ADAPTIVE NUMERICAL SCHEMES FOR A PARABOLIC PROBLEM WITH BLOW-UP

RAÚL FERREIRA, PABLO GROISMAN AND JULIO D. ROSSI

Abstract. In this paper we present adaptive procedures for the numerical study of positive solutions of the following problem,

$$\begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times [0,T), \\ u_x(0,t) = 0 & t \in [0,T), \\ u_x(1,t) = u^p(1,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,1), \end{cases}$$

with $p > 1$. We describe two methods, the first one refines the mesh in the region where the solution becomes bigger in a precise way that allows us to recover the blow-up rate and the blow-up set of the continuous problem. The second one combine the ideas used in the first one with the moving mesh methods and moves the last points when necessary. This scheme also recovers the blow-up rate and set. Finally we present numerical experiments to illustrate the behavior of both methods.

1. Introduction.

In this paper we deal with numerical approximations for the following problem,

$$\begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times [0,T), \\ u_x(0,t) = 0 & t \in [0,T), \\ u_x(1,t) = u^p(1,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,1). \end{cases} \tag{1.1}$$

We assume that u_0 is positive and compatible in order to have a regular solution. If $p > 1$ it is well known that every positive solution becomes unbounded in finite time, a phenomena that is known as blow-up. If T is the maximum time of existence of the solution u then

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_\infty = +\infty,$$

see [24], [28], [31] and also [17], [23], [27], [30] for general references on blow-up problems. For (1.1) it is proved in [13], [18] that the blow-up

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rate is given by,

$$\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2(p-1)}}.$$

The blow-up set, $B(u)$, i.e., the set of points $x \in [0,1]$ where $u(x,t)$ becomes unbounded, is given by,

$$B(u) = \{1\}.$$

This phenomena is called single point blow-up, see [21].

Here we are interested in numerical approximations of (1.1). Since the solution u develops a singularity in finite time, it is an interesting question what can be said about numerical approximations of this kind of problems. For previous work on numerical approximations of blowing up solutions we refer to [1], [2], [6], [7], [10], [11], [15], [22], [25], [26] the survey [5] and references therein. For approximations of (1.1) we refer to [3], [4] and [14]. Those papers deal with the behaviour of a semidiscrete numerical approximation in a fixed mesh of size h. It is observed there that significant differences appear between the continuous and the discrete problems. First, it is proved that positive solutions of the numerical problem blow up if and only if $p > 1$, see [14], this is the same blow-up condition that holds for the continuous problem. Next, it is shown in [3] and [4] that the blow-up rate for the numerical approximation, U, is given by

$$\|U(t)\|_\infty \sim (T_h - t)^{-\frac{1}{p-1}}.$$

Therefore the blow-up rate does not coincide with the expected for the continuous problem. Concerning the blow-up set it holds,

$$B(U) = [1-Lh, 1] \text{ if } p > 1,$$

see [4], [16]. The constant L depends only on p, L is the only integer that verifies

$$\frac{L+1}{L} < p \le \frac{L}{L-1}.$$

Hence the numerical blow-up set can be larger than a single point, $x = 1$, but as L does not depend on h,

$$B(U) \to B(u), \qquad \text{as } h \to 0.$$

Collecting all these results, we observe that when computing numerical approximations of a blow-up problem with a fixed mesh significant differences appear concerning the behaviour of the numerical and the continuous solutions. For this problem the blow-up rate and the blow-up set can be different. We conclude that the usual methods with a fixed grid are not well suited for the problem under consideration.

Therefore an adaptive mesh refinement is necessary. Some references that use adaptive numerical methods are [3], [7] and [10].

In this paper we introduce two adaptive methods that give the right blow-up rate and the exact blow-up set. Our main interest is to provide rigorous proofs of these facts.

The first method, that we will call *adding points method*, is based on a semidiscretization in space and adds points near the boundary, $x = 1$, when the numerical solution becomes large. This adaptive procedure leads to a non-uniform mesh that concentrates near the singularity and gives the precise blow-up rate and set.

The second one, that we will call *moving points method*, also uses a semidiscretization in space but this time we move the last K points near the boundary when the numerical solution becomes large. The number of moving points, K, depends only on the exponent p, in fact we will choose $K = [1/(p-1)]$. This procedure is inspired in the moving mesh algorithms developed in [8], [9], [10], [20]. In our case we take advantage of the a priori knowledge of the spatial location of the singularity at $x = 1$ and instead of moving the whole mesh continuously as time evolves, we concentrate only the last K points near the boundary, leaving the rest of the mesh fixed. This allows us to use a unified approach to analyze rigorously both schemes simultaneously. One advantage of this moving procedure is keeping the size of the ODE problem to be solved constant in time, while the adding points methods enlarges the number of equations as time evolves.

Both numerical schemes are based on the scale invariance of the heat equation in the half line with the nonlinear boundary condition placed at $x = 0$, $-u_x(0,t) = u^p(0,t)$. If $u(x,t)$ is a solution then $u_\lambda(x,t) = \lambda^{\frac{1}{2(p-1)}} u(\lambda^{\frac{1}{2}} x, \lambda t)$ also is a solution. In [18] it is proved that there exists a self-similar blowing up solution in the half-line of the form

$$u_S(x,t) = (T-t)^{-\frac{1}{2(p-1)}} \varphi(\xi), \qquad \xi = x(T-t)^{-\frac{1}{2}}.$$

For an explicit form of the profile φ, see [18]. This solution $u_S(x,t)$ gives the behaviour near the blow-up time T for solutions of (1.1) in the following sense

$$u(x,t) \sim (T-t)^{-\frac{1}{2(p-1)}} \varphi(\xi), \tag{1.2}$$

for $x = 1 - \xi(T-t)^{1/2}$, $|\xi| \le C$. So the behavior near the blow-up point $(1,T)$ is given by the self-similar solution in the half-line. The numerical schemes presented here use this fact to add or move points near $x = 1$ trying to reproduce the scaling invariance in the half-line.

The main goal of this paper is to present rigorous estimates for both methods simultaneously showing that they give the right blow-up rate and set. Our result reads as follows and holds for the method that adds points as well as for the method that moves the last points.

Theorem 1.1. *Let u be an increasing in x, smooth solution of* (1.1) *and u_h be the numerical solution obtained by any of the adaptive schemes described in section 2. Then, for every $\tau > 0$, the numerical solution u_h converges to the continuous one uniformly in $[0,1] \times [0, T-\tau]$, in fact there exists a constant $C = C(\tau)$ such that*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \leq Ch^{\frac{3}{2}}.$$

The numerical solution u_h blows up if and only if $p > 1$ in the sense that there exists a finite time T_h with

$$\lim_{t \nearrow T_h} u_h(1,t) = +\infty.$$

This numerical blow-up time converges to the continuous one when $h \to 0$, in fact there exist $\alpha > 0$ and $C > 0$ such that

$$|T_h - T| \leq Ch^\alpha.$$

Moreover, the numerical blow-up rate is given by

$$\lim_{t \nearrow T_h} (T_h - t)^{\frac{1}{2(p-1)}} \|u_h(\cdot, t)\|_\infty = \Gamma = \varphi(0)$$

and the numerical blow-up set is

$$B(u_h) = \{1\}.$$

Organization of the paper: In section 2 we describe the numerical adaptive procedures. In sections 3 to 6 we develop the proofs of the main result, Theorem 1.1. To begin the analysis we prove in section 3 that numerical approximations converge uniformly in sets of the form $[0,1] \times [0, T-\tau]$. In section 4 we prove that the scheme reproduces the blow-up rate. In section 5 we find the numerical blow-up set that coincides with the continuous one. In section 6 we prove that the numerical blow-up time converges to the continuous one. Finally, in section 7 we present some numerical experiments comparing the performance of both methods and make a few comments on possible extensions of the ideas developed here to several space dimensions. We leave for the Appendix the proof of a technical result needed in section 6.

