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Adaptive output-feedback control for a class of uncertain stochastic non-linear systems with time delays

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In this paper, we investigate the adaptive output-feedback stabilisation for a class of stochastic non-linear systems with time-varying time delays. First, we give some sufficient conditions to ensure the existence and uniqueness of the solution process for stochastic non-linear systems with time delays, and introduce a new stability notion and the related criterion. Then, for a class of stochastic non-linear systems with time-varying time delays, uncertain parameters in both drift and diffusion terms, and general constant virtual control coefficients, we present a systematic design procedure for a memoryless adaptive output-feedback control law by using the backstepping method. It is shown that under the control law based on a memoryless observer, the closed-loop equilibrium of interest is globally stable in probability, and moreover, the solution process can be regulated to the origin almost surely.

Keywords: stochastic non-linear systems; time-delay systems; output-feedback stabilisation; memoryless; adaptive control; virtual control coefficients

1. Introduction

Time delay phenomena exist in many mechanical, physical, biological, medical, and economical systems (Kolmanovskii and Myshkis 1999). The existence of time delay is often a source of instability and poor performance. Since stochastic modelling has come to play an important role in many branches of science and engineering application, the stability analysis and robust control for time-delay stochastic systems have received much attention (Verriest and Florchinger 1995; Mao et al. 1998; Xie and Xie 2000; Xu and Chen 2002; Xie et al. 2003; Fu et al. 2005; Lu et al. 2005; Shu and Wei 2005; Rodkina and Basin 2006 and the references therein). Most of these existing papers focus on stability analysis or H_∞ analysis (Verriest and Florchinger 1995; Mao et al. 1998; Shu and Wei 2005; Rodkina and Basin 2006), or robust stabilisation of linear stochastic time-delay systems (Xie and Xie 2000; Xu and Chen 2002; Lu et al. 2005), and only a few on the construction of stabilisation controller of non-linear stochastic time-delay systems (Xie et al. 2003; Fu et al. 2005). It is known that for non-linear systems, there is no general controller construction method, and thus, how to design a controller constructively is the key issue. Backstepping method provides an effective and constructive design tool for a class of low-triangle non-linear systems. Based on this method, a decentralised output-feedback stabilisation controller dependent on time delays was designed for a class of large-scale strick-feedback stochastic non-linear systems with time delays, in which the diffusion terms are independent of time delays (Xie et al. 2003). In Fu et al. (2005), the problem of the fourth-moment exponential output-feedback stabilisation was considered for a class of stochastic non-linear systems with constant time delays, in which the diffusion terms were assumed to be independent of time delays, and the drift terms were of globally linear growth.

The goal of this paper is to design an adaptive output-feedback controller constructively for a more general class of stochastic non-linear systems with time delays. First, we give some sufficient conditions to ensure the existence and uniqueness of the solution process for stochastic non-linear systems with time delays. In order to discuss the stability of stochastic non-linear systems with time delays, we introduce a new stability notion and the related criterion. Then, we propose a systematic procedure to design a memoryless adaptive output-feedback control law for a class of stochastic non-linear systems with time-varying time delays, uncertain parameters in

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both drift and diffusion terms, and general constant virtual control coefficients.

The rest of the paper is organised as follows. First in §2, we provide some notations and preliminary results. Then in §3, the problem to be investigated is presented. In §4, we present the design of observer and later in §5, we give the output-feedback control design procedure. To illustrate the effectiveness of our results obtained in previous sections, a numerical example is discussed in §6. In the final section, we give some concluding remarks.

2. Notations and preliminary results

Throughout this paper, the following notations are adopted:

- $\mathbb{R}_+$ denotes the set of all non-negative real numbers; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times r}$ denotes the real $n\times r$ matrix space;
- $\mathrm{Tr}(X)$ denotes the trace for square matrix X; $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimal and maximal eigenvalues of symmetric real matrix X, respectively;
- $|X|$ denotes the Euclidean norm of a vector X and the corresponding induced norm for matrices is denoted by $\|X\|_M$; $\|X\|_F$ denotes the Frobenius norm of X defined by $\|X\|_F=\sqrt{Tr(X^TX)}$;
- $\mathcal{C}([-d,0];\mathbb{R}^n)$ denotes the space of continuous $\mathbb{R}^n$-valued functions on $[-d,0]$ endowed with the norm $\|\cdot\|$ defined by $\|f\|=\sup_{x\in[-d,0]}|f(x)|$ for $f\in\mathcal{C}([-d,0];\mathbb{R}^n)$; $\mathcal{C}^b_{\mathcal{F}_0}([-d,0];\mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable bounded $\mathcal{C}([-d,0];\mathbb{R}^n)$-valued random variables $\xi=\{\xi(\theta): -d\le\theta\le 0\}$;
- $\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathcal{C}^{2,1}((\mathbb{R}^n\times[-d,\infty);\mathbb{R}_+))$ denotes the family of all non-negative functions $V(x,t)$ on $\mathbb{R}^n\times[-d,\infty)$ which are $\mathcal{C}^2$ in x and $\mathcal{C}^1$ in t; $\mathcal{C}^{2,1}$ denotes the family of all functions which are $\mathcal{C}^2$ in the first argument and $\mathcal{C}^1$ in the second argument;
- $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+\to\mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)$: $\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, which are of $\mathcal{K}$ for each fixed t, and decrease to zero as $t\to\infty$ for each fixed s.

Consider an n-dimensional stochastic time-delay system

$$dx(t)=f(x(t),x(t-d(t)),t)dt + g(x(t),x(t-d(t)),t)dw_t,\quad \forall t\ge 0,\tag{1}$$

with initial data $\{x(\theta): -d\le\theta\le 0\}=\xi\in\mathcal{C}^b_{\mathcal{F}_0}\times([-d,0];\mathbb{R}^n)$, where $d(t)$: $\mathbb{R}_+\to[0,d]$ is a Borel measurable function; f: $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^n$ and g: $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^{n\times r}$ are locally Lipschitz; w_t is an r-dimensional standard Brownian motion defined on the complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},P)$, with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\ge0}$ being a filtration, and P being a probability measure.

Define a differential operator $\mathcal{L}$ as follows:

$$\mathcal{L}V=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(x(t),x(t-d(t)),t)+\frac{1}{2}Tr\left\{g^T\frac{\partial^2V}{\partial x^2}g\right\},$$

where $V(x,t)\in\mathcal{C}^{2,1}$.

The following theorem provides a sufficient condition to ensure the existence and uniqueness of global solution for the system (1), which is an extension of Has'minskii (1980, Theorem 4.1 of Chapter 3).

Theorem 1: *For system* (1), *assume that both terms $f(x,y,t)$ and $g(x,y,t)$ are locally Lipschitz in (x,y), and $f(0,0,t),g(0,0,t)$ are bounded uniformly in t. If there exists a function $V(x,t)\in\mathcal{C}^{2,1}(\mathbb{R}^n\times[-d,\infty);\mathbb{R}_+)$ such that for some constant $K>0$ and any $t\ge0$,*

$$\mathcal{L}V\le K(1+V(x(t),t)+V(x(t-d(t)),t-d(t))),\tag{2}$$

$$\lim_{|x|\to\infty}\inf_{t\ge0}V(x,t)=\infty,\tag{3}$$

then, there exists a unique solution on $[-d,\infty)$ for any initial data $\{x(\theta): -d\le\theta\le0\}=\xi\in\mathcal{C}^b_{\mathcal{F}_0}([-d,0];\mathbb{R}^n)$.

Proof: It can be proved by a method similar to the time-invariant delay case in Mao (2002), and is thus omitted. □

In order to discuss the stability of stochastic non-linear systems with time delays, we introduce the following stability notion.

Definition 1: The equilibrium $x=0$ of system (1) with $f(0,0,t)\equiv0$, $g(0,0,t)\equiv0$ is said to be globally stable in probability if for any $\epsilon>0$, there exists a function $\gamma(\cdot)\in\mathcal{K}$ such that

$$P\{|x(t)|\le\gamma(\|\xi\|)\}\ge1-\epsilon,\qquad\forall t\ge0,\quad \forall\xi\in\mathcal{C}^b_{\mathcal{F}_0}([-d,0];\mathbb{R}^n)\backslash\{0\},\tag{4}$$

where $\|\xi\|=\sup_{\theta\in[-d,0]}|x(\theta)|$.

