desired by increasing c_0, and/or κ_0, g_0. A practical limit to the increase of these gain coefficients is that, in the presence of an error in the initial state measurement, they increase $z(0)$ and the performance deteriorates. As for the pure-feedback systems mentioned in Remark 6.1, the feasibility region may, in general, decrease as c_i, κ_i, g_i increase. □

Remark 7.3: The above bounds are readily modified to also cover the case when $\tilde{\epsilon}(0) \neq 0$. For example, the $\mathcal{L}_\infty$ bound (7.8) is augmented by the term

$$\frac{1}{2\sqrt{c_0}}\left[\frac{\gamma}{4c_0}\left(\frac{1}{\kappa_0} + \frac{2\gamma^2}{g_0\nu^2}\right) + \frac{2\gamma^2}{g_0\nu}\right]^{1/2}|\tilde{\epsilon}(0)|. \quad (7.12)$$

If we set both filter initial conditions to zero, namely, $\chi(0) = 0$ and $\chi_0(0) = 0$, we get $\tilde{\epsilon}(0) = z(0)$. (This initialization is always exact because it does not depend on the measured state.) In this case, (7.12) shows that, for $z(0) \neq 0$ the performance bound (7.8) is

$$|z(t)| \le \frac{|\tilde{\theta}(0)|}{2\sqrt{c_0}}\left(\frac{1}{\kappa_0} + \frac{2\gamma^2}{g_0\nu^2}\right)^{1/2} + \frac{1}{2\sqrt{c_0}}\left[\frac{\gamma}{4c_0}\left(\frac{1}{\kappa_0} + \frac{2\gamma^2}{g_0\nu^2}\right) + \frac{2\gamma^2}{g_0\nu}\right]^{1/2}|z(0)| + |z(0)|e^{-c_0 t}. \quad (7.13)$$

□

Lemma 7.2: For the adaptive system (2.1), (3.1), (5.1), (5.2), (5.7), the following inequalities hold

$$i) \qquad \|\,|\chi|_\mathcal{F}\|_\infty \le \frac{1}{2\sqrt{c_0\kappa_0}} \quad (7.14)$$

$$ii) \qquad \|\epsilon\|_\infty \le \frac{|\tilde{\theta}(0)|}{2\sqrt{c_0\kappa_0}} \quad (7.15)$$

$$iii) \qquad \|\epsilon\|_2 \le \sqrt{\frac{1}{2\gamma} + \frac{\nu}{2\gamma}\frac{1}{4c_0\kappa_0}}|\tilde{\theta}(0)|. \quad (7.16)$$

Proof: i) By Lemma A.1–i), and since $\chi(0) = 0$, inequality (6.1) is rewritten as

$$|\chi(t)|_\mathcal{F}^2 \le |\chi(0)|_\mathcal{F}^2 e^{-2c_0 t} + \int_0^t e^{-2c_0(t-\tau)}\frac{1}{2\kappa_0}\,d\tau \le \frac{1}{4c_0\kappa_0} \quad (7.17)$$

and (7.14) follows.

ii) Now (5.5) implies $\|\epsilon\|_\infty \le \|\,|\chi|_\mathcal{F}\|_\infty\|\tilde{\theta}\|_\infty \le (1/2\sqrt{c_0\kappa_0})|\tilde{\theta}(0)|$ which proves (7.15).

iii) The bound on the $\mathcal{L}_2$ norm of ϵ is obtained using

$$\int_0^\infty |\epsilon(\tau)|^2\,d\tau \le \int_0^\infty \frac{|\epsilon(\tau)|^2}{1+\nu|\chi|_\mathcal{F}^2}(1+\nu\|\,|\chi|_\mathcal{F}\|_\infty^2)\,d\tau \le (1+\nu\|\,|\chi|_\mathcal{F}\|_\infty^2)\left\|\frac{\epsilon}{\sqrt{1+\nu|\chi|_\mathcal{F}^2}}\right\|_2. \quad (7.18)$$

By substituting (7.7) and (7.14) into (7.18) we prove (7.16). □

Remark 7.4: With the bounds (7.14)–(7.16) for the z-swapping scheme we can tighten the bounds on $\|\dot{\hat{\theta}}\|_2$ and $\|\dot{\hat{\theta}}\|_\infty$ in Lemma 7.1 and make them valid for the unnormalized update laws with $\nu = 0$. It is straightforward to show that $1/\nu$ in (7.2)–(7.3) can be replaced by $\min\{1/\nu, 1/4c_0\kappa_0\}$. The same is true for (7.8)–(7.9). We can also show that for the x-swapping scheme $1/\nu$ can be replaced by $\min\{1/\nu, (p/2\lambda)[\lambda_{\max}(P_0)/\lambda_{\min}(P_0)]\}$. □