2. Adaptive numerical schemes

In this section we present the numerical methods. We consider a semidiscrte scheme, that is, we discretize the space variable, keeping t continuous. For the spatial discretization we propose piecewise linear finite elements with mass lumping.

Consider a partition (that can be non-uniform), $x_1, ..., x_{N+1}$ of $(0,1)$ of size h ($h = \max(x_i - x_{i-1})$) and its associated standard piecewise linear finite element space V_h. Let $\{\varphi_j\}_{1\le j\le N}$ be the usual Lagrange basis of V_h. The semidiscrete approximation $u_h(t) \in V_h$ obtained by the finite element method with mass lumping is defined by:

$$\int_0^1 (u_h' v)^I dx + \int_0^1 (u_h)_x v_x dx = u^p(1,t)v(1,t), \qquad \forall v \in V_h, \forall t \in (0,T)$$

where the superindex I denotes the Lagrange interpolation. We denote with $U(t) = (u_1(t), \dots, u_{N+1}(t))$ the values of the numerical approximation at the nodes x_k, at time t. Writing,

$$u_h(t) = \sum_{j=1}^{N} u_j(t)\varphi_j,$$

a simple computation shows that $U(t)$ satisfies the system of ordinary differential equations (see [12]):

$$\begin{cases} MU'(t) = -AU(t) + BU^p(t), \\ U(0) = u_0^I, \end{cases}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial data, u_0.

Adding points method.

First, let us describe a method that adds points near the boundary. Let us pay special attention to the numerical solution at the point $x = 1$ (at this point is where the continuous solution develops the singularity). Writing the equation satisfied by u_{N+1} explicitly we obtain the following ODE,

$$u_{N+1}'(t) = \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t),$$

where $h_N = 1 - x_N$. Now, let us describe our adaptive procedure. As we want the numerical blow-up rate to be

$$u_h(1,t) \sim (T_h - t)^{-\frac{1}{2(p-1)}}$$

we impose that the last node $u_{N+1}(t) = u_h(1,t)$ satisfies

$$c_1 u_{N+1}^{2p-1}(t) \le u_{N+1}'(t) \le c_2 u_{N+1}^{2p-1}(t). \tag{2.1}$$

Imposing (2.1) is equivalent to,

$$c_1 u_{N+1}^{2p-1}(t) \le \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t) \le c_2 u_{N+1}^{2p-1}(t). \tag{2.2}$$

As it is proved in [14] that the scheme with a fixed mesh blows up at the last node, we have that

$$R(t; h_N) = \frac{\frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t)}{u_{N+1}^{2p-1}(t)} \to 0 \qquad \text{as } t \text{ increases.}$$

Let t_1 be the first time such that $R(t; h_N) = c_1$, that is the first time where $R(t; h_N)$ meets the tolerance c_1. At that time t_1 we add a point z between x_N and $x_{N+1} = 1$ to the mesh and give the value of $u_h(z, t_1)$ such that the slope of the line between $(z, u_h(z, t_1))$ and $(1, u_h(1, t_1))$ is the same as the slope between $(x_N, u_h(x_N, t_1))$ and $(1, u_h(1, t_1))$, see Figure 1.

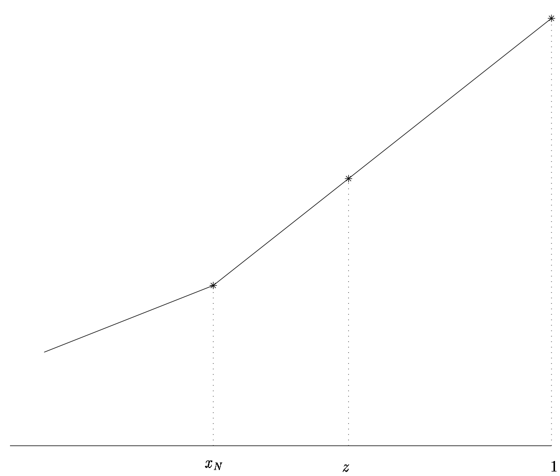

Figure 1.

Hence we have a new value for the length of the last interval, $[z, 1]$,

$$h_{N,1} = 1 - z < h_N = 1 - x_N.$$

We remark that $u_h(z, t_1) = u_z(t_1)$ satisfies,

$$\frac{1}{h_N}(u_N(t_1) - u_{N+1}(t_1)) = \frac{1}{h_{N,1}}(u_z(t_1) - u_{N+1}(t_1)).$$

Let us see what happens with $R(t_1; h_{N,1})$.

$$\begin{aligned} R(t_1; h_{N,1}) &= \frac{\frac{2}{h_{N,1}^2}(u_z(t_1) - u_{N+1}(t_1)) + \frac{2}{h_{N,1}} u_{N+1}^p(t_1)}{u_{N+1}^{2p-1}(t_1)} \\ &= \frac{\frac{2}{h_{N,1}}(u_z(t_1) - u_{N+1}(t_1)) + 2u_{N+1}^p(t_1)}{h_{N,1}\ u_{N+1}^{2p-1}(t_1)} \\ &= \frac{\frac{2}{h_N}(u_N(t_1) - u_{N+1}(t_1)) + 2u_{N+1}^p(t_1)}{h_{N,1}\ u_{N+1}^{2p-1}(t_1)} \\ &= \frac{h_N}{h_{N,1}} R(t_1; h_N) = \frac{h_N}{h_{N,1}}\ c_1 > c_1. \end{aligned}$$

Therefore with this new mesh $x_1, ..., x_N, z, x_{N+1}$ we have that

$$R(t_1; h_{N,1}) > c_1,$$

and we can apply the method with initial data $u_h(x, t_1)$ in the new mesh to continue. This gives a solution u_h that verifies (2.2) in a time interval $[t_1, t_2]$ where at time t_2, the function $R(t_2; h_{N,1})$ reaches the tolerance c_1. At that time t_2 we have to add another point in the last interval. As before this increases $R(t_2; h_{N,2})$ and we can continue with initial data $u_h(x, t_2)$ in a new mesh that is the old one plus the point that we have added near the boundary $x = 1$. This procedure generates an increasing sequence of times t_i and a decreasing sequence $h_i = h_{N,i}$, at which the tolerance $R(t_i; h_i) = c_1$ is reached, a sequence of added points accumulating at $x = 1$ and a numerical solution $u_h(x, t)$.