Remark 1: Definition 1 can be regarded as an extension of the stability notions without time delays in Krstić and Deng (1998) and Deng et al. (2001). Compared with the stability notions in Kolmanovskii and Nosov (1986) and Kolmanovskii and Myshkis (1999), it has the following advantages: (i) it focuses on the global case, which is essential in the stabilisation of stochastic non-linear systems (Pan 2002); (ii) the presentation is based on class $\mathcal{K}$ functions rather than the ε-δ format, which shows a clearer connection between conventional deterministic stability results (in the style of Khalil 2002) and stochastic stability ones; and (iii) it makes the role of the initial condition explicit instead of a kind of qualitative description in Kolmanovskii and Nosov (1986) and Kolmanovskii and Myshkis (1999).

The following theorem gives some sufficient conditions ensuring global stability in probability.

Theorem 2: *For system* (1), *assume that both terms* $f(x, y, t)$ *and* $g(x, y, t)$ *are locally Lipschitz in* (x, y) *and* $f(0, 0, t) \equiv 0$, $g(0, 0, t) \equiv 0$. *If there exists a function* $V(x, t) \in C^{2,1}(\mathbb{R}^n \times [-d, \infty); \mathbb{R}_+)$ *and two* $\mathcal{K}_\infty$ *functions* α_1 *and* α_2 *such that*

$$\alpha_1(|x(t)|) \le V(x(t), t) \le \alpha_2\left(\sup_{-d \le s \le 0} |x(t+s)|\right), \tag{5}$$

$$\mathcal{L}V \le -W(x(t)), \tag{6}$$

where $W(x)$ *is continuous and non-negative, then,* (i) *there exists a unique solution on* $[-d, \infty)$; *and* (ii) *the solution* $x = 0$ *of the system* (1) *is globally stable in probability*, *and moreover*,

$$P\left\{\lim_{t\to\infty} W(x(t)) = 0\right\} = 1.$$

Proof: Conclusion (i) can be proved directly by Theorem 1, and conclusion (ii) can be proved in a way similar to the proofs of Deng et al. (2001, Theorem 2.1) and Mao (2002, Theorem 2.1). Hence the details are omitted here. □

Remark 2: For any $t \ge 0$, let $\phi_t(\theta) = x(t+\theta)$, $\theta \in [-d, 0]$. Then, $\phi_t \in \mathcal{C}([-d, 0]; \mathbb{R}^n)$ and conditions (5) and (6) are equivalent to

$$\alpha_1(|\phi_t(0)|) \le \bar{V}(\phi_t, t) \le \alpha_2(\|\phi_t\|),$$

$$\mathcal{L}\bar{V} \le -W(\phi_t(0)),$$

for a continuous functional $\bar{V}(\psi, t) \in \mathcal{C}([-d, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}_+$, which is called Lyapunov–Krasovskii functional (Kolmanovskii and Nosov 1986). In fact, for any $\psi \in \mathcal{C}([-d, 0]; \mathbb{R}^n)$, we can define $\bar{V}(\psi, t) := V(\psi(0), t)$, where V satisfies (5) and (6). For simplicity, in the rest of this paper, the notation ϕ_t and $\bar{V}$ are not introduced and the function $V(x,t)$ can be considered as a Lyapunov–Krasovskii functional.

Remark 3: In Mao (1999, 2002), a Lyapunov–Razumikhin function is used to analyse the stability of stochastic systems with time delays. A Lyapunov–Razumikhin function is a common Lyapunov function (positive definite and radially unbounded) and its derivative along the solution trajectory is required to be negative (definite) when some Razumikhin condition holds, while a Lyapunov–Krasovskii functional has a more relaxed upper bound, and its derivative along the solution trajectory needs to be negative (definite) in all directions. It is well known, as far as the stability of time-delay systems is concerned, that the Razumikhin method can be regarded as a special case of the method of Lyapunov–Krasovskii functionals (Kolomanovskii and Myshkis 1999, §4.8, p. 254 and Pepe and Jiang 2006). In this paper, the stability analysis is based on the Lyapunov–Krasovskii functionals, and moreover, by the backstepping method, a Lyapunov–Krasovskii functional and an adaptive control law are constructed simultaneously to achieve our stabilisation result.

3. Problem formulation

Consider the following stochastic non-linear system with time delays:

$$\left.\begin{aligned}
dx_1(t) &= (m_1 x_2(t) + f_1(y(t), y(t-d(t)), t))\,dt \\
&\quad + g_1(y(t), y(t-d(t)), t)\,dw_t, \\
&\ \ \vdots \\
dx_{n-1}(t) &= (m_{n-1} x_n(t) + f_{n-1}(y(t), \\
&\quad y(t-d(t)), t))\,dt + g_{n-1}(y(t), \\
&\quad y(t-d(t)), t)\,dw_t, \\
dx_n(t) &= (m_n u(t) + f_n(y(t), y(t-d(t)), t))\,dt \\
&\quad + g_n(y(t), y(t-d(t)), t)\,dw_t, \\
y(t) &= x_1(t),
\end{aligned}\right\} \tag{7}$$

where $x = [x_1, \ldots, x_n]^T$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ represent the state vector, the control input, the measurement output, respectively; $d(t)$: $\mathbb{R}_+ \to [0, d]$ is the time-varying time delay satisfying $\dot{d}(t) \le \gamma < 1$ for a known constant γ, and the initial condition $\{x(\theta): -d \le \theta \le 0\} = \xi \in \mathcal{C}^b_{\mathcal{F}_0} \times ([-d, 0]; \mathbb{R}^n)$ is unknown; the virtual control coefficients (Krstić et al. 1995) $m_i \ne 0$, $i = 1, \ldots, n$, are known constants; $f_i \in \mathbb{R}$, $g_i^T \in \mathbb{R}^r$, $i = 1, \ldots, n$, are uncertain locally Lipschitz continuous functions; $w_t \in \mathbb{R}^r$ is an r-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$, with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\ge 0}$ being a filtration and P being a probability measure.

To facilitate control system design, the following assumption is made:

A1: There are unknown constants $l_i^* > 0$, $h_i^* > 0$, and known smooth functions $\varphi_{id} \geq 0$, $\varphi_i \geq 0$, $\psi_{id} \geq 0$, $\psi_i \geq 0$, $i = 1, \ldots, n$, such that

$$|f_i(y(t), y(t-d(t)), t)| \leq l_i^* \varphi_{id}(|y(t-d(t))|) + l_i^* \varphi_i(|y(t)|),$$

$$|g_i(y(t), y(t-d(t)), t)| \leq h_i^* \psi_{id}(|y(t-d(t))|) + h_i^* \psi_i(|y(t)|).$$

Without loss of generality, we assume that $\varphi_i(0) = \varphi_{id}(0) = \psi_i(0) = \psi_{id}(0) = 0$ for any $i = 1, \ldots, n$.

Remark 4: In Fu et al. (2005), the drift terms $f_i(y(t), y(t-d(t)), t) = f_i(y(t)) + h_i(y(t-d))$ are known and depend on constant time delays, where $h_i(\cdot)$ is of globally linear growth; the diffusion terms $g_i(y(t), y(t-d(t)), t) = g_i(y(t))$ are known and are independent of time delays. In Xie et al. (2003), the diffusion terms are also known and independent of time delays. In this paper, the drift and diffusion terms are all dependent on time-varying time delays and uncertain parameters.

Remark 5: For the unknown covariance case, one can formulate the noise term as $\Sigma(t)dw_t$ (Deng et al. 2001), where w_t is a standard Brownian motion and $\Sigma(t)$ is an unknown bounded deterministic function. In this case, by modifying Assumption A1 as follows:

$$|g_i(y(t), y(t-d(t)), t)\Sigma(t)| \leq h_i^* \psi_{id}(|y(t-d(t))|) + h_i^* \psi_i(|y(t)|),$$

one can easily generalise our results to the unknown covariance case.

The control objective of this paper is to constructively design an adaptive output-feedback controller:

$$\dot{\chi}(t) = \varpi(\chi(t), y(t)),$$
$$u(t) = \mu(\chi(t), y(t)),$$

such that the closed-loop equilibrium of interest is globally stable in probability, and moreover, the solution process can be regulated to the origin almost surely, i.e. $P\{\lim_{t\to\infty} |x(t)| = 0\} = 1$.