Theorem 7.2: In the adaptive system (2.1), (3.1) with the z-swapping identification scheme (5.1), (5.2), (5.7) the $\mathcal{L}_2$ norm of z is bounded by

$$\|z\|_2 \le \frac{|\tilde{\theta}(0)|}{\sqrt{2c_0}}\left(\frac{\gamma}{g_0\nu} + \frac{\gamma}{2c_0^2\kappa_0\nu} + \frac{\nu}{2\kappa_0\gamma}\right)^{1/2} + \frac{|\tilde{\theta}(0)|}{\sqrt{2\gamma}} + \frac{1}{\sqrt{c_0}}|z(0)|. \quad (7.19)$$

Proof: We will calculate the $\mathcal{L}_2$ norm bound for z as $\|z\|_2 \le \|\epsilon\|_2 + \|\psi\|_2$, where

$$\psi \triangleq \chi_0 - \chi^T\hat{\theta}. \quad (7.20)$$

A bound on $\|\epsilon\|_2$ is given by (7.16). To obtain a bound on $\|\psi\|_2$, we examine

$$\dot{\psi} = A_z(z, \hat{\theta}, t)\psi - D(z, \hat{\theta}, t)^T\dot{\hat{\theta}} - \chi^T\dot{\hat{\theta}}. \quad (7.21)$$

By using (3.3) and repeating the sequence of inequalities (3.11), we derive

$$\frac{d}{dt}\left(\frac{1}{2}|\psi|^2\right) \le -c_0|\psi|^2 + \frac{1}{4g_0}|\dot{\hat{\theta}}|^2 - \psi^T\chi^T\dot{\hat{\theta}} \le -\frac{c_0}{2}|\psi|^2 - \frac{c_0}{2}\left|\psi - \frac{1}{c_0}\chi^T\dot{\hat{\theta}}\right|^2 + \frac{1}{4g_0}|\dot{\hat{\theta}}|^2 + \frac{1}{2c_0}|\chi^T\dot{\hat{\theta}}|^2 \quad (7.22)$$

which gives

$$\frac{d}{dt}\left(\frac{1}{2}|\psi|^2\right) \le -\frac{c_0}{2}|\psi|^2 + \frac{1}{2}\left(\frac{1}{2g_0}|\dot{\hat{\theta}}|^2 + \frac{1}{c_0}|\chi^T\dot{\hat{\theta}}|^2\right). \quad (7.23)$$

By applying Lemma A.1.(ii) to (7.23), we arrive at

$$\|\psi\|_2 \le \frac{1}{\sqrt{c_0}}\left(|\psi(0)| + \frac{1}{\sqrt{2g_0}}\|\dot{\hat\theta}\|_2 + \frac{1}{\sqrt{c_0}}\|\chi^T\dot{\hat\theta}\|_2\right). \quad (7.24)$$

By substituting (7.3) and (7.14) into (7.24) we get

$$\begin{aligned}\|\psi\|_2 &\le \frac{1}{\sqrt{c_0}}\left(|\psi(0)| + \frac{1}{\sqrt{2g_0}}\|\dot{\hat\theta}\|_2 + \frac{1}{\sqrt{c_0}}\|\,|\chi|_{\mathcal F}\|_\infty\|\dot{\hat\theta}\|_2\right)\\ &\le \frac{|\tilde\theta(0)|}{2\sqrt{c_0}}\sqrt{\frac{\gamma}{\nu}}\left(\frac{1}{\sqrt{g_0}} + \frac{1}{\sqrt{2c_0^2\kappa_0}}\right) + \frac{1}{\sqrt{c_0}}|z(0)| \quad (7.25)\end{aligned}$$

where we have assumed that $\chi_0(0) = z(0)$. Combining this and (7.16), and rearranging the terms, we obtain (7.19). □

The form of the bound (7.19) is favorable because it is linear in $|\tilde\theta(0)|$. It may not be possible to make the $\mathcal{L}_2$ norm of z as small as desired by c_0 alone because of the term $|\tilde\theta(0)|/\sqrt{2\gamma}$. With the standard initialization $z(0) = 0$, however, a possibility to improve the $\mathcal{L}_2$ performance is by simultaneously increasing c_0, g_0 and γ.