It remains to be more concrete on the election of the sequence h_i, c_1 and c_2. In fact if we take,

$$c_1 = \Gamma^{-2(p-1)}(2(p-1))^{-1},$$

and h_i and c_2 such that

$$\frac{h_i}{h_{i+1}} \to 1 \qquad \text{and} \qquad c_2 = c_2(t) \sim c_1, \tag{2.3}$$

integrating (2.1) we get that

$$\begin{aligned} \Gamma(T_h - t)^{-\frac{1}{2(p-1)}} &\geq u_{N+1}(t) = \max u_h(\cdot, t) \\ &\geq \left(2(p-1)\int_t^{T_h} c_2(s)\, ds\right)^{-\frac{1}{2(p-1)}}, \end{aligned}$$

Hence, as $t \nearrow T_h$,

$$u_{N+1}(t) = \max u_h(\cdot, t) \sim \Gamma(T_h - t)^{-\frac{1}{2(p-1)}},$$

and we have recovered the continuous asymptotic behaviour (1.2). So we want that h_i and c_2 verify (2.3). On the other hand, as we have mentioned in the introduction, the continuous problem verifies that

$$u(x,t) \sim (T-t)^{-\frac{1}{2(p-1)}} \varphi(\xi)$$

for $x = 1 - \xi(T-t)^{1/2} \sim C(u(1,t))^{-(p-1)}$. In order to recover this self-similar scaling we choose h_i such that

$$h_{i+1}(u_{N+1}(t_i))^{p-1} = \frac{2}{c_1} - \frac{A}{u_{N+1}(t_i)},$$

this choice make

$$c_2 \sim c_1 \left(\frac{1}{1 - \frac{c_1 A}{2u_{N+1}(t_i)}} \right),$$

and hence,

$$1 - x_N = h_{i+1} \sim \frac{2}{c_1}(u_{N+1}(t_i))^{-(p-1)} \sim C(T_h - t)^{1/2}.$$

This procedure gives an approximation of the curve $x = 1-\xi(T-t)^{1/2}$. Notice that if h_{i+1} and c_2 are given as above, relations (2.3) hold. So we recover the self-similar behavior (1.2).

Moving points method.

As we mentioned in the introduction we may use a procedure inspired in the moving mesh method. Let us describe briefly the main ingredients of such type of schemes, referring to [10] for details. Following [10], the numerical solution is defined on a new mesh $x_i(t)$ that is obtained from a differentiable mesh transformation $x = x(\xi, t)$ with $x_i(t) = x(i/N, t)$. For this approach a new partial differential equation for $x(\xi, t)$ is solved numerically simultaneously with the original equation for $u(x, t)$ imposing a self-similar invariance on the resulting problem. In our case we impose that the blowing up end, $x_N = 1$, is fixed and move the rest of the points according to the previously described procedure, obtaining a $2N$ system of ODE to be solved. There are various ways to move the meshes, we refer to [10] for the details.

This type of procedure may loose some accuracy as it concentrates the points near $x = 1$ as the solution becomes bigger leaving "holes" near the other end $x = 0$. Our strategy is to take advantage of the fact that we know a priori that the singular set is $x = 1$ and hence to reproduce the main features of the continuous solution: the blow-up rate, the structure of the solution near the blow-up time (approximate self-similar behavior) and the blow-up set. We only need to move the last K points near $x = 1$. In fact we use the ideas developed in the *adding points method* to move the points not in a continuous way but at certain times where $R(t, h_N) = c_1$.

Next we describe our *moving points method* that, as we mentioned before, instead of adding a point when $R(t, h_N) = c_1$, moves the last K points near the boundary $x = 1$.

Let us begin with a mesh composed by two types of nodes, a uniform mesh of size $h = 1/N$ (the fact that this mesh is uniform is not essential for our analysis) and K nodes placed between $x_N = (N-1)h$ and $x_{N+1} = 1$, that we are going to move when appropriate. The number of moving nodes, K, depends only on p, we choose $K = [1/(p-1)]$. The motivation for this choice of K comes from the desired localization of the blow-up set at $x = 1$, see Section 5. Let us call $0 = x_1 < .... < x_N < x_{N+1} = 1$ the fixed mesh and $x_N < y_1 < .... < y_K < x_{N+1} = 1$ the moving nodes. As before, we use a semidiscretization in space, using the mesh composed by the x_i and the y_i together, and arrive to a system of ODE

$$\begin{cases} MU'(t) = -AU(t) + BU^p(t), \\ U(0) = u_0^I. \end{cases}$$

Again, we pay special attention to the numerical solution at the point $x = 1$. Writing the equation satisfied by u_{N+1} explicitly we obtain the following ODE,

$$u'_{N+1}(t) = \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t),$$

where $h_N = 1 - y_K$. Now we proceed as follows to obtain an adaptive procedure: as before we want the numerical blow-up rate to be

$$u_h(1,t) \sim (T_h - t)^{-\frac{1}{2(p-1)}}.$$

Hence we impose that the last node $u_{N+1}(t) = u_h(1,t)$ satisfies (2.1), this is equivalent to

$$c_1 u_{N+1}^{2p-1}(t) \le \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t) \le c_2 u_{N+1}^{2p-1}(t). \tag{2.4}$$

In this case $h_N = 1 - y_K$. We have that

$$R(t; h_N) = \frac{\frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t)}{u_{N+1}^{2p-1}(t)} \to 0 \qquad \text{as } t \text{ increases.}$$

Let t_1 be the first time such that $R(t; h_N) = c_1$, that is the first time where $R(t; h_N)$ meets the tolerance c_1.

We remark that our criteria to stop and modify the mesh is the same as before.

Assume, to simplify the exposition, that $K = 1$, that is, we only have one moving point, y_K between x_{N-1} and $x_N = 1$. At that time t_1 we move the point y_K between x_N and $x_{N+1} = 1$ to the point z and give the value of $u_h(z, t_1)$ such that the slope of the line between $(z, u_h(z, t_1))$ and $(1, u_h(1, t_1))$ is the same as the slope between $(x_N, u_h(x_N, t_1))$ and $(1, u_h(1, t_1))$, see Figure 2.

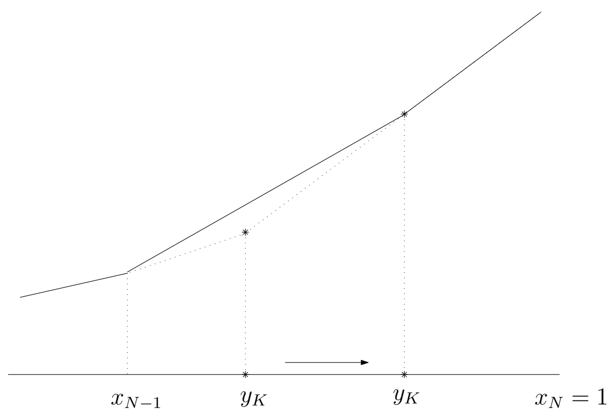

Figure 2.

Hence, exactly as before, we have a new value for the length of the last interval, $[z, 1]$,

$$h_{N,1} = 1 - z < h_N = 1 - y_K.$$

As before, $R(t_1; h_{N,1})$ verifies,

$$R(t_1; h_{N,1}) = \frac{h_N}{h_{N,1}} R(t_1; h_N) = \frac{h_N}{h_{N,1}} \, c_1 > c_1.$$

Therefore with this new mesh $x_1, ..., x_N, z = y_K, x_{N+1}$ we have that

$$R(t_1; h_{N,1}) > c_1,$$

and we can apply the method with initial data $u_h(x, t_1)$ in the new mesh to continue.

We remark that the number of nodes does not increase. We have only moved the node y_K from its previous position to its new position, z. Hence, the size of the ODE system does not increase.