4. Observer design

Since x_1 is measurable, it needs only to estimate the states $x_2, x_3, \ldots, x_n$. Thus, without adopting a full-order observer as in Xie et al. (2003), Fu et al. (2005), and Hua et al. (2005), here we use a reduced-order one:

$$\Sigma_0 : \begin{cases} \dot{\hat{x}}_i(t) = m_{i+1}(\hat{x}_{i+1}(t) + a_{i+1}y(t)) \\ \qquad\quad -a_i m_1(\hat{x}_1(t) + a_1 y(t)), \\ \qquad\quad i = 1, \ldots, n-2, \\ \dot{\hat{x}}_{n-1}(t) = m_n u(t) - a_{n-1}m_1(\hat{x}_1(t) + a_1 y(t)), \end{cases}$$

where $a_1, \ldots, a_{n-1}$ are constants such that the following matrix:

$$A = \begin{bmatrix} -m_1 a_1 & m_2 & 0 & 0 & 0 \\ -m_1 a_2 & 0 & m_3 & 0 & 0 \\ \vdots & & & \ddots & \\ -m_1 a_{n-2} & 0 & 0 & 0 & m_{n-1} \\ -m_1 a_{n-1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

is stable, i.e. the polynomial $\lambda^{n-1} + a_1 m_1 \lambda^{n-2} + a_2 m_1 m_2 \lambda^{n-1} + \cdots + a_{n-1} m_1 m_2 \cdots m_{n-1}$ is Hurwitz.[1] Let $l^* = \max\{1, l_i^*, h_i^*, 1 \leq i \leq n\}$,

$$F(y(t), y(t-d(t)), t) = [f_2 - a_1 f_1, \ldots, f_n - a_{n-1} f_1]^T,$$

$$G(y(t), y(t-d(t)), t) = \left[g_2^T - a_1 g_1^T, \ldots, g_n^T - a_{n-1} g_1^T\right]^T,$$

and $\tilde{x} = [\tilde{x}_2, \ldots, \tilde{x}_n]^T$ with the components $\tilde{x}_i$ $(i = 2, \ldots, n)$ given as follows:

$$\begin{cases} \tilde{x}_2(t) = \dfrac{1}{l^*}(x_2(t) - \hat{x}_1(t) - a_1 y(t)), \\ \quad\vdots \\ \tilde{x}_n(t) = \dfrac{1}{l^*}(x_n(t) - \hat{x}_{n-1}(t) - a_{n-1} y(t)). \end{cases}$$

Then, the evolution behaviour of the state estimation error $\tilde{x}$ can be described by

$$d\tilde{x}(t) = \left[A\tilde{x}(t) + \frac{1}{l^*}F(y(t), y(t-d(t)), t)\right]dt + \frac{1}{l^*}G(y(t), y(t-d(t)), t)dw_t, \tag{8}$$

and the complete system can be expressed as

$$\left.\begin{aligned} d\tilde{x}(t) &= \left[A\tilde{x}(t) + \frac{1}{l^*}F(y(t), y(t-d(t)), t)\right]dt \\ &\quad + \frac{1}{l^*}G(y(t), y(t-d(t)), t)dw_t, \\ dy(t) &= [m_1\hat{x}_1(t) + a_1 m_1 y(t) + l^* m_1 \tilde{x}_2(t) \\ &\quad + f_1(y(t), y(t-d(t)), t)]dt \\ &\quad + g_1(y(t), y(t-d(t)), t)dw_t, \\ d\hat{x}_1(t) &= [m_2\hat{x}_2(t) + a_2 m_2 y(t) - a_1 m_1(\hat{x}_1(t) \\ &\quad + a_1 y(t))]dt, \\ &\ \ \vdots \\ d\hat{x}_{n-2}(t) &= [m_{n-1}\hat{x}_{n-1}(t) + a_{n-1} m_{n-1} y(t) \\ &\quad - a_{n-2} m_1(\hat{x}_1(t) + a_1 y(t))]dt, \\ d\hat{x}_{n-1}(t) &= [m_n u(t) - a_{n-1} m_1(\hat{x}_1(t) + a_1 y(t))]dt. \end{aligned}\right\} \tag{9}$$

Remark 6: The observer Σ_0 is independent of time delays, i.e. memoryless, which results in more complex error dynamics. As we all know, a memoryless control law is more desired in practical engineering for its lower storage demand and higher reliability.

Remark 7: Since the virtual control coefficients are constants and may be different, we design a new observer which depends on the virtual control coefficients, and is different from the observer designed for each subsystem in Liu et al. (2007), but has the similar error dynamics except for the different matrix A. In the following, we can see clearly the effects of these coefficients on the controller design.

5. Adaptive control

In this section, we give the adaptive control design for the system (7) by the backstepping method. At first, we introduce a new state transformation

$$z_1 = y, \quad z_{i+1} = \hat{x}_i - \phi_i\left(\bar{\hat{x}}_i, \hat{l}\right), \quad i = 1, \ldots, n,$$

where $\bar{\hat{x}}_i = [y, \hat{x}_1, \ldots, \hat{x}_{i-1}]^T$ and $\phi_i(\bar{\hat{x}}_i, \hat{l})$, $i = 1, \ldots, n$, are smooth virtual controls to be designed, $\hat{l}(t)$ is a parameter to be designed. Then, by Itô formula, we have

$$\begin{aligned} dz_1(t) = {} & [m_1 z_2(t) + m_1\phi_1(t) + a_1 m_1 y + l^* m_1 \tilde{x}_2(t) \\ & + f_1(y(t), y(t-d(t)), t)]dt \\ & + g_1(y(t), y(t-d(t)), t)dw_t, \end{aligned} \tag{10}$$

$$\begin{aligned} dz_{i+1}(t) = {} & d\hat{x}_i(t) - \frac{\partial \phi_i}{\partial y} dy(t) - \sum_{k=1}^{i-1} \frac{\partial \phi_i}{\partial \hat{x}_k} d\hat{x}_k(t) - \frac{\partial \phi_i}{\partial \hat{l}} \dot{\hat{l}} dt \\ & - \frac{1}{2}\frac{\partial^2 \phi_i}{\partial y^2} g_1 g_1^T dt \\ = {} & \Bigg(m_{i+1} z_{i+2}(t) + m_{i+1}\phi_{i+1}(t) + a_{i+1} m_{i+1} y(t) \\ & + \sum_{j=1}^{4} \Omega_{i+1,j}(t) \Bigg) dt + \Phi_{i+1}(t) dw_t, \\ & i = 1, \ldots, n-1, \end{aligned} \tag{11}$$

where

$$\begin{aligned} & z_{n+1} = 0, \quad \phi_n = u, \quad a_n = 0, \\ & \Omega_{i+1,1} = -a_i m_1(\hat{x}_1 + a_1 y) - \frac{\partial \phi_i}{\partial y} m_1(\hat{x}_1 + a_1 y) \\ & \qquad - \sum_{k=1}^{i-1} \frac{\partial \phi_i}{\partial \hat{x}_k}(m_{k+1}\hat{x}_{k+1} + a_{k+1} m_{k+1} y \\ & \qquad - a_k m_1(\hat{x}_1 + a_1 y)), \\ & \Omega_{i+1,2} = -\frac{\partial \phi_i}{\partial y} l^* m_1 \tilde{x}_2, \\ & \Omega_{i+1,3} = -\frac{\partial \phi_i}{\partial y} f_1 - \frac{1}{2}\frac{\partial^2 \phi_i}{\partial y^2} g_1 g_1^T, \\ & \Omega_{i+1,4} = -\frac{\partial \phi_i}{\partial \hat{l}} \dot{\hat{l}}, \\ & \Phi_{i+1} = -\frac{\partial \phi_i}{\partial y} g_1, \qquad 1 \le i \le n-1. \end{aligned}$$

Now, we start the backstepping design procedure.