VIII. Rapprochement with Linear Designs

A connection of the adaptive nonlinear ISS-design presented in this paper with linear estimation-based designs will become clearer when the ISS-controller of Section III is replaced by the weaker SG-controller developed in this section. The only difference between the two controllers is that the SG-controller employs weaker nonlinear damping functions s_i. For example, for an uncertain term $\theta\varphi(x_1)$ in the first equation of the plant, the ISS and SG nonlinear damping functions are respectively

$$s_1^{\rm ISS}(x_1) = \varphi(x_1)^2 \quad \text{and}$$

$$s_1^{\rm SG}(x_1) = \left[\frac{\varphi(x_1) - \varphi(0)}{x_1}\right]^2 \triangleq \omega(x_1)^2. \quad (8.1)$$

In this way the growth of $s_1^{\rm SG}$ is reduced by a factor of x_1^2. In the process of backstepping this reduction is even more pronounced. However, the SG-controller can no longer guarantee the ISS property with respect to $\tilde\theta$ and $\dot{\hat\theta}$. Instead, we reveal a small gain property and prove boundedness with a linear-like Gronwall lemma argument. The main interest in the SG-controller is that for linear systems it becomes linear in x and in that sense is similar to linear estimation-based designs. In contrast, the ISS-controller for linear systems remains nonlinear.

To derive the nonlinear damping expressions for the SG-controller we rewrite the regressor vectors w_i as follows

$$w_i(\bar z_i, \hat\theta, t) = w_i(0, \hat\theta, t) + \omega_i(\bar z_i, \hat\theta, t)^T\bar z_i \quad (8.2)$$

where $\bar z_i \triangleq [z_1, \cdots, z_i]^T$, and $\omega_i : \mathbb{R}^i \times \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}^{i\times p}$ is a matrix-valued function smooth in the first two arguments and continuous and bounded in the third argument (with a slight abuse of notation relative to (3.1) we now express w_i as a function of $\bar z_i$, $\hat\theta$, t). Thus we have

$$W = W_0 + [\omega_1^T\bar z_1, \cdots, \omega_n^T\bar z_n] \quad (8.3)$$

where W_0 denotes $W(0, \hat\theta, t)$. Likewise, we rewrite D from (3.3)

$$D = D_0 + [0, \delta_2^T\bar z_1, \cdots, \delta_n^T\bar z_{n-1}] \quad (8.4)$$

where $\delta_i : \mathbb{R}^{i-1} \times \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}^{(i-1)\times p}$ are matrix-valued functions smooth in the first two arguments and continuous and bounded in the third argument, and D_0 denotes $D(0, \hat\theta, t)$.

The SG-controller has the same form (3.1) as the ISS-controller, but its nonlinear damping functions are defined as

$$s_i = \kappa_i|\omega_i|_{\mathcal F}^2 + g_i|\delta_i|_{\mathcal F}^2. \quad (8.5)$$

As in linear estimation-based adaptive control, the SG-controller employs an identifier with normalized update laws.

Theorem 8.1: All the signals in the adaptive system consisting of the plant (2.1), SG-controller (3.1) with (8.5), z-swapping filters (5.1), (5.2), and either the gradient (5.7) or the least-squares (5.8) normalized update law ($\nu > 0$), are globally uniformly bounded for all $t \ge 0$, and $\lim_{t\to\infty} z(t) = 0$. This means, in particular, that global asymptotic tracking is achieved: $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$. Furthermore, if $\lim_{t\to\infty} r(t) = 0$ and $\phi(0) = 0$ then $\lim_{t\to\infty} x(t) = 0$.

Proof: Using (8.3) we write (5.2) as

$$\dot\chi^T = A_z\chi^T + W_0^T + [\omega_1^T\bar z_1, \cdots, \omega_n^T\bar z_n]^T. \quad (8.6)$$

In a fashion similar to (6.1) we compute

$$\begin{aligned}\frac{d}{dt}\left(\frac12|\chi|_{\mathcal F}^2\right) &\le -c_0|\chi|_{\mathcal F}^2 - \sum_{i=1}^n\kappa_i|\omega_i|_{\mathcal F}^2|\chi_i|^2\\ &\quad + \sum_{i=1}^n\bar z_i^T\omega_i\chi_i + \operatorname{tr}\{W_0^T\chi\}\\ &\le -\frac{c_0}{2}|\chi|_{\mathcal F}^2 + \sum_{i=1}^n\frac{1}{4\kappa_i}|\bar z_i|^2 + \frac{1}{2c_0}|W_0|_{\mathcal F}^2\\ &\le -\frac{c_0}{2}|\chi|_{\mathcal F}^2 + \frac{1}{4\kappa_0}|z|^2 + \frac{1}{2c_0}|W_0|_{\mathcal F}^2. \quad (8.7)\end{aligned}$$