This gives us a solution u_h that verifies (2.2) in a time interval $[t_1, t_2]$ where at time t_2, the function $R(t_2; h_{N,1})$ reaches the tolerance c_1. At that time t_2 we have to move again the moving point, y_K, in the last interval. As before this increases $R(t_2; h_{N,2})$ and we can continue with initial data $u_h(x,t_2)$. This procedure generates an increasing sequence of times t_i and a decreasing sequence $h_i = h_{N,i}$, at which the tolerance $R(t_i;h_i) = c_1$ is reached, a sequence of moving points accumulating at $x = 1$ and a numerical solution $u_h(x,t)$. It remains to be more concrete on the election of the sequence h_i, c_1 and c_2. In fact if we take, exactly as before,

$$c_1 = \Gamma^{-2(p-1)}(2(p-1))^{-1},$$

and h_i such that

$$\frac{h_i}{h_{i+1}} \to 1 \qquad \text{and} \qquad c_2 = c_2(t) \sim c_1,$$

we get that

$$u_{N+1}(t) = \max u_h(\cdot,t) \sim \Gamma(T_h - t)^{-\frac{1}{2(p-1)}},$$

and we have recovered the continuous asymptotic behavior (1.2). On the other hand, to recover the self-similar behavior, we choose h_i as in the adding points method, that is such that

$$h_{i+1}(u_{N+1}(t_i))^{p-1} = \frac{2}{c_1} - \frac{A}{u_{N+1}(t_i)},$$

so

$$c_2 \sim c_1 \left(\frac{1}{1 - \frac{c_1 A}{2u_{N+1}(t_i)}} \right).$$

Hence,

$$1 - y_K = h_{i+1} \sim \frac{2}{c_1}(u_{N+1}(t_i))^{-(p-1)} \sim C(T_h - t)^{1/2}.$$

This procedure gives an approximation of the curve $x = 1 - \xi(T-t)^{1/2}$. Notice that if h_{i+1} and c_2 are given as above, relations (2.3) hold. So we recover the self-similar behavior (1.2).

In case we have $K > 1$ we proceed in the same way, but in this case we have to move K points near $x_{N+1} = 1$. Observe that the scaling factor that we are using places y_K at z, we move the rest of the points lying between x_N and 1 such that the mesh remains uniform in the interval $[y_1, 1]$. This imposes on y_j, $j = 1, \dots, K$ the same behavior described above for y_K.

Let us remark that the criteria that we use to modify the mesh is the same in the *adding points method* and in the *moving points method.* This allows us to make a unified approach in the course of the proofs contained in the following sections.

3. Convergence of the numerical schemes

In this section we prove a uniform convergence result. We use ideas from [14] and we include the arguments here to make the paper self-contained. For any $\tau > 0$ we want that $u_h \to u$ (when $h \to 0$) uniformly in $[0,1] \times [0, T-\tau]$. This is a natural requirement since on such an interval the exact solution is regular.

In particular, uniform convergence can be obtained by using standard inverse inequalities. In the following Theorem we give a proof of the L^2 convergence for a problem like (1.1) considering mass lumping. As a corollary, we will obtain uniform convergence for problem (1.1).

Theorem 3.1. *Let u be the solution of* (1.1) *and let u_h its semidiscrete approximation obtained by any of the adaptive schemes described in Section 2. If $u \in C^{2,1}([0,1] \times [0, T-\tau])$ for some $\tau > 0$ then, there exists a constant C depending on τ such that:*

$$\|u - u_h\|_{L^\infty([0,T-\tau],L^2)} \le Ch^2.$$

Proof. We fix our attention on the convergence of the method in a time interval of the form $[t_i, t_{i+1}]$, that is between two consecutive refinement times. Next we will show that this is enough for our purposes. Let us begin by $t \in (0, t_1)$.

In this proof we use the notation $L^2 = L^2((0,1))$ that refers to the L^2 norm in the x variable for each t (we will use analogous notations for other norms below) and u' for the derivative respect to time, u_t.

As u is a solution of (1.1) it satisfies

$$\int_0^1 u'v + \int_0^1 u_x v_x = u^p(1,t)v(1,t) \qquad \forall v \in H^1.$$

The numerical scheme is equivalent to

$$\int_0^1 ((u_h)'v)^I + \int_0^1 u_x v_x = u_h^p(1,t)v(1,t) \qquad \forall v \in V_h.$$

Hence we have that $e = u - u_h$ satisfies the following error equation,

$$\int_0^1 (e'v)^I \, dx + \int_0^1 e_x v_x \, dx = (u^p - u_h^p)(1,t)v(1,t) + \int_0^1 ((u'v)^I - u'v) \, dx$$

for all $v \in V_h$. Writing

$$e = u - u_h + u^I - u^I = u - u^I + \eta$$

and using known error estimates for Lagrange interpolation it rest to estimate $\eta = u^I - u_h$.

First, it is easy to see that,

$$\int_0^1 (u-u^I)_x v_x = 0 \qquad \forall v \in V_h,$$

and therefore, replacing in the error equation we have an equation for η,

$$\begin{aligned}\int_0^1 (\eta' v)^I + \int_0^1 \eta_x v_x &= (u^p(1,t) - u_h^p(1,t))v(1,t)\\ &+ \int_0^1 ((u'v)^I - u'v) - \int_0^1 ((u'-(u^I)')v)^I \qquad \forall v \in V_h.\end{aligned}$$

In particular if we choose $v = \eta \in V_h$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\int_0^1 (\eta^2)^I) + \int_0^1 (\eta_x)^2 = & \ (u^p - u_h^p)(1,t)\eta(1,t)\\ & + \int_0^1 ((u'\eta)^I - u'\eta) = I + II.\end{aligned}$$

First, let us estimate I.

$$|I| = |(u^p(1,t) - u_h^p(1,t))\eta(1,t)| = \left|((u^I)^p(1,t) - u_h^p(1,t))\eta(1,t)\right|.$$

Hence we get that

$$|I| \le C|\eta(1,t)|^2.$$

Using the well known inequality,

$$|\eta(1,t)|^2 \le (C + \frac{C}{\varepsilon})\|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))},$$

we have that

$$|I| \le C_\varepsilon \|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))}.$$

In order to get a bound for II we decompose it in the following form,

$$II = \int_0^1 ((u'\eta)^I - u'\eta)\,dx = \int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx + \int_0^1 ((u')^I\eta - u'\eta)\,dx.$$

We proceed as before, for each subinterval I_j of the partition we know that,

$$\begin{aligned}\|((u')^I\eta)^I - (u')^I\eta\|_{L^1(I_j)} \le & \ Ch^2\|((u')^I\eta)_{xx}\|_{L^1(I_j)}\\ \le & \ Ch^2\|((u')^I)_x\eta_x\|_{L^1(I_j)}\end{aligned}$$

because $(u')^I$ and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|((u')^I)_x\|_{L^2} \le C\|u'\|_{H^1}$ we obtain,

$$\int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx \le Ch^2 \int_0^1 |((u')^I)_x||\eta_x|\,dx$$

$$\le Ch^2\|((u')^I)_x\|_{L^2((0,1))}\|\eta_x\|_{L^2} \le Ch^4\|u'\|^2_{H^1} + \tfrac{1}{4}\|\eta_x\|^2_{L^2}.$$

It rests to estimate the second summand of II. We have,

$$\int_0^1 ((u')^I - u')\eta\,dx \le \|(u')^I - u'\|_{L^2}\|\eta\|_{L^2}$$

$$\le \|(u')^I - u'\|^2_{L^2} + \|\eta\|^2_{L^2} \le Ch^4\|u'\|^2_{H^2} + \|\eta\|^2_{L^2}.$$

Collecting all the previous bounds we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + \int_0^1 |\eta_x|^2 \le C_\varepsilon\|\eta\|^2_{L^2((0,1))} + C\varepsilon\|\eta_x\|^2_{L^2((0,1))}$$

$$+ \; Ch^4\|u'\|^2_{H^2} + \|\eta\|^2_{L^2} + Ch^4\|u'\|^2_{H^1} + \tfrac{1}{4}\|\eta_x\|^2_{L^2}.$$

We choose ε such that $C\varepsilon = 1/4$ and we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + \frac{1}{2}\int_0^1 |\eta_x|^2 \le C\|\eta\|^2_{L^2((0,1))} + Ch^4\|u'\|^2_{H^2}.$$