Step 1: Recall that in §4, the parameters a_i are designed such that A is stable. We know that there exists a positive definite matrix P such that

$$A^T P + PA = -I.$$

Let

$$\begin{aligned} V_1 = {} & \frac{\delta_1}{2}\left(\tilde{x}(t)^T P \tilde{x}(t)\right)^2 + \frac{1}{4} y^4(t) + \frac{1}{2\lambda_0}\left(\hat{l}(t) - l\right)^2 \\ & + \frac{1}{1-\gamma}\int_{t-d(t)}^{t} S(y(s))ds, \end{aligned}$$

where $\delta_1 > 0$, $\lambda_0 > 0$ are design parameters; l is an unknown constant such that $l \ge \max\{l^{*4/3}, h_1^{*4}\}$, $\hat{l} = \hat{l}(t)$ is governed by the update law $\dot{\hat{l}} = \varpi_n(\bar{\hat{x}}_n, \hat{l})$ and to be designed to counteract the parameter uncertainties; and $S(\cdot)$ is a positive continuous function to be determined.

Notice that $\dot{d}(t) \le \gamma < 1$, it follows from (8), (10) and Itô formula that

$$\begin{aligned} \mathcal{L}V_1 = {} & -\delta_1 \tilde{x}^T P \tilde{x} |\tilde{x}|^2 + \frac{2\delta_1}{l^*} \tilde{x}^T P \tilde{x}\left(F^T P \tilde{x}\right) \\ & + \frac{\delta_1}{l^{*2}} Tr\left\{(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P \tilde{x} P) G G^T\right\} \\ & + y^3(m_1 z_2 + m_1 \phi_1 + a_1 m_1 y + l^* m_1 \tilde{x}_2 + f_1) \\ & + \frac{3}{2} y^2 g_1 g_1^T + \frac{1}{\lambda_0}\left(\hat{l} - l\right)\dot{\hat{l}} \\ & + \frac{1}{1-\gamma} S(y(t)) - \frac{1 - \dot{d}(t)}{1-\gamma} S(y(t-d(t))) \\ \le {} & -\delta_1 \tilde{x}^T P \tilde{x} |\tilde{x}|^2 + \frac{2\delta_1}{l^*} \tilde{x}^T P \tilde{x}\left(F^T P \tilde{x}\right) \\ & + \frac{\delta_1}{l^{*2}} Tr\left\{(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P \tilde{x} P) G G^T\right\} \\ & + y^3(m_1 z_2 + m_1 \phi_1 + a_1 m_1 y + l^* m_1 \tilde{x}_2 + f_1) \\ & + \frac{3}{2} y^2 g_1 g_1^T + \frac{1}{\lambda_0}\left(\hat{l} - l\right)\dot{\hat{l}} \\ & + \frac{1}{1-\gamma} S(y(t)) - S(y(t-d(t))). \end{aligned} \tag{12}$$

For simplicity, here and hereafter, the argument t of all states, such as $\tilde{x}(t)$, $y(t)$ and $z_2(t)$, is omitted except for the case in $S(\cdot)$ or specialisation.

Since φ_i, φ_{id}, ψ_i, ψ_{id}, $i = 1, \ldots, n$, are smooth and vanish at zero, there exist smooth non-negative functions $\bar{\varphi}_i$, $\bar{\varphi}_{id}$, $\bar{\psi}_i$ and $\bar{\psi}_{id}$ such that

$$\begin{cases} \varphi_i^4(|y(t)|) \le \bar{\varphi}_i(y(t))\, y^4(t), \\ \psi_i^4(|y(t)|) \le \bar{\psi}_i(y(t))\, y^4(t), \end{cases} \tag{13}$$

$$\begin{cases} \varphi_{id}^4(|y(t-d(t))|) \le \bar{\varphi}_{id}(y(t-d(t)))\, y^4(t-d(t)), \\ \psi_{id}^4(|y(t-d(t))|) \le \bar{\psi}_{id}(y(t-d(t)))\, y^4(t-d(t)). \end{cases} \tag{14}$$

Thus, by Assumption A1 and Young inequality,[2] we obtain

$$y^3 m_1 z_2 \le \frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y^4 + \frac{1}{4\varepsilon_1^4} z_2^4,$$

$$\begin{aligned} y^3 l^* m_1 \tilde{x}_2 &\le \frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y^4 l^{*4/3} + \frac{1}{4\varepsilon_1^4}\tilde{x}_2^4 \\ &\le \frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y^4 l + \frac{1}{4\varepsilon_1^4}\tilde{x}_2^4, \end{aligned}$$

$$\begin{aligned} y^3 f_1 &\le |y^3| l_1^* (\varphi_{1d} + \varphi_1) \\ &\le \frac{3}{2}\eta_0^{4/3} l y^4 + \frac{1}{4\eta_0^4}\varphi_{1d}^4 + \frac{1}{4\eta_0^4}\overline{\varphi}_1 y^4, \end{aligned}$$

$$\begin{aligned} \frac{3}{2} y^2 g_1 g_1^T &\le 3|y|^2 h_1^{*^2}(\psi_{1d}^2 + \psi_1^2) \\ &\le \frac{3}{\eta_1} y^4 l + \frac{3}{2}\eta_1 \psi_{1d}^4 + \frac{3\eta_1}{2}\overline{\psi}_1 y^4, \end{aligned}$$

$$\begin{aligned} \frac{2\delta_1}{l^*}\tilde{x}^T P\tilde{x}(F^T P\tilde{x}) &\le 2\delta_1 \|P\|_M^2 |\tilde{x}|^3 \frac{1}{l^*}|F| \\ &\le 2\delta_1 \|P\|_M^2 \left[\frac{3}{4}\varepsilon^{4/3}|\tilde{x}|^4 + \frac{1}{4\varepsilon^4}\left|\frac{1}{l^*}F\right|^4\right] \\ &\le \frac{3\delta_1}{2}\|P\|_M^2 \varepsilon^{4/3}|\tilde{x}|^4 \\ &\quad + C_1 \sum_{i=2}^n \left(\overline{\varphi}_i y^4 + a_{i-1}^4 \overline{\varphi}_1 y^4\right) \\ &\quad + C_1 \sum_{i=2}^n \left(\varphi_{id}^4 + a_{i-1}^4 \varphi_{1d}^4\right), \end{aligned}$$

$$\begin{aligned} &\frac{\delta_1}{l^{*^2}} Tr\{(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P)GG^T\} \\ &= \frac{2\delta_1}{l^{*^2}} Tr\left[G^T P\tilde{x}\left(G^T P\tilde{x}\right)^T\right] \\ &\quad + \frac{\delta_1}{l^{*^2}}\left(\tilde{x}^T P\tilde{x}\right) Tr\left(G^T P G\right) \\ &\le \frac{2\delta_1}{l^{*^2}}\left\|G^T P\tilde{x}\right\|_F^2 + \frac{\delta_1}{l^{*^2}}\left|\tilde{x}^T P\tilde{x}\right| \\ &\quad \times \lambda_{\max}(P)\|G\|_F^2 \\ &\le \frac{2\left[2\delta_1\|P\|_M^2 + \delta_1\lambda_{\max}^2(P)\right]}{l^{*^2}} \\ &\quad \times \sum_{i=2}^n \left(|g_i|^2 + a_{i-1}^2 |g_1|^2\right)|\tilde{x}|^2 \\ &\le C_2 \sum_{i=2}^n \left(\psi_{id}^4 + a_{i-1}^4 \psi_{1d}^4\right) \\ &\quad + C_2 \sum_{i=2}^n \left(\overline{\psi}_i y^4 + a_{i-1}^4 \overline{\psi}_1 y^4\right) + C_3 |\tilde{x}|^4, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{16\delta_1\|P\|_M^2 (n-1)}{\varepsilon^4}, \\ C_2 &= 16\left(2\delta_1\|P\|_M^2 + \delta_1\lambda_{\max}^2(P)\right)(n-1)\epsilon, \\ C_3 &= \frac{2\delta_1\|P\|_M^2 + \delta_1\lambda_{\max}^2(P)}{2\epsilon}, \end{aligned}$$