On the other hand, using (8.4) we write (7.21) as

$$\dot\psi = A_z\psi - (\chi + D_0)^T\dot{\hat\theta} - [0, \delta_2^T\bar z_1, \cdots, \delta_n^T\bar z_{n-1}]^T\dot{\hat\theta} \quad (8.8)$$

and compute

$$\begin{aligned}\frac{d}{dt}\left(\frac12|\psi|^2\right) &\le -c_0|\psi|^2 - \sum_{i=1}^n g_i|\delta_i|_{\mathcal F}^2\psi_i^2\\ &\quad - \psi^T(\chi + D_0)^T\dot{\hat\theta} - \sum_{i=1}^n\bar z_{i-1}^T\delta_i\psi_i\dot{\hat\theta}\\ &\le -\frac{c_0}{2}|\psi|^2 + \frac{1}{2c_0}|(\chi + D_0)^T\dot{\hat\theta}|^2\\ &\quad + \sum_{i=1}^n\frac{1}{4g_i}|\bar z_{i-1}|^2|\dot{\hat\theta}|^2\\ &\le -\frac{c_0}{2}|\psi|^2 + \frac{1}{c_0}|\chi|_{\mathcal F}^2|\dot{\hat\theta}|^2\\ &\quad + \frac{1}{4g_0}|z|^2|\dot{\hat\theta}|^2 + \frac{1}{c_0}|D_0|_{\mathcal F}^2|\dot{\hat\theta}|^2. \quad (8.9)\end{aligned}$$

The system (8.7), (8.9) is summarized as

$$\frac{d}{dt}(|\chi|_{\mathcal{F}}^2) \le -c_0|\chi|_{\mathcal{F}}^2 + \frac{1}{2\kappa_0}|z|^2 + \frac{1}{c_0}|W_0|_{\mathcal{F}}^2 \tag{8.10}$$

$$\frac{d}{dt}(|\psi|^2) \le -c_0|\psi|^2 + \frac{2}{c_0}|\dot{\hat{\theta}}|^2|\chi|_{\mathcal{F}}^2 + \frac{1}{2g_0}|\dot{\hat{\theta}}|^2|z|^2 + \frac{2}{c_0}|D_0|_{\mathcal{F}}^2|\dot{\hat{\theta}}|^2. \tag{8.11}$$

From Lemma 5.1 we have $\hat{\theta} \in \mathcal{L}_\infty[0, t_f)$, so $|W_0|_{\mathcal{F}}^2 < k$ and $|D_0|_{\mathcal{F}}^2 < k$, where k denotes a generic positive finite constant. From Lemma 5.1 we also have $\dot{\hat{\theta}}, \epsilon/\sqrt{1+\nu|\chi|_{\mathcal{F}}^2} \in \mathcal{L}_2[0, t_f) \cap \mathcal{L}_\infty[0, t_f)$. Let us denote by l_1 a generic function in $\mathcal{L}_1[0, t_f) \cap \mathcal{L}_\infty[0, t_f)$. Since $\epsilon = z + \psi$, then we have

$$\begin{aligned}|z|^2 &\le 2\frac{|\epsilon|^2}{1+\nu|\chi|_{\mathcal{F}}^2}(1+\nu|\chi|_{\mathcal{F}}^2) + 2|\psi|^2 \\ &\le 2l_1(1+\nu|\chi|_{\mathcal{F}}^2) + 2|\psi|^2.\end{aligned} \tag{8.12}$$

Thus (8.10)–(8.11) become

$$\frac{d}{dt}(|\chi|_{\mathcal{F}}^2) \le -(c_0 - l_1)|\chi|_{\mathcal{F}}^2 + \frac{1}{\kappa_0}|\psi|^2 + k \tag{8.13}$$

$$\frac{d}{dt}(|\psi|^2) \le -(c_0 - l_1)|\psi|^2 + l_1|\chi|_{\mathcal{F}}^2 + k. \tag{8.14}$$

This is a loop with small gain because $|\chi|_{\mathcal{F}}^2$ appears multiplied by l_1 in (8.14). To finish the proof we define the "superstate"

$$X \triangleq |\chi|_{\mathcal{F}}^2 + \frac{2}{\kappa_0 c_0}|\psi|^2 \tag{8.15}$$

differentiate it and substitute (8.13)–(8.14). After straightforward rearrangements and majorizations, we get

$$\dot{X} \le -\left(\frac{c_0}{2} - l_1\right)X + k. \tag{8.16}$$

By applying the Gronwall lemma we conclude that X is uniformly bounded on $[0, t_f)$. In view of (5.5), ϵ is bounded, which along with the boundedness of ψ, proves that z is bounded. Thus $t_f = \infty$. The rest of the proof is the same as for Theorem 6.1, and again uses Lemma 4.2 for proving convergence. □

In contrast to the ISS design, the global result has been established only with normalized update laws. The issue of normalization in adaptive nonlinear design was discussed in [26].

Performance bounds similar to those in Section VII for the ISS design are not available for the SG design, nor is it clear how to develop its x-swapping version.

In Section IX we compare the linear SG design with the new ISS design for a linear plant.