Since $\int_0^1 (\eta^2)^I\,dx \sim \|\eta\|^2_{L^2}$ we can apply Gronwall's Lemma to obtain for $t \in [0, T_1]$,

$$\|\eta(t)\|_{L^2} + \left(\int_0^{t_1} \|\eta_x\|^2_{L^2}\,dt\right)^{1/2} \le Ce^{C(t_1)}h^2.$$

In particular,

$$\|\eta\|_{L^2} \le C(u, t_1)h^2.$$

and hence,

$$\|e\|_{L^2} \le \|u - u^I\|_{L^2} + \|\eta\|_{L^2} \le C(u, t_1)h^2.$$

This proves the result in $[0, t_1]$. Assume that the result is true in $[0, t_i]$. The same arguments used before prove that the bound holds up to t_{i+1}. The observation that for h small enough there exists i_0 such that $t_{i_0} > T - \tau$ finishes the proof. □

As a corollary of Theorem 3.1 we can prove the following result that gives uniform, L^∞, convergence in an interval of the form $[0, T - \tau]$.

Corollary 3.1. *Let u be the solution of (1.1) and u_h its numerical approximation. Given $\tau > 0$, there exists a constant C depending on $\|u\|$ in $C^{2,1}([0,1] \times [0, T-\tau])$ such that, for h small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \le Ch^{\frac{3}{2}}.$$

Proof. It is known that before the blow up time u is regular, more precisely, $u \in C^{2,1}([0,1] \times [0, T-\tau])$. Let $g(u)$ be a globally Lipschitz function which agrees with $f(u) = u^p$ for $u \le 2M$ where $M = \|u\|_{L^\infty([0,1]\times[0,T-\tau])}$. Let $\overline{u}$ and $\overline{u}_h$ be the exact and approximate solutions of a problem like (1.1) with $f(u) = u^p$ replaced by $g(u)$. By uniqueness $u = \overline{u}$ in $[0,1] \times [0, T-\tau]$. A bound for $\|\overline{u} - \overline{u}_h\|_{L^\infty}$ can be obtained from Theorem 3.1. Indeed, it is enough to bound $\|\overline{u}^I - \overline{u}_h\|_{L^\infty}$, and using a standard inverse inequality (see [12]) we have,

$$\begin{aligned} \|\overline{u}^I - \overline{u}_h\|_{L^\infty} \le\ & Ch^{-\frac{1}{2}} \|\overline{u}^I - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)} \\ \le\ & Ch^{-\frac{1}{2}} \left\{ \|\overline{u}^I - \overline{u}\|_{L^\infty([0,T-\tau],L^2)} \right. \\ & \left. +C\|\overline{u} - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)} \right\} \le Ch^{\frac{3}{2}} \end{aligned}$$

with C depending on u and the constant in Theorem 3.1 and so on τ.

Consequently, for h small enough $|\overline{u}_h| \le 2M$. Therefore $u_h^p = f(\overline{u}_h) = g(\overline{u}_h)$ and so $\overline{u}_h$ is the finite element approximation of u and, by uniqueness $\overline{u}_h = u_h$ which concludes the proof. □

To finish this Section, we state a Lemma that says that the maximum of $U(t)$ is attained at the last node, x_{N+1}.

Lemma 3.1. *If $U(0)$ is increasing, then the numerical solution $U(t)$ is increasing for every time t. Hence it satisfies*

$$\max u_h(\cdot, t) = u_{N+1}(t),$$

for every t.

Proof. As $U(0)$ is increasing then $u_{i+1}(0) > u_i(0)$, let us see that this holds for every $t > 0$. Assume not, then there exists a first positive time t_0 and an index i such that $u_{i+1}(t_0) = u_i(t_0)$. From the equations satisfied by u_{i+1} and u_i we have that at that time t_0 it holds $u'_{i+1}(t_0) - u'_i(t_0) > 0$ a contradiction that proves the result. □

4. Numerical blow-up.

In this section we prove that solutions of the numerical problems blow up if and only if $p > 1$ and we find the blow-up rate in this case.

Lemma 4.1. *If $p > 1$ then every positive solution of any of the numerical schemes blows up. Moreover, if $U(0)$ is increasing, then*

$$\lim_{t \nearrow T_h} (T_h - t)^{\frac{1}{2(p-1)}} \|U(t)\|_\infty = \Gamma,$$

where T_h is the blow-up time of U.

Proof. If $U(0)$ is increasing, by Lemma 3.1 $U(t)$ has its maximum at $u_{N+1}(t)$. At a modifying time t_i we have that $R(h_i, t_i) = c_1$ and we modify the mesh to get

$$c_1 < R(h_{i+1}, t_i) = \frac{h_i}{h_{i+1}} R(h_i, t_i) = \frac{h_i}{h_{i+1}} c_1$$

We obtain that

$$\frac{u'_{N+1}}{u_{N+1}^{2p-1}}(t) \geq c_1,$$

for every t. By integration we get that, as $p > 1$, u_{N+1} (and hence u_h) cannot be global. Moreover, if we call T_h the blow-up time, using that $h_i/h_{i+1} \to 1$ $(i \to \infty)$, we have

$$\lim_{t \to T_h} \frac{u'_{N+1}}{u_{N+1}^{2p-1}}(t) = c_1. \tag{4.1}$$

Integrating (4.1), we get

$$(c_1 - \varepsilon_2(t))(T_h - t) \leq \int_t^{T_h} \frac{u'_{N+1}}{u_{N+1}^{2p-1}}(s)\, ds \leq (c_1 + \varepsilon_1(t))(T_h - t).$$

where $\varepsilon_i(t) \to 0$ as $t \nearrow T_h$. Changing variables we obtain

$$(c_1 - \varepsilon_2(t))(T_h - t) \leq \int_{u_{N+1}(t)}^{+\infty} \frac{1}{s^{2p-1}}\, ds \leq (c_1 + \varepsilon_1(t))(T_h - t).$$

Hence

$$\lim_{t \to T_h} (T_h - t)^{\frac{1}{2(p-1)}} u_{N+1}(t) = (2(p-1)c_1)^{\frac{-1}{2(p-1)}} = \Gamma,$$

as expected from the description of the method. □

We remark that this rate coincides with the blow-up rate of the continuous problem.

5. Numerical blow-up set.

Now we turn our interest to the blow-up set of the numerical solution. We want to look at the set of points, x, such that $u_h(x,t) \to +\infty$ as $t \nearrow T_h$.

In [3] and [16] it is proved that for a fixed mesh the numerical blow-up set is given by $B(U) = [1 - Lh, 1]$, where L depends only on p. Notice that as L does not depend on h, the blow-up set converges to the continuous blow-up set $B(u) = \{1\}$.

We remark that for the methods introduced in this paper at least K points are collapsed near $x = 1$ when the solution blows up, therefore the parameter h_i goes to zero and formally the blow-up set for the method must be $B(U) = \{1\}$. In order to prove this, we take a point $\bar{x} < 1$ and we claim that the numerical solution is bounded in $[0, \bar{x}]$.

Lemma 5.1. *Let $\bar{x} < 1$ then for all nodes $x_i < \bar{x}$ the numerical solution is bounded, i.e., there exists $C = C(\bar{x}) > 0$ such that*

$$u_h(x,t) \le C,$$

for all $x \in [0, \bar{x}]$, $t \in [0, T_h)$.

Proof. First we focus on the adding points method. The proof for the moving points method is slightly different.