φ_{id} and ψ_{id} are shortened for $\varphi_{id}(|y(t-d(t))|)$, $\psi_{id}(|y(t-d(t))|)$, respectively, and ε_1, η_0, η_1, ε, ϵ are positive design constants to be specified. These, together with (12), give

$$\begin{aligned} \mathcal{L}V_1 \le\ & y^3\Bigg[m_1\phi_1 + a_1 m_1 y + \frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y + \frac{1}{4\eta_0^4}\overline{\varphi}_1 y \\ &+ \frac{3\eta_1}{2}\overline{\psi}_1 y + C_1 \sum_{i=2}^n \left(\overline{\varphi}_i + a_{i-1}^4 \overline{\varphi}_1\right) y \\ &+ C_2 \sum_{i=2}^n \left(\overline{\psi}_i + a_{i-1}^4 \overline{\psi}_1\right) y\Bigg] + \frac{1}{\lambda_0}\left(\hat{l} - l\right)\dot{\hat{l}} \\ &+ l\left(\frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y^4 + \frac{3}{2}\eta_0^{4/3} y^4 + \frac{3}{\eta_1} y^4\right) + \frac{1}{4\varepsilon_1^4} z_2^4 \\ &- \delta_1 \tilde{x}^T P\tilde{x}|\tilde{x}|^2 + \tilde{c}|\tilde{x}|^4 + \frac{1}{4\varepsilon_1^4}\tilde{x}_2^4 \\ &+ \Delta_d(|y(t-d(t))|) + \Psi_{1d}(|y(t-d(t))|) \\ &+ \frac{1}{1-\gamma} S(y(t)) - S(y(t-d(t))), \end{aligned} \tag{15}$$

where

$$\tilde{c} = \frac{3\delta_1}{2}\|P\|_M^2 \varepsilon^{3/4} + C_3,$$

$$\begin{aligned} \Delta_d(|y(t-d(t))|) = C_1 \sum_{i=2}^n &\left(\varphi_{id}^4(|y(t-d(t))|)\right. \\ &\left.+ a_{i-1}^4 \varphi_{1d}^4(|y(t-d(t))|)\right) \\ + C_2 \sum_{i=2}^n &\left(\psi_{id}^4(|y(t-d(t))|)\right. \\ &\left.+ a_{i-1}^4 \psi_{1d}^4(|y(t-d(t))|)\right), \end{aligned}$$

$$\begin{aligned} \Psi_{1d}(|y(t-d(t))|) &= \frac{1}{4\eta_0^4}\varphi_{1d}^4(|y(t-d(t))|) \\ &\quad + \frac{3}{2}\eta_1 \psi_{1d}^4(|y(t-d(t))|). \end{aligned}$$

Define the virtual parameter update law and the virtual control as follows:

$$\varpi_1 = \lambda_0\left(\frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y^4 + \frac{3}{2}\eta_0^{4/3} y^4 + \frac{3}{\eta_1} y^4\right), \tag{16}$$

$$\begin{aligned} \phi_1(y,\hat{l}) = -\frac{1}{m_1}\Bigg[&\beta_1 y + v(y) y + a_1 m_1 y + \frac{3}{4}\varepsilon_1^{4/3} m_1^{4/3} y \\ &+ \frac{1}{4\eta_0^4}\overline{\varphi}_1 y + \frac{3\eta_1}{2}\overline{\psi}_1 y + C_1 \sum_{i=2}^n \left(\overline{\varphi}_i + a_{i-1}^4 \overline{\varphi}_1\right) y \\ &+ C_2 \sum_{i=2}^n \left(\overline{\psi}_i + a_{i-1}^4 \overline{\psi}_1\right) y \\ &+ \hat{l}\left(\frac{3}{4}\varepsilon_1^{\frac{4}{3}} m_1^{\frac{4}{3}} y + \frac{3}{2}\eta_0^{\frac{4}{3}} y + \frac{3}{\eta_1} y\right)\Bigg], \end{aligned} \tag{17}$$

where $\beta_1 > 0$ is a design parameter, and $\nu(\cdot)$ is a smooth non-negative function to be designed later. Obviously, $\phi_1(0, \hat{l}) = 0$ for all $\hat{l} \in \mathbb{R}$.

It follows from (15), (16) and (17) that

$$\begin{aligned}\mathcal{L}V_1 \le& -\beta_1 y^4 - \nu(y)y^4 + \frac{1}{\lambda_0}\big(\hat{l}-l\big)\big(\dot{\hat{l}} - \varpi_1\big)\\&+\frac{1}{4\varepsilon_1^4}z_2^4 - \delta_1 \tilde{x}^T P\tilde{x}|\tilde{x}|^2 + \tilde{c}|\tilde{x}|^4 + \frac{1}{4\varepsilon_1^4}\tilde{x}_2^4\\&+\Delta_d(|y(t-d(t))|) + \Psi_{1d}(|y(t-d(t))|)\\&+\frac{1}{1-\gamma}S(y(t)) - S(y(t-d(t))). \qquad (18)\end{aligned}$$

Remark 8: Owing to the appearance of Itô correction term (Kallenberg 2002) a quartic Lyapunov–Krasovskii functional is to be constructed, which is different from the quadratic Lyapunov–Krasovskii functional in the deterministic cases. To deal with the time-varying time delays, a time-varying term with the regulation factor $1/(1-\gamma)$ is to be designed, which is similar to the deterministic case, but the design of integrand $S(\cdot)$ is more complex due to the appearance of the stochastic disturbance.

Step k ($k = 2, \ldots, n$). At this step, we can obtain an inequality similar to (18). For clarity, it is summarised in the following Lemma.

Lemma 1: *For every $k = 1, \ldots, n$, there exist smooth functions $\varpi_i, \phi_i, (1 \le i \le k)$ and positive constants β_i such that $\phi_i(0, \hat{l}) = 0$ for all $\hat{l} \in \mathbb{R}$ and that along the solutions of* (9), *$V_k = V_1 + \frac{1}{4}\sum_{j=2}^k z_j^4$ satisfies*

$$\begin{aligned}\mathcal{L}V_k \le& -\sum_{j=1}^k \beta_j z_j^4 - \nu(y)y^4\\&+\left[\frac{1}{\lambda_0}\big(\hat{l}-l\big) - \sum_{j=2}^k z_j^3 \frac{\partial\phi_{j-1}}{\partial\hat{l}}\right]\big(\dot{\hat{l}} - \varpi_k\big)\\&+\frac{1}{4\varepsilon_k^4}z_{k+1}^4 - \delta_1\tilde{x}^T P\tilde{x}|\tilde{x}|^2 + \tilde{c}|\tilde{x}|^4\\&+\sum_{j=1}^k \frac{1}{4\varepsilon_j^4}\tilde{x}_2^4 + \Gamma_k(y(t))\\&+\Delta_d(|y(t-d(t))|) + \Psi_{kd}(|y(t-d(t))|)\\&+\frac{1}{1-\gamma}S(y(t)) - S(y(t-d(t))), \qquad (19)\end{aligned}$$

where

$$\begin{aligned}\Gamma_k(y(t)) =& \left(\frac{k-1}{4\eta_0^4}\overline{\varphi}_1(y(t)) + \frac{3(k-1)}{2}\eta_1\overline{\psi}_1(y(t))\right.\\&\left.+\sum_{j=2}^k \frac{b_j}{2}\overline{\psi}_1(y(t))\right)y^4(t),\end{aligned}$$

$$\begin{aligned}\Psi_{kd}(|y(t-d(t))|) =& \frac{k}{4\eta_0^4}\varphi_{1d}^4(|y(t-d(t))|)\\&+\frac{3k}{2}\eta_1\psi_{1d}^4(|y(t-d(t))|)\\&+\sum_{j=2}^k \frac{b_j}{2}\psi_{1d}^4(|y(t-d(t))|).\end{aligned}$$

Proof: See Appendix A. □

At the last step that $k = n$, we obtain the parameter update law and the control law

$$\dot{\hat{l}}(t) = \varpi_n(y(t), \ldots, z_n(t), \hat{l}(t)), \qquad (20)$$

$$u(t) = \phi_n(y(t), \ldots, z_n(t), \hat{l}(t)), \qquad (21)$$

where

$$\varpi_n = \varpi_{n-1} + \lambda_0 z_n^4 \vartheta_n,$$

$$\begin{aligned}\phi_n =& -\frac{1}{m_n}\left[\beta_n z_n + a_n m_n y + \Omega_{n1} + \frac{1}{4\varepsilon_{n-1}^4}z_n\right.\\&\left.+\left(\hat{l} - \lambda_0\sum_{j=2}^{n-1} z_j^3\frac{\partial\phi_{j-1}}{\partial\hat{l}}\right)z_n\vartheta_n - \frac{\partial\phi_{n-1}}{\partial\hat{l}}\varpi_n\right],\end{aligned}$$

$$\begin{aligned}\vartheta_n =& \frac{1}{b_n}\left(\frac{\partial^2\phi_{n-1}}{\partial y^2}\right)^2 z_n^2 + \frac{3}{2}\eta_0^{4/3}\left[\left(\frac{\partial\phi_{n-1}}{\partial y}\right)^2 + 1\right]^{2/3}\\&+\frac{3}{4}\varepsilon_n^{4/3}m_1^{4/3}\left[\left(\frac{\partial\phi_{n-1}}{\partial y}\right)^2 + 1\right]^{2/3} + \frac{3}{\eta_1}\left(\frac{\partial\phi_{n-1}}{\partial y}\right)^4.\end{aligned}$$