IX. Examples and Discussion

The first example in this section illustrates the performance properties of the ISS design on the relative-degree two plant

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta\varphi(x_1) \\ \dot{x}_2 &= u\end{aligned} \tag{9.1}$$

with the objective to regulate x to zero (without a reference model). The second example makes a comparison between the ISS design and the SG design.

We define the error variables

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= x_2 - \alpha_1(x_1, \hat{\theta}).\end{aligned} \tag{9.2}$$

The two-step ISS-controller design

$$\alpha_1 = -c_1 z_1 - \kappa_1\varphi^2 z_1 - \hat{\theta}\varphi$$

$$u = -z_1 - c_2 z_2 - \kappa_2\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2\varphi^2 z_2 - g_2\left(\frac{\partial\alpha_1}{\partial\hat{\theta}}\right)^2 z_2 + \frac{\partial\alpha_1}{\partial x_1}(x_2 + \hat{\theta}\varphi) \tag{9.3}$$

results in the error system

$$\dot{z} = A_z(z, \hat{\theta})z + \begin{bmatrix}\varphi \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi\end{bmatrix}\tilde{\theta} + \begin{bmatrix}0 \\ -\frac{\partial\alpha_1}{\partial\hat{\theta}}\end{bmatrix}\dot{\hat{\theta}} \tag{9.4}$$

where

$$A_z = \begin{bmatrix}-c_1 - \kappa_1\varphi^2 & 1 \\ -1 & -c_2 - \kappa_2\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2\varphi^2 - g_2\left(\frac{\partial\alpha_1}{\partial\hat{\theta}}\right)^2\end{bmatrix}. \tag{9.5}$$

The z-swapping identifier is designed with the following filters

$$\dot{\chi}_0 = A_z(z, \hat{\theta})\chi_0 + \begin{bmatrix}\varphi \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi\end{bmatrix}\hat{\theta} - \begin{bmatrix}0 \\ -\frac{\partial\alpha_1}{\partial\hat{\theta}}\end{bmatrix}\dot{\hat{\theta}} \tag{9.6}$$

$$\dot{\chi}^T = A_z(z, \hat{\theta})\chi^T + \begin{bmatrix}\varphi \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi\end{bmatrix} \tag{9.7}$$

which are used to implement the augmented error

$$\epsilon = z + \chi_0 - \chi^T\hat{\theta} \tag{9.8}$$

and the gradient update law

$$\dot{\hat{\theta}} = \gamma\frac{\chi\epsilon}{1+\nu\chi\chi^T}. \tag{9.9}$$

The x-swapping identifier is designed with the following filters

$$\dot{\Omega}_0 = -(\bar{a} + \lambda\varphi^2)(\Omega_0 - x_1) + x_2, \qquad \Omega_0 \in \mathbb{R} \tag{9.10}$$

$$\dot{\Omega} = -(\bar{a} + \lambda\varphi^2)\Omega + \varphi, \qquad \Omega \in \mathbb{R} \tag{9.11}$$

which are used to implement the equation error

$$\epsilon = x_1 - \Omega_0 - \Omega\hat{\theta} \tag{9.12}$$

and the gradient update law

$$\dot{\hat{\theta}} = \gamma\frac{\Omega\epsilon}{1+\nu\Omega^2}. \tag{9.13}$$

This reveals that the x-swapping approach is uncertainty specific in the sense that only the terms φ_i multiplying the unknown parameter θ need to be filtered. This opens a possibility for a reduction in the dynamic order of the identifier.

In simulations, the only difference between the z-swapping and the x-swapping approach was in the value of γ needed to achieve the same speed of adaptation—higher value was needed in the z-swapping case. Since the responses were similar we show them only for the z-swapping scheme.

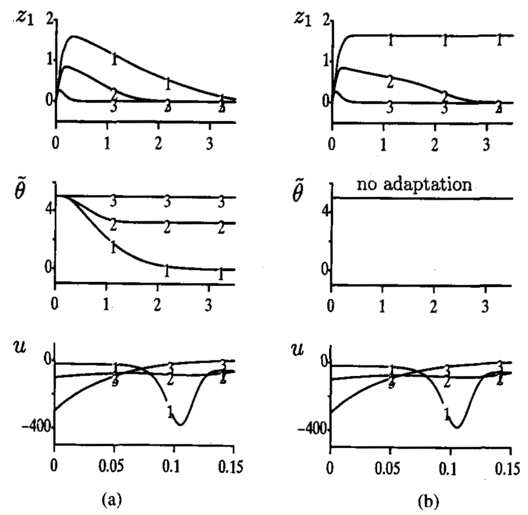

Fig. 1. Dependence of the transients on c_0 with $\kappa_0 = g_0 = 1$. (Note an expanded time scale for control u.) (a) $\gamma = 10$; (b) $\gamma = 0$. 1: $c_0 = 1$; 2: $c_0 = 5$; 3: $c_0 = 15$.