Let L be the first integer such that $L - 1/2(p-1) > -1$, i.e. $L = [1/2(p-1)]$, this implies that the function $g(t) = t^{-1/2(p-1)}$ is L times integrable.

From Lemma 4.1 we have that

$$\max u_h(\cdot, t) = u_{N+1}(t) \sim C(T_h - t)^{-\frac{1}{2(p-1)}}.$$

Since we adapt near $x = 1$ collapsing (at least) L points in $\{1\}$ as the solution gets large, we can consider a node $x_{i_L} > \bar{x}$ such that at time t_{i_L} there exits $L + 1$ nodes between $\bar{x}$ and x_{i_L}.

On the other hand, as we modify the location of the nodes only for points between x_N and $x_{N+1} = 1$, we obtain that for the points x_i, $i = 1, \cdots, N-1$ the mesh is fixed. Therefore from the equation satisfied by these nodes we have that for $t > t_{i_L}$

$$u_i'(t) \le C \max\,(u_{i+1}, u_{i-1}) \le C(T_h - t)^{-\frac{1}{2(p-1)}},$$

and then for $i = 1, \cdots, N-2$ we get

$$u_i(t) \le C(T_h - t)^{-\frac{1}{2(p-1)}+1}.$$

This bound implies that

$$u_i'(t) \le C \max\,(u_{i+1}, u_{i-1}) \le C(T_h - t)^{-\frac{1}{2(p-1)}+1},$$

for $i = 1, \cdots, N-2$. Integrating again we get

$$u_i(t) \le C(T_h - t)^{-\frac{1}{2(p-1)}+2},$$

for every $i = 1, \cdots, N-3$. Repeating the same argument L times we have that $u_i(t)$ is bounded for all $i = 1, \cdots, N-L-1$.

Hence, as there are L nodes between $\bar{x}$ and x_{N+1}, for t close to T_h, we get that for all nodes $x_i < \bar{x}$, $u_i(t)$ is bounded.

In the case of the moving points method the idea of the proof is the same but we deal with the K moving points between x_N and $x_{N+1} = 1$. In fact, arguing as before, for u_K the value of the numerical solution at the moving node y_K, we have

$$u_K'(t) \le C(t)(T_h - t)^{-\frac{1}{2(p-1)}}.$$

The behavior of $C(t)$ depends on our choice of h in the adaptive procedure, hence

$$C(t) \sim (T_h - t)^{-\frac{1}{2}}.$$

Therefore,

$$u_K(t) \le C(T_h - t)^{-\frac{1}{2(p-1)}+\frac{1}{2}}.$$

For u_{K-1} we get

$$u_{K-1}'(t) \le C(t)(T_h - t)^{-\frac{1}{2(p-1)}+\frac{1}{2}} \le C(T_h - t)^{-\frac{1}{2(p-1)}}.$$

Integrating we get

$$u_{K-1}(t) \le C(T_h - t)^{-\frac{1}{2(p-1)}+1}.$$

Iterating this procedure K times we get that the value of the numerical solution at y_1 is bounded. Since y_1 goes to 1 as $t \to T_h$ and the numerical solution is increasing in space we conclude that $u_h(\bar{x}, t)$ is bounded. □

We remark that this result implies that $B(U) = \{1\}$.

6. Convergence of the numerical blow-up times.

In this section we prove that the numerical blow-up times, T_h, converges to the continuous one, T, when h goes to zero.

Lemma 6.1. *The numerical blow-up times converge to the continuous one when $h \to 0$, in fact there exist $\alpha > 0$ and $C > 0$ such that*

$$|T_h - T| \le Ch^\alpha.$$

Proof. The idea of the proof is as follows, first we get bounds for the first time t_0 such that the error verifies

$$\|(u_h - u)(t)\|_{L^\infty([0,1])} \leq 1, \qquad \text{for all } t \in [0, t_0], \tag{6.1}$$

then we use this bound to prove our convergence result.

To get a bound for t_0 let us look at our scheme between two consecutive refinement times, from section 2 we have,

$$MU' = -AU + BU^p. \tag{6.2}$$

If we call $z_i(t) = u(x_i, t)$ we get that $Z = (z_1, \dots, z_{N+1})$ satisfies

$$MZ' \leq -AZ + BZ^p + Ch(T-t)^\theta, \tag{6.3}$$

where θ depends only on p, in fact $\theta = -2 - 1/2(p-1)$. Let us remark that $C(T-t)^\theta$ is a bound for the first four spatial derivatives of $u(x,t)$ a regular solution of (1.1).

Subtracting (6.2) from (6.3) we have, for $E = U - Z$,

$$\begin{aligned} ME' \leq & \ -AE + B(Z^p - U^p) + Ch(T-t)^\theta \\ \leq & \ -AE + Bp\xi^{p-1}E + Ch(T-t)^\theta, \end{aligned} \tag{6.4}$$

where ξ is a point between u_{N+1} and z_{N+1}. Using that $t \in [0, t_0]$, (6.1), and the blow-up rate we get

$$\xi^{p-1} \leq C(T-t)^{-\frac{1}{2}}.$$

Hence (6.4) gives

$$ME' \leq -AE + BC\frac{E}{(T-t)^{\frac{1}{2}}} + Ch(T-t)^\theta, \tag{6.5}$$

This inequality is a discretization of the problem

$$\begin{cases} E_t \leq E_{xx} + C(T-t)^\theta h & (x,t) \in (0,1) \times [0,T), \\ E_x(0,t) \leq 0 & t \in [0,T), \\ E_x(1,t) \leq C(T-t)^{-1/2}E(1,t) & t \in [0,T), \\ E(x,0) \leq Ch & x \in (0,1). \end{cases} \tag{6.6}$$

In the appendix we construct a supersolution for problem (6.6) such that

$$\overline{E}(x,t) \leq Ch(T-t)^{-\gamma} \tag{6.7}$$

and with the first four spatial derivatives positives in $[0,1]$.

Now we take $\overline{e}_i(t) = \overline{E}(x_i, t)$ and we get a supersolution for (6.5). We use a comparison argument, see [14], between two consecutive refinement times to obtain that

$$e_i(t) \leq \overline{e}_i(t).$$

Therefore, using (6.7) we conclude that

$$e_i(t) \le C(T-t)^{-\gamma} h.$$

The same arguments applied to $E = Z - U$ gives the lower bound

$$e_i(t) \ge C(T-t)^{-\gamma} h.$$

Therefore

$$|e_i(t)| \le C(T-t)^{-\gamma} h.$$

We get a bound for t_0 as follows, let t_* be the first time such that $C(T-t_*)^{-\gamma} h = 1$. That is, $T - t_* = Ch^{1/\gamma}$. As $t_0 \ge t_*$ we have

$$T - t_0 \le Ch^{1/\gamma}.$$

This means that the first time where the error is of size one is less than a power of the parameter h. With this bound for $T - t_0$ we can get the desired result as follows

$$|T_h - T| \le |T - t_0| + |T_h - t_0| \le Ch^{1/\gamma} + |T_h - t_0|.$$

Now we observe that the blow-up rate for u_h gives that

$$\begin{aligned} |T_h - t_0| \le & \; C(u_{N+1}(t_0))^{-2(p-1)} \le C(z_{N+1}(t_0)+1)^{-2(p-1)} \\ \le & \; C(T - t_0) \le Ch^{\frac{1}{\gamma}}, \end{aligned}$$

and the result follows. □

Let us observe that we can obtain a convergence result using these ideas. However, the methods used in Section 3 provided a better estimate.