Thus, by taking the following Lyapunov–Krasovskii functional candidate:

$$V_n = V_{n-1}\big(\tilde{x}, y, \ldots, z_{n-1}, \hat{l}, t\big) + \frac{1}{4}z_n^4,$$

we have

$$\begin{aligned}\mathcal{L}V_n \le& -\sum_{j=1}^n \beta_j z_j^4 - \nu(y)y^4 - \left[\delta_1\lambda_{\min}(P) - \tilde{c} - \sum_{j=1}^n \frac{1}{4\varepsilon_j^4}\right]|\tilde{x}|^4\\&+\Gamma_n(y(t)) + \Delta_d(|y(t-d(t))|) + \Psi_{nd}(|y(t-d(t))|)\\&+\frac{1}{1-\gamma}S(y(t)) - S(y(t-d(t))). \qquad (22)\end{aligned}$$

We are now in a position to choose the function $\nu(\cdot)$ in (17) so as to obtain a desired control law of the form (20)–(21).

First, the positive function $S(\cdot)$ is designed such that the term $S(y(t-d(t)))$ is to counteract the time-delay terms and (14). Thus, define $S(\cdot)$ as follows:

$$S(y(t)) = \overline{S}(y(t))\, y^4(t),$$

where

$$\overline{S}(y(t)) = C_1 \sum_{i=2}^{n} \left(\overline{\varphi}_{id}(y(t)) + a_{i-1}^4 \overline{\varphi}_{1d}(y(t))\right) + C_2 \sum_{i=2}^{n} \left(\overline{\psi}_{id}(y(t)) + a_{i-1}^4 \overline{\psi}_{1d}(y(t))\right) + \frac{n}{4\eta_0^4}\overline{\varphi}_{1d}(y(t)) + \frac{3n}{2}\eta_1 \overline{\psi}_{1d}(y(t)) + \sum_{i=2}^{n} \frac{b_i}{2}\overline{\psi}_{1d}(y(t)).$$

Then, we design the smooth non-negative function $\nu(\cdot)$ to cancel the term $S(y(t))$. To this end, define $\nu(\cdot)$ as

$$\nu(y(t)) = \frac{1}{1-\gamma}\overline{S}(y(t)) + \frac{n-1}{4\eta_0^4}\overline{\varphi}_1(y(t)) + \frac{3(n-1)}{2}\eta_1 \overline{\psi}_1(y(t)) + \sum_{j=2}^{n} \frac{b_j}{2}\overline{\psi}_1(y(t)). \tag{23}$$

Choose parameters ϵ, ε, δ_1, ε_j, $j=1,\dots,n$, such that

$$c_0 = \delta_1 \lambda_{\min}(P) - \tilde{c} - \sum_{j=1}^{n} \frac{1}{4\varepsilon_j^4} > 0$$

with β_j, λ_0, η_0, η_1,b_j being any positive constants. Then, it follows from (22) and (23) that

$$\mathcal{L}V_n \leq -\sum_{j=1}^{n} \beta_j z_j^4 - c_0 |\tilde{x}|^4. \tag{24}$$

With (24) and Theorem 2, we obtain the following stability result.

Theorem 3: *The closed-loop system has a unique solution on* $[-d, \infty)$ *and the closed-loop equilibrium of interest is globally stable in probability, and moreover, the solution process can be regulated to the origin almost surely, i.e.* $P\{\lim_{t\to\infty} |x(t)| = 0\} = 1$.

Remark 9: From the above design procedure, we can see that the virtual control coefficients m_i, $i=1,\dots,n$, and the upper bound of the change rate of time delays γ have important impact on the control effort. To keep the control effort within the certain range, the virtual control coefficients cannot be arbitrarily small and the upper bound of the change rate of time delays γ cannot be arbitrarily close to 1, which should be considered in practical engineering design.

6. Simulation

In this section, we will give a numerical example to illustrate the efficiency of our results obtained in previous sections.

Consider the following stochastic system with time delays

$$\begin{cases} dx_1(t) = \left[x_2(t) + \frac{1}{2}\theta_{11} x_1^2(t-d(t))\sin(t)\right]dt + \frac{1}{4}\theta_{12} x_1(t)\, dw_t, \\ dx_2(t) = \left[2u(t) + \frac{1}{2}\theta_{21} x_1(t)\right]dt + \frac{1}{4}\theta_{22} x_1(t-d(t))\, dw_t, \\ y(t) = x_1(t), \end{cases} \tag{25}$$

where $d(t) = \frac{1}{2}(1+\sin(t))$.

With the notation of Assumption A1, we can take

$$\varphi_1(|y(t)|) = 0, \quad \varphi_{1d}(|y(t-d(t))|) = \frac{1}{2}y^2(t-d(t)),$$
$$\psi_1(|y(t)|) = \frac{1}{4}|y(t)|, \quad \psi_{1d}(|y(t-d(t))|) = 0,$$
$$\varphi_2(|y(t)|) = \frac{1}{2}|y(t)|, \quad \varphi_{2d}(|y(t-d(t))|) = 0,$$
$$\psi_2(|y(t)|) = 0, \quad \psi_{2d}(|y(t-d(t))|) = \frac{1}{4}|y(t-d(t))|.$$

Let $l^* \geq \max\{1, |\theta_{11}|, |\theta_{12}|, |\theta_{21}|, |\theta_{22}|\}$ and $l \geq \max\{l^{*4/3}, |\theta_{12}|^4\}$. Design the state-observer as

$$\dot{\hat{x}}_1(t) = 2u(t) - (\hat{x}_1(t) + y(t)).$$

Then, the parameter update law ϖ_2, the virtual control ϕ_1 and control u are

$$\dot{\hat{l}} = \varpi_2 = \varpi_1 + \lambda_0 z_2^4 \vartheta_2,$$

$$\phi_1 = -\left[\beta_1 + 2\left(\frac{C_2}{256} + \frac{3\eta_1}{512} + \frac{b_2}{512}\right) + 1 + \frac{3}{4}\varepsilon_1^{4/3} + \frac{3\eta_1}{512} + \frac{C_1}{16} + \frac{C_2}{256} + \hat{l}\left(\frac{3}{4}\varepsilon_1^{4/3} + \frac{3}{2}\eta_0^{4/3} + \frac{3}{\eta_1}\right)\right]y - 2\left(\frac{C_1}{16} + \frac{2}{64\eta_0^4}\right)y^5,$$

$$u = -\frac{1}{2}\left[\beta_2 z_2 - \hat{x}_1 - y - \frac{\partial \phi_1}{\partial y}(\hat{x}_1 + y) + \frac{1}{4\varepsilon_1^4} z_2 + \hat{l} z_2 \vartheta_2 - \frac{\partial \phi_1}{\partial \hat{l}}\varpi_2\right],$$

where

$$\varpi_1 = \lambda_0 \left(\frac{3}{4}\varepsilon_1^{4/3} + \frac{3}{2}\eta_0^{4/3} + \frac{3}{\eta_1}\right)y^4,$$

$$\vartheta_2 = \frac{1}{b_2}\left(\frac{\partial^2 \phi_1}{\partial y^2}\right)^2 z_2^2 + \frac{3}{2}\eta_0^{4/3}\left[\left(\frac{\partial \phi_1}{\partial y}\right)^2 + 1\right]^{2/3} + \frac{3}{4}\varepsilon_2^{4/3}\left[\left(\frac{\partial \phi_1}{\partial y}\right)^2 + 1\right]^{2/3} + \frac{3}{\eta_1}\left(\frac{\partial \phi_1}{\partial y}\right)^4.$$

Let $\delta_1 = 0.5$, $\varepsilon = 1$, $\epsilon = 0.04$, $\varepsilon_1 = \varepsilon_2 = 4$, $b_2 = \eta_0 = 1$, $\eta_1 = 100$, $\beta_1 = \beta_2 = 0.01$, $\lambda_0 = 20$. Then we obtain $C_1 = 2$ and $C_2 = 150$. Assume $\theta_{11} = \frac{1}{2}$, $\theta_{12} = \frac{1}{2}$, $\theta_{21} = \frac{1}{2}$, $\theta_{22} = \frac{1}{4}$. Then, we have the simulation results: Figures 1 and 2 for initial conditions $x_1(0) = 0.0005$, $x_2(0) = 1$, $\hat{x}_1(0) = 0$, $\hat{l}(0) = 1$.

