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Adaptive Predictive Control using Multiple Models, Switching and Tuning

Leonardo Giovanini, Andrzej W. Ordys, and Michael J. Grimble

Abstract: In this work, a new method of design adaptive controllers for SISO systems based on multiple models and switching is presented. The controller selects the model from a given set, according to a switching rule based on output prediction errors. The goal is to design, at each sample instant, a predictive control law that ensures the robust stability of the closed-loop system and achieves the best performance for the current operating point. At each sample the proposed control scheme identifies a set of linear models that best characterizes the dynamics of the current operating region. Then, it carries out an automatic reconfiguration of the controller to achieve the best possible performance whilst providing a guarantee of robust closed-loop stability. The results are illustrated by simulations a nonlinear continuous and stirred tank reactor.

Keywords: Adaptive control, infinite controller cover set, multiple models, multi-objective optimization, predictive control.

1. INTRODUCTION

The control of dynamical systems in the presence of large uncertainties and constraints is of great interest for many applications. Such problems emerge when there are large variations of parameters due to changes in the operating region. They arise from the fact that processes are nonlinear, and nowadays are operated in wider regions subject to larger disturbances, frequent set point changes and failures in the system. In such cases, the controller has to determine the specific situation that exists at any instant and take the appropriate control action. Accomplishing this rapidly, accurately and in stable fashion is the objective in control design.

Model predictive control (MPC) is one of the few techniques able to cope with constraints and modelling errors in an explicit manner. There are many ways of considering modelling errors. The most popular approach is the minimization of worst-case controller cost [1]. Lee and Yu [2] summarized the development of so called *min-max* algorithms and pointed out that the closed-loop implementation provides a good performance and they also discussed computationally tractable approximations. The main disadvantage of the *minmax* approach is that control performance may be too conservative in some cases [2]. Since all possible plant dynamics are considered equally likely at every sample, the controller only focuses on the worst-case. Alternative formulations of robust MPC based on a generalized objective function [3], cost function constraints [4] and multiobjective optimization [5] have being presented lately. To overcome this limitation, several authors have proposed nonlinear MPC algorithms based on neural networks [6], fuzzy logic [7], Hammerstein [8] and Wiener models [9,10], Volterra series [11], successive linearization [12], input-output models [13] and multiple models [14].

The problem described in the above is one of adaptive control in which, typically, controller parameters are adjusted on the basis of plant parameter estimates. However, if conventional adaptive control is used, experience indicates that the presence of large parameter errors will generally result in slow convergence with large tracking errors during the transient phase. An alternative approach involves the use of multiple models, switching and tuning. This technique was introduced in the early 1990's [15] and later developed in [16-18].

A standard switching controller consists of an inner loop where the candidate controller is connected in closed-loop with the system, and an outer loop where based on a performance criterion and input-output data the supervisor decides which controller to select and when to switch to a different one. The supervisor then selects the candidate associated with the model that minimizes a performance index. Implementation and analysis of the switching control scheme is often simplified by considering a finite set of candidate controllers. This set is called a *controller cover set*

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[16]. In this framework, robustness and performance issues are addressed off-line when the controller cover set is designed. If the controller cover set consists of a small number of controllers, each one stabilizing a wide set of models, then stability is generally rapidly achieved, even before a large amount of information has been accrued, but in the long run the resulting performance is typically low. In contrast, if the controller cover consists of a large number of controllers, each one tailored