7. Numerical experiments.

In this section we present numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one perform numerical computations. For the numerical experiments we use an adaptive ODE solver to integrate in time. We take as initial datum $u_0(x) = (x^2+1)/2$. In Figures 1, 2 below we compare the adding points scheme with one with a fixed mesh of the same size $h = 1/50$, for $p = 2$ and $p = 1.5$. We present the evolution of $\ln(u_h(1,t))$ as a function of $-\ln(T_h - t)$. The slope of the obtained curves measures the blow-up rate. As expected these slopes for $p = 2$ are $1/2$ with the adding points scheme and 1 with a fixed mesh, while for $p = 1.5$ are 1 and 2 respectively.

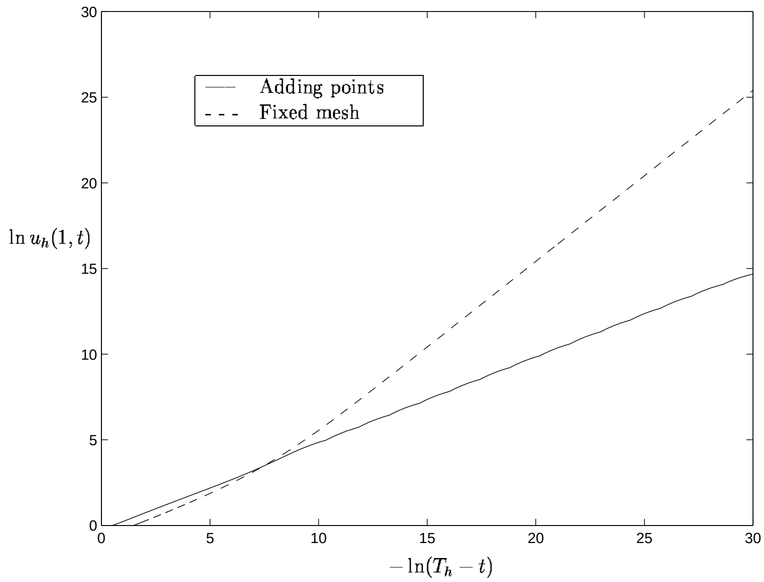

Figure 1. Blow-up rates. $p = 2$.

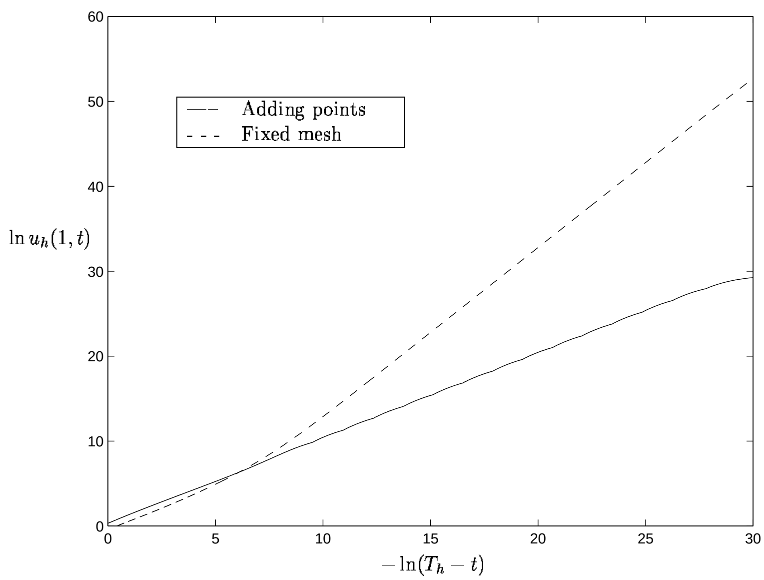

Figure 2. Blow-up rates. $p = 1.5$.

In Figures 3, 4 we show the performance of the moving points method, using the same initial datum and the same powers ($p = 2$ and $p = 1.5$) used above. Again we can observe that the slopes of the resulting lines reproduce the expected blow-up rates.

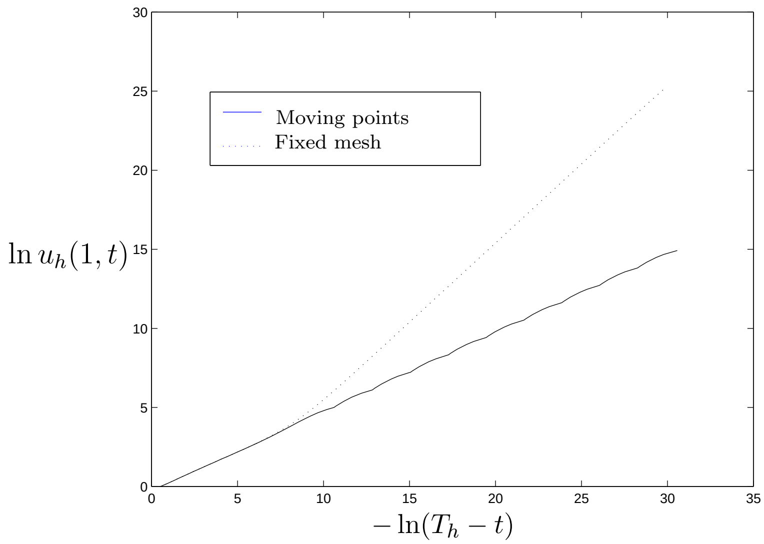

Figure 3. Blow-up rates. $p = 2$.

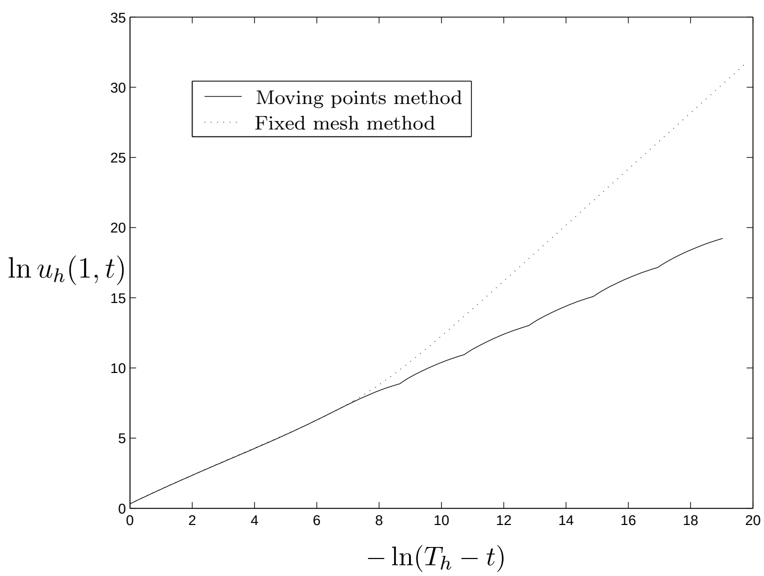

Figure 4. Blow-up rates. $p = 1.5$.

Finally Figures 5, 6 show the evolution of the moving nodes in the case of the moving points method for $p = 2$ (just one node is moved) and $p = 1.5$ (two moving nodes) respectively. In these figures it can be observed that the moving nodes satisfy

$$-\ln(T_h - t) \sim 2(-\ln(1 - y_i)) + C_i$$

which means that they obey the scaling invariance of the problem, that is,

$$\frac{1-y_i}{\sqrt{T_h-t}} \sim \xi_i,$$

and so this method recovers the self-similar behavior of the continuous solution as it was proved.