From Figures 1 and 2, we can see that under the constructed controller, the solution process of the closed-loop system converges to zero almost surely. We can also see that a little larger control effort is needed at the beginning, especially for the larger initial values. Generally, when there exist time delays and stochastic

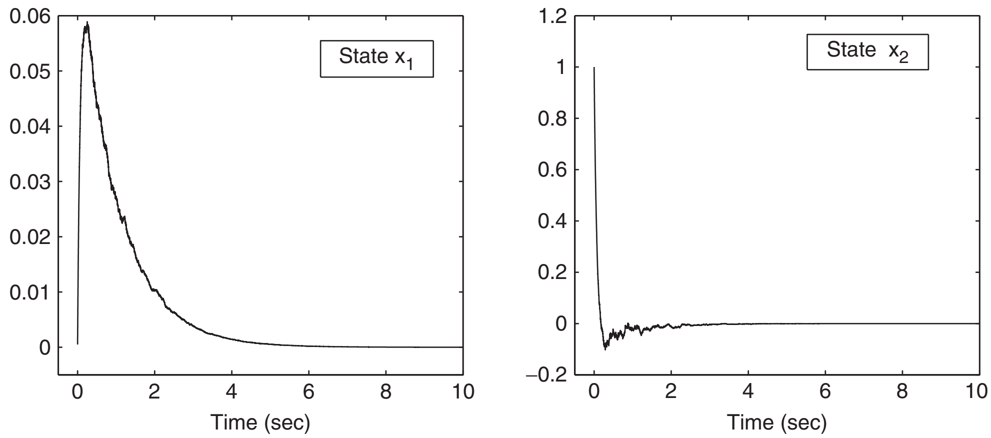

Figure 1. States of the closed-loop system.

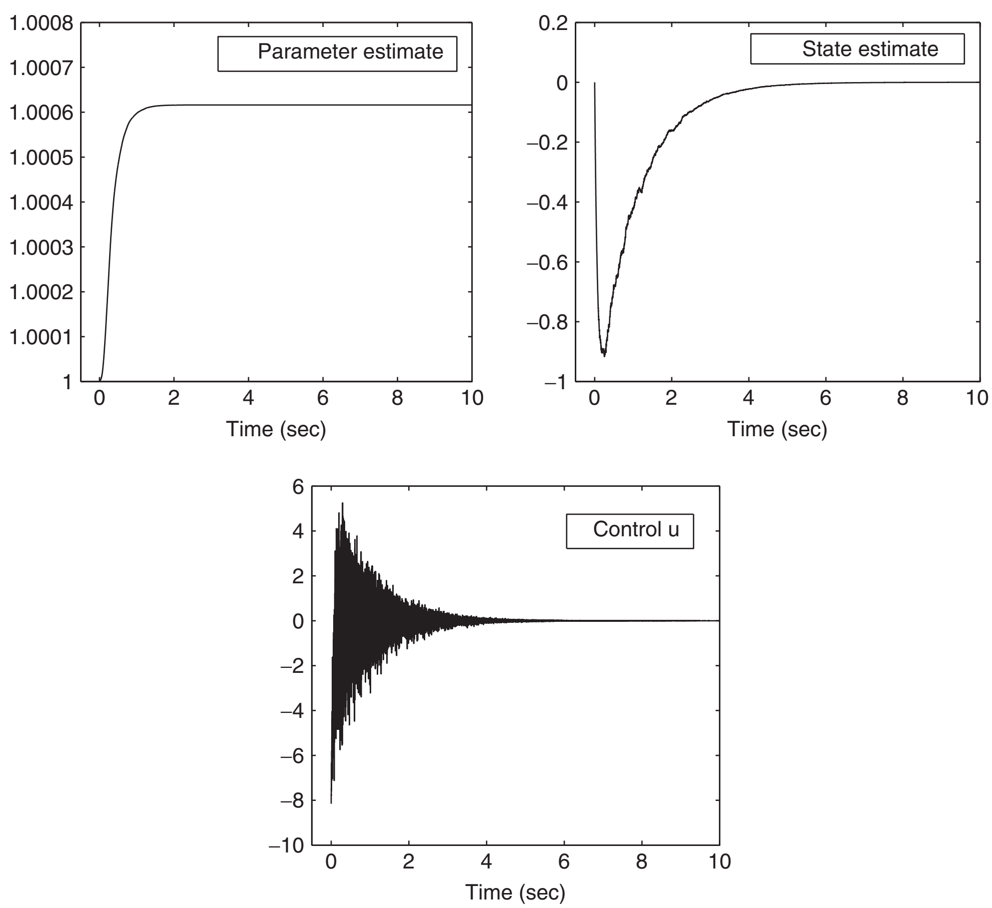

Figure 2. Parameter estimate, state estimate and control of the closed-loop system.

disturbances, the effort of a controller designed based on the backstepping method is bigger than the common case, to which attention should be paid in practical use.

7. Concluding remarks

In this paper, we have studied the adaptive output-feedback stabilisation for a class of stochastic non-linear systems with time delays. Our main contributions are three-fold: (i) global adaptive stabilisation controller design has been investigated for stochastic non-linear systems with time-varying time delays and the design is constructive; (ii) different from the existing work of stabilisation for stochastic non-linear systems with time delays, the existence and uniqueness of the solution of the closed-loop system have been investigated; (iii) in the stochastic time-delay systems investigated in this paper, uncertain parameters in both drift and diffusion terms are allowed and handled by the means of adaptive control techniques.

The method used in this paper can be modified or extended to investigate the adaptive control of stochastic time-delay systems driven by noise of unknown covariance, and to large-scale stochastic multi-time-delay non-linear systems. In the results presented here, the control is independent of noise. As for the noise-dependent case, the controller construction of non-linear stochastic time-delay systems is hard and complicated, and will be a good topic for further research.

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Notes

1. As it is well known, there exist constants $a_1', \dots, a_{n-1}'$ such that the polynomial $\lambda^{n-1} + a_1'\lambda^{n-2} + a_2'\lambda^{n-1} + \cdots + a_{n-1}'$ is Hurwitz. In this case, we can take $a_1 = (a_1')/(m_1)$, $a_2 = (a_2')/(m_1 m_2), \dots, a_{n-1} = (a_{n-1}')/(m_1 m_2 \cdots m_{n-1})$.
2. For any two given real vectors x and y with the same dimension, $x^T y \le \epsilon^p/p|x|^p + 1/(q\epsilon^q)|y|^q$, where $\epsilon > 0$, $p > 1$, $q > 1$, and $p^{-1} + q^{-1} = 1$.
3. When $k = n-1$, we have $z_{k+2} = 0$ in (A1). Thus define

$$\phi_n = -\frac{1}{m_n}\left[\beta_n z_n + a_n m_n y + \Omega_{n1} + \frac{1}{4\varepsilon_{n-1}^4} z_n + \left(\hat{l} - \lambda_0 \sum_{j=2}^{n-1} z_j^3 \frac{\partial \phi_{j-1}}{\partial \hat{l}}\right) z_n \vartheta_n - \frac{\partial \phi_{n-1}}{\partial \hat{l}} \varpi_n\right].$$

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Appendix

A. *Proof of Lemma 1*

As shown at Step 1 of § 5, Lemma 1 holds for $k=1$. Now, we demonstrate Lemma 1 by induction. Assume that Lemma 1 is true for Step k, we will show that Lemma 1 is still true for Step $k+1$. To this end, consider the following function:

$$V_{k+1} = V_k\left(\tilde{x}, y, \ldots, z_k, \hat{l}, t\right) + \tfrac{1}{4} z_{k+1}^4.$$

It follows from (11) that

$$\mathcal{L}V_{k+1} = \mathcal{L}V_k + z_{k+1}^3\left[m_{k+1}(z_{k+2}+\phi_{k+1}+a_{k+1}y) + \sum_{j=1}^{4}\Omega_{k+1,j}\right] + \frac{3}{2}z_{k+1}^2\left(\frac{\partial\phi_k}{\partial y}\right)^2 g_1 g_1^T. \tag{A1}$$

As in Step 1, by Assumption A1, Young inequality and (13), we have

$$\begin{aligned}
z_{k+1}^3 m_{k+1} z_{k+2} &\le \frac{3}{4}\varepsilon_{k+1}^{4/3} m_{k+1}^{4/3} z_{k+1}^4 + \frac{1}{4\varepsilon_{k+1}^4} z_{k+2}^4,\\
z_{k+1}^3\Omega_{k+1,2} &\le |z_{k+1}^3|\left|\frac{\partial\phi_k}{\partial y}\right| |l^* m_1 \tilde{x}_2|\\
&\le \frac{3}{4}\, l\,\varepsilon_{k+1}^{4/3}\, m_1^{4/3}\left[\left(\frac{\partial\phi_k}{\partial y}\right)^2 + 1\right]^{2/3} z_{k+1}^4 + \frac{1}{4\varepsilon_{k+1}^4}\tilde{x}_2^4,\\
z_{k+1}^3\Omega_{k+1,3} &\le |z_{k+1}^3|\left|\frac{\partial\phi_k}{\partial y}\right||f_1| + \frac{1}{2}|z_{k+1}^3|\left|\frac{\partial^2\phi_k}{\partial y^2}\right||g_1|^2\\
&\le l\left[\frac{3}{2}\left[\left(\frac{\partial\phi_k}{\partial y}\right)^2+1\right]^{2/3}\eta_0^{4/3} + z_{k+1}^2\left(\frac{\partial^2\phi_k}{\partial y^2}\right)^2\frac{1}{b_{k+1}}\right] z_{k+1}^4 + \frac{1}{4\eta_0^4}\varphi_{1d}^4\\
&\quad + \frac{b_{k+1}}{2}\psi_{1d}^4 + \frac{1}{4\eta_0^4}\overline{\varphi}_1 y^4 + \frac{b_{k+1}}{2}\overline{\psi}_1 y^4,\\
\frac{3}{2}z_{k+1}^2\left(\frac{\partial\phi_k}{\partial y}\right)^2 g_1 g_1^T &\le 3 z_{k+1}^2\left(\frac{\partial\phi_k}{\partial y}\right)^2 h_1^{*^2}(\psi_{1d}^2 + \psi_1^2)\\
&\le l\frac{3}{\eta_1}\left(\frac{\partial\phi_k}{\partial y}\right)^4 z_{k+1}^4 + \frac{3}{2}\eta_1\psi_{1d}^4 + \frac{3}{2}\eta_1\overline{\psi}_1 y^4,
\end{aligned}$$

where and whereafter, ε_j, b_j, $j=2,\ldots,n$, are positive design constants to be specified. These together with (19) and (A1) lead to

$$\begin{aligned}
\mathcal{L}V_{k+1} \le& -\sum_{j=1}^{k}\beta_j z_j^4 - \nu(y)y^4 + \left[\frac{1}{\lambda_0}\left(\hat{l}-l\right) - \sum_{j=2}^{k} z_j^3\frac{\partial\phi_{j-1}}{\partial\hat{l}}\right]\left(\dot{\hat{l}} - \varpi_k\right)\\
&- z_{k+1}^3\frac{\partial\phi_k}{\partial\hat{l}}\dot{\hat{l}} - \delta_1\tilde{x}^T P\tilde{x}|\tilde{x}|^2 + \tilde{c}|\tilde{x}|^4 + \sum_{j=1}^{k+1}\frac{1}{4\varepsilon_j^4}\tilde{x}_2^4\\
&+ l\left[\frac{1}{b_{k+1}}\left(\frac{\partial^2\phi_k}{\partial y^2}\right)^2 z_{k+1}^6 + \frac{3}{2}\eta_0^{4/3}\left[\left(\frac{\partial\phi_k}{\partial y}\right)^2+1\right]^{2/3} z_{k+1}^4\right.\\
&\left.+ \frac{3}{4}\varepsilon_{k+1}^{4/3} m_{k+1}^{4/3}\left[\left(\frac{\partial\phi_k}{\partial y}\right)^2+1\right]^{2/3} z_{k+1}^4 + \frac{3}{\eta_1}\left(\frac{\partial\phi_k}{\partial y}\right)^4 z_{k+1}^4\right]\\
&+ z_{k+1}^3\left[m_{k+1}\phi_{k+1} + a_{k+1}m_{k+1}y + \Omega_{k+1,1}\right.\\
&\left.+ \frac{3}{4}\varepsilon_{k+1}^{4/3} m_{k+1}^{4/3} z_{k+1} + \frac{1}{4\varepsilon_k^4} z_{k+1}\right]\\
&+ \frac{1}{4\varepsilon_{k+1}^4} z_{k+2} + \Delta_d(|y(t-d(t))|)\\
&+ \Psi_{k+1,d}(|y(t-d(t))|) + \Gamma_{k+1}(y(t)).
\end{aligned} \tag{A2}$$

Define the virtual parameter update law and the virtual control as follows:[3]

$$\varpi_{k+1} = \varpi_k + \lambda_0 z_{k+1}^4\vartheta_{k+1}, \tag{A3}$$

$$\begin{aligned}
\phi_{k+1}(\overline{\hat{x}}_{k+1}, \hat{l}) = -\frac{1}{m_{k+1}}\Bigg[&\beta_{k+1}z_{k+1} + a_{k+1}m_{k+1}y + \Omega_{k+1,1}\\
&+ \frac{3}{4}\varepsilon_{k+1}^{4/3} m_{k+1}^{4/3} z_{k+1} + \frac{1}{4\varepsilon_k^4} z_{k+1}\\
&+ \left(\hat{l} - \lambda_0\sum_{j=2}^{k} z_j^3\frac{\partial\phi_{j-1}}{\partial\hat{l}}\right) z_{k+1}\vartheta_{k+1}\\
&- \frac{\partial\phi_k}{\partial\hat{l}}\varpi_{k+1}\Bigg],
\end{aligned} \tag{A4}$$

where $k=1,\ldots,n-1$, $\overline{\hat{x}}_{k+1} = (y, \hat{x}_1, \ldots, \hat{x}_k)$, and

$$\begin{aligned}
\vartheta_{k+1} =& \frac{1}{b_{k+1}}\left(\frac{\partial^2\phi_k}{\partial y^2}\right)^2 z_{k+1}^2 + \frac{3}{2}\eta_0^{4/3}\left[\left(\frac{\partial\phi_k}{\partial y}\right)^2+1\right]^{2/3}\\
&+ \frac{3}{4}\varepsilon_{k+1}^{4/3} m_1^{4/3}\left[\left(\frac{\partial\phi_k}{\partial y}\right)^2+1\right]^{2/3} + \frac{3}{\eta_1}\left(\frac{\partial\phi_k}{\partial y}\right)^4.
\end{aligned}$$

Obviously, $\phi_{k+1}(0, \hat{l}) = 0$ for all $\hat{l}\in\mathbb{R}$.

It follows from (A2), (A3) and (A4) that for $k=1,\ldots,n-1$,

$$\begin{aligned}
\mathcal{L}V_{k+1} \le& -\sum_{j=1}^{k+1}\beta_j z_j^4 - \nu(y)y^4 + \left[\frac{1}{\lambda_0}(\hat{l}-l) - \sum_{j=2}^{k+1} z_j^3\frac{\partial\phi_{j-1}}{\partial\hat{l}}\right]\left(\dot{\hat{l}} - \varpi_{k+1}\right)\\
&+ \frac{1}{4\varepsilon_{k+1}^4} z_{k+2}^4 - \delta_1\tilde{x}^T P\tilde{x}|\tilde{x}|^2 + \tilde{c}|\tilde{x}|^4 + \sum_{j=1}^{k+1}\frac{1}{4\varepsilon_j^4}\tilde{x}_2^4\\
&+ \Gamma_{k+1}(y(t)) + \Delta_d(|y(t-d(t))|) + \Psi_{k+1,d}(|y(t-d(t))|).
\end{aligned}$$

Therefore, the proof is completed. □