Example 8.1 (ISS-Performance): We consider system (9.1) with nonlinearity $\varphi(x_1) = x_1^2$. The simulations were carried out with nominal values $c_1 = c_2 = c_0 = \kappa_1 = \kappa_2 = \kappa_0 = g_2 = g_0 = 1$, $\gamma = 10$, $\theta = 5$, $\hat\theta(0) = 0$ which were judged to give representative responses. All simulations are with following initial conditions: $x(0) = -\chi_0\ (0) = [0,\ 10]^T$, $\chi(0) = 0$ (to set $\tilde\epsilon(0) = 0$).

Fig. 1(a) illustrates Theorems 7.1 and 7.2. The design parameter c_0 can be used for systematically improving the transient performance. Up to a certain point the error transients and the control effort in Fig. 1(a) are simultaneously decreasing as c_0 increases. Beyond that point the control effort starts increasing. The control u is given in an expanded time scale in order to clearly display the main qualitative differences among the three cases. Fig. 1(b) illustrates Corollary 3.1. When adaptation is switched off, the states are uniformly bounded and converge to (or remain inside) a compact residual set. Corollary 3.1 does not describe the behavior inside the residual set, which may contain multiple equilibria, limit cycles, etc. For this example (but not in general), there is an asymptotically stable equilibrium at the origin for any value of the parameter error. For small values of c_0, this equilibrium has a basin of attraction which is strictly inside the residual set. For higher values of c_0 the global asymptotic stability is achieved.

Fig. 2 shows the influence of κ_0 on transients. According to Theorem 7.1 and Remark 7.4, the peak values can be decreased by increasing κ_0, which is confirmed by the plot. The $\mathcal{L}_2$ performance may not be improved, however, by increasing κ_0 because the κ-terms slow down the adaptation and make the transients longer. The effect of the g-terms was shown to be significant only for very small c_0 and κ_0 or for very large γ.

Fig. 3 demonstrates the influence of the adaptation gain γ on transients. Due to the slow initial adaptation, which should be attributed not only to the normalized gradient update law but also to the fact that the regressor is filtered, there is a clear separation of action of the nonadaptive controller, which at the beginning brings the state z quickly to the residual set, and the adaptive controller which takes over to drive the state to the origin. The property that the $\mathcal{L}_\infty$ bounds are increasing functions of γ, to be expected from Theorem 7.1, was exhibited in simulations only with extremely high values of γ. This indicates that some of the bounds derived are not very tight over the entire range of design parameter values.

Finally, an explanation is in order about the initial condition $x(0)$ in our simulations. We used $x(0) = [0,\ 10]^T$, and hence $z(0) = [0,\ 10]^T$ which is independent of the design gains c_0, κ_0, g_0. This is why the peak of z_1 decreases monotonically as any of these gains increases. If, instead, we used $x_1(0) \neq 0$, then, according to Remark 7.2, we would have added an appropriately initialized reference model (with $r(t) \equiv 0$). In this way, bad transients would be eliminated by following a less aggressive path to the origin. □

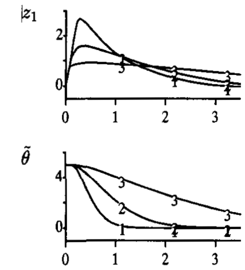

Fig. 2. Dependence of the transients on κ_0 with $c_0 = g_0 = 1$, $\gamma = 10$. 1: $c_0 = 1$; 2: $c_0 = 5$; 3: $c_0 = 15$.

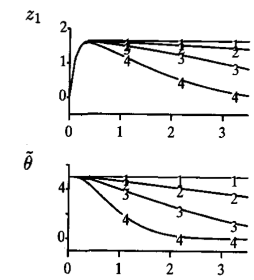

Fig. 3. Dependence of the transients on γ with $c_0 = g_0 = \kappa_0 = 1$. 1: $c_0 = 1$; 2: $c_0 = 5$; 3: $c_0 = 15$.

Example 8.2 (ISS vs. SG): Let us consider system (9.1) with $\varphi(x_1) = x_1$. For this linear system we make a comparison between the ISS and SG designs. The only difference is that the terms $\kappa_1\varphi^2, \kappa_2(\partial\alpha_1/\partial x_1)^2\varphi^2, g_2(\partial\alpha_1/\partial\hat\theta)^2$ in the ISS design are, respectively, replaced by $\kappa_1, \kappa_2(\partial\alpha_1/\partial x_1)^2, g_2$ in the SG design. The same design coefficients and initial conditions are used as in Example 8.1, except for $\theta = 3$. The adaptation gains, $\gamma = 5$ for the ISS design, and $\gamma = 1.5$ for the SG design, are chosen so that the rate of parameter convergence is the same for both designs. The control law of the SG design is linear in x and nonlinear in $\hat\theta$.

Fig. 4 shows the difference in performance between the two designs. The ISS design uses larger control effort and achieves better attenuation of the z_1-transient. The dashed responses illustrate the underlying nonadaptive behavior ($\gamma = 0$). While

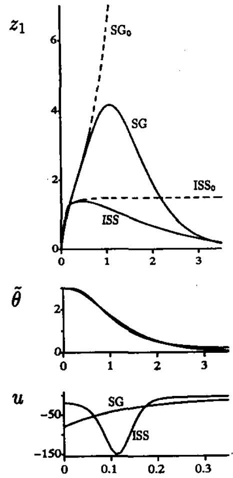

Fig. 4. ISS design vs. SG design. The dashed lines show $z_1(t)$ when $\gamma = 0$.(Note an expanded time scale for control u.)

the response of the SG design exhibits linear exponential instability, the response of the ISS design, according to Corollary 3.1, is bounded. Hence, there is a clear trade-off of performance improvement versus control effort between the new ISS design and the linear adaptive designs such as the SG design. □

X. Conclusions

Recent Lyapunov-based recursive designs of adaptive controllers for nonlinear systems transformable into the parametric-strict-feedback form [12], [8], [19], [36] achieve global stability and tracking, but do not allow a choice of parameter update laws. In these designs the wealth of knowledge about standard identifiers is not utilized because the identifier does not appear as a separate module of the adaptive system.

A complete separation of the controller and identifier modules is one of the main accomplishments of this paper. It has been achieved by a new nonlinear controller with an input-to-state stability property with respect to the parameter estimation error and its derivative as disturbance inputs. This strong ISS-controller remains nonlinear even when the plant is linear. For comparison with linear estimation-based designs, a weaker SG-controller is introduced, resulting in a small-gain rather than the ISS property. For linear plants this controller is linear.

As a separate module, the ISS-controller can be connected with the standard unnormalized or normalized gradient or least-squares identifiers, while the SG-controller requires normalization. The connection of the controller and identifier modules is made possible by a nonlinear extension of the well known swapping lemma.

In addition to the global boundedness and tracking, the new design also provides explicit bounds on the transient performance, which can be utilized for its systematic improvement.

The results of this paper assume that the full state is available for feedback. Relying on the experience gained with recent recursive output-feedback designs, such as [22], [14], it is expected that the estimation-based design of this paper will be extended to nonlinear systems in the output-feedback form.

The applicability of various designs, Lyapunov-based or estimation-based, will ultimately depend on their robustness with respect to unmodeled phenomena. This is another important topic of current research.

Appendix

Lemma A.1: Let $v, \rho\colon \mathbb{R}_+ \to \mathbb{R}$, $c, b > 0$. If

$$\dot{v} \le -cv + b\rho^2, \qquad v(0) \ge 0 \tag{A.1}$$

i) then

$$v(t) \le v(0)e^{-ct} + b\int_0^t e^{-c(t-\tau)}\rho(\tau)^2\, d\tau. \tag{A.2}$$

ii) If, in addition, $\rho \in \mathcal{L}_2$, then $v \in \mathcal{L}_\infty \cap \mathcal{L}_1$ and

$$\|v\|_1 \le \frac{1}{c}(v(0) + b\|\rho\|_2^2). \tag{A.3}$$

Proof: i) Upon multiplication of (A.1) by e^{ct}, it becomes

$$\frac{d}{dt}(v(t)e^{ct}) \le b\rho(t)^2 e^{ct}. \tag{A.4}$$

Integrating (A.4) over $[0, t]$, we arrive at (A.2).

ii) Noting that (A.2) implies that

$$v(t) \le v(0)e^{-ct} + b\sup_{\tau\in[0,\, t]}\{e^{-c(t-\tau)}\}\int_0^t \rho(\tau)^2\, d\tau \tag{A.5}$$

we conclude that $v \in \mathcal{L}_\infty$. By integrating (A.2) over $[0, t]$, we get

$$\begin{aligned}\int_0^t v(\tau)\, d\tau &\le \int_0^t v(0)e^{-c\tau}\, d\tau + b\int_0^t\left[\int_0^\tau e^{-c(\tau-s)}\rho(s)^2\, ds\right] d\tau\\ &\le \frac{1}{c}v(0) + b\int_0^t e^{cs}\rho(s)^2\left(\int_s^t e^{-c\tau}\, d\tau\right) ds\\ &\le \frac{1}{c}v(0) + b\int_0^t e^{cs}\rho(s)^2\frac{1}{c}e^{-cs}\, ds\\ &\le \frac{1}{c}\left[v(0) + b\int_0^t \rho(\tau)^2\, d\tau\right]\end{aligned} \tag{A.6}$$

which proves (A.3). □

Lemma A.2: Let $T_i\colon u \mapsto \zeta_i$, $i = 1, 2$ be linear time-varying operators defined by

$$\dot{\zeta}_i = A_i(t)\zeta_i + u \tag{A.7}$$

where $A_i\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ are continuous, bounded and exponentially stable. Suppose $\tilde{\theta}\colon \mathbb{R}_+ \to \mathbb{R}^p$ is differentiable, $\phi\colon \mathbb{R}_+ \to \mathbb{R}^{p\times m}$ is piecewise continuous and bounded, and $M\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is bounded and has a bounded derivative on $\mathbb{R}_+$. If $\dot{\tilde{\theta}} \in \mathcal{L}_2$ then

$$T_1[\phi^T]\tilde{\theta} \in \mathcal{L}_2 \Rightarrow T_2[M\phi^T]\tilde{\theta} \in \mathcal{L}_2. \tag{A.8}$$

If moreover, $M(t)$ is nonsingualr $\forall t$, and M^{-1} is bounded and has a bounded derivative on $\mathbb{R}_+$ then (A.8) holds in both directions.

Proof: Suppose that $T_1[\phi^T]\tilde{\theta} \in \mathcal{L}_2$. By Lemma 4.2, $T_1[\phi^T\tilde{\theta}] - T_1[\phi^T]\tilde{\theta} \in \mathcal{L}_2$ and therefore $\zeta_1 \triangleq T_1[\phi^T\tilde{\theta}] \in \mathcal{L}_2$. We will show first that $\zeta_2 \triangleq T_2[M\phi^T\tilde{\theta}] \in \mathcal{L}_2$. By substituting $\phi^T\tilde{\theta} = \dot{\zeta}_1 - A_1(t)\zeta_1$ into the variation of constants formula and applying partial integration we calculate

$$\begin{aligned}\zeta_2(t) &= \Phi_2(t, 0)\zeta_2(0) + \int_0^t \Phi_2(t, \tau)M(\tau)\phi^T(\tau)\tilde{\theta}(\tau)\,d\tau \\ &= \Phi_2(t, 0)\zeta_2(0) + \int_0^t \Phi_2(t, \tau)M(\tau) \\ &\quad \cdot [\dot{\zeta}_1(\tau) - A_1(\tau)\zeta_1(\tau)]\,d\tau \\ &= \Phi_2(t, 0)\zeta_2(0) + M(t)\zeta_1(t) - \Phi_2(t, 0)M(0)\zeta_1(0) \\ &\quad + \int_0^t \Phi_2(t, \tau)[\dot{M}(\tau) + A_2(\tau)M(\tau) - M(\tau)A_1(\tau)] \\ &\quad \cdot \zeta_1(\tau)\,d\tau \end{aligned} \tag{A.9}$$

where $\Phi_2(t, \tau)$ is the state transition matrix of $A_2(t)$ that satisfies $|\Phi_2(t, \tau)|_2 \leq k\mathrm{e}^{-\alpha(t-\tau)}, k, \alpha > 0$. It is clear that $\Phi_2(t, 0)\zeta_2(0)+M(t)\zeta_1(t)-\Phi_2(t, 0)M(0)\zeta_1(0) \in \mathcal{L}_2$ because $\Phi_2(t, 0)$ is exponentially decaying, $M(t)$ is bounded and $\zeta_1 \in \mathcal{L}_2$. Since

$$\begin{aligned}&\left|\int_0^t \Phi_2(t, \tau)[\dot{M}(\tau) + A_2(\tau)M(\tau) - M(\tau)A_1(\tau)]\zeta_1(\tau)\,d\tau\right|^2 \\ &\leq \|\dot{M} + A_2M - MA_1\|_\infty^2 k^2 \int_0^t e^{-2\alpha(t-\tau)}|\zeta_1(\tau)|^2\,d\tau \end{aligned} \tag{A.10}$$

then similarly to (4.16)–(4.17) from the proof of Lemma 4.2, we can show that the expression (A.10) is in $\mathcal{L}_2$. Thus $\zeta_2 = T_2[M\phi^T\tilde{\theta}] \in \mathcal{L}_2$. By Lemma 4.2, $T_2[M\phi^T\tilde{\theta}] - T_2[M\phi^T]\tilde{\theta} \in \mathcal{L}_2$ and therefore $T_2[M\phi^T]\tilde{\theta} \in \mathcal{L}_2$. The proof of the other direction of (A.8) when $M(t)$ is nonsingular $\forall t$, and M^{-1} is bounded and has a bounded derivative on $\mathbb{R}_+$ is identical. □

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