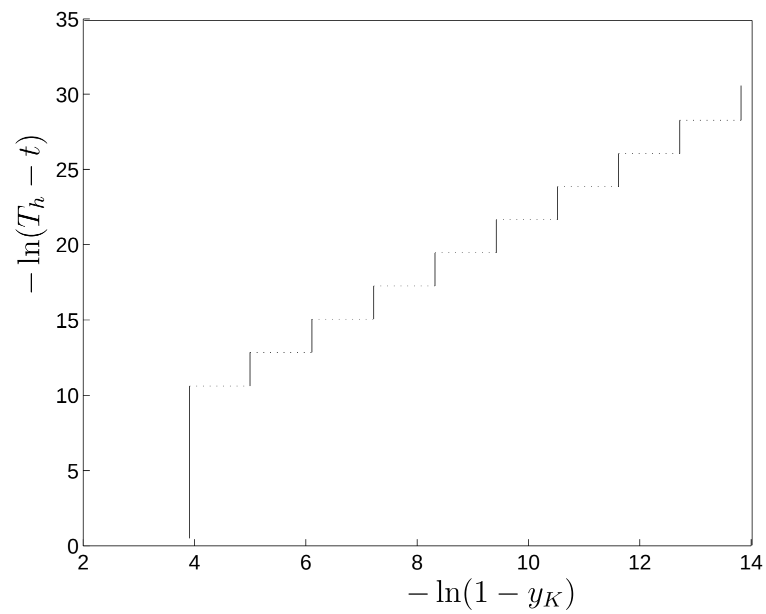

Figure 5. Position of the moving point. $p = 2$.

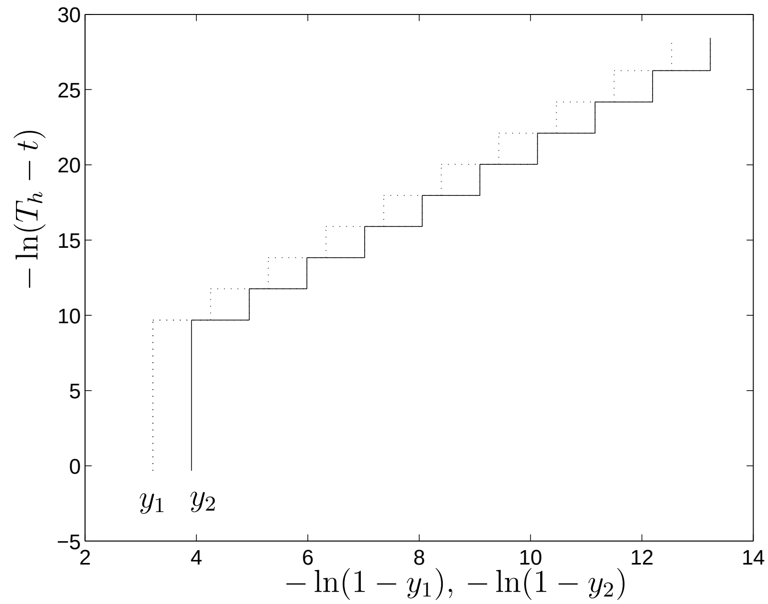

Figure 6. Position of the two moving points. $p = 1.5$.

Comments and Extensions.

Moving mesh methods, based in moving mesh partial differential equations are also expected to reproduce these asymptotic behaviors, in spite of the evidence is just heuristical and experimental. Rigorous proofs are not available. However, the moving mesh algorithms can be applied to a large family of problems. The idea behind these procedures is to take advantage of the self-similar scaling of the problems under consideration. This seems to us to be the more natural way to face blow-up problems. Our results can be viewed as a preliminary step to obtain rigorous proofs for these kind of schemes.

The results contained in the paper are restricted to one space dimension. We may try to extend these ideas to several space dimensions. If Ω is a bounded domain we face the problem $u_t = \Delta u$ in $\Omega \times (0,T)$ with a boundary condition of the form $\frac{\partial u}{\partial \eta} = u^p$ on $\partial\Omega \times (0,T)$ and an initial datum $u_0(x)$. See [13], [17], [21], [28] for references that includes the blow-up rate and set. There is a natural extension of the ideas developed here to deal with the higher dimensional case. However, in order to extend any of the two adaptive procedures described in Section 2 we need to have an a priori knowledge of the spatial location of the blow-up set. This is the case for example if the domain under consideration is a square $\Omega = [0,1] \times [0,1]$ for certain initial conditions that forces the blow-up set to be one of the corners (see [19]). Assume that we know where a single point blow-up set is located and that the maximum of the solution is placed at that point for every time (as in the case of the square described in [19]), then we place a node at that point and look for the asymptotic behavior of that node. It turns out that we get a condition of the form $c_1 \leq R(t, h_{N,1}, h_{N,2}, ..., h_{N,J})$. When $R(t, \cdot) = c_1$ we modify the mesh locally near the blow-up point, adding or moving the adjacent nodes, by performing a contraction by a factor $r < 1$. A simple calculation shows that when such a contraction is performed the new value of $R(t, \cdot)$ increases, allowing us to proceed just as described in Section 2.

However, if the maximum of the solution moves from one node to another then we have to perform adaptive procedures at different points that may compensate as time evolves and hence our proofs are no longer valid.

8. Appendix.

In this appendix we find a supersolution for

$$\begin{cases} E_t = E_{xx} + C(T-t)^{\theta} h & (x,t) \in (0,1) \times [0,T), \\ E_x(0,t) = 0 & t \in [0,T), \\ E_x(1,t) = C(T-t)^{-1/2} E(1,t) & t \in [0,T), \\ E(x,0) = Ch & x \in (0,1), \end{cases}$$

such that

$$\overline{E}(x,t) \leq Ch(T-t)^{-\gamma}$$

and with the first four spatial derivatives positives in $[0,1]$.

Let us look for a supersolution of the form

$$\overline{E}(x,t) = Ch(T-t)^{\theta} a(x,t),$$

with $a(x,t)$ a solution of

$$\begin{cases} a_t = a_{xx} & (x,t) \in (0,1) \times [0,T), \\ a_x(0,t) = 0 & t \in [0,T), \\ a_x(1,t) = C(T-t)^{-1/2} a(1,t) & t \in [0,T), \\ a(x,0) = a_0(x) & x \in (0,1). \end{cases} \tag{8.1}$$

In order to get a supersolution, $\overline{E}$, with the first four spatial derivatives positives, we impose that $a_0(x)$ is a smooth compatible initial datum with the first four spatial derivatives positives. The positivity of the derivatives is preserved for every $t \in [0,T)$.

Now let us see that there exists r such that

$$a(x,t) \leq \frac{C}{(T-t)^r}. \tag{8.2}$$

To this end we want to construct a supersolution to (8.1) such that (8.2) holds. This can be easily done by the following procedure: take $v(x,t)$ a solution of (1.1) with boundary condition given by $v_x(1,t) = v^q(1,t)$, with q small, and initial datum v_0 such that $v(x,t)$ blows up exactly at time T, see [29] for a proof of the fact that for every time T there exist an initial datum such that $v(x,t)$ blows up at time T. The blow-up rate for solutions of (1.1), see [18], [21], gives that $v(x,t)$ verifies

$$v(1,t) \sim \frac{L}{(T-t)^{\frac{1}{2(q-1)}}},$$

with $L^{q-1} \to +\infty$ as $q \searrow 1$, see [18] for an explicit formula for $L(q)$. Let us fix q such that $L^{q-1}(q) > C$. With this choice, v is a supersolution of (8.1). In fact,

$$v_x(1,t) = v^q(1,t) = v^{q-1}v(1,t) \geq \frac{L^{q-1}}{(T-t)^{\frac{1}{2}}} v(1,t),$$

as we wanted to prove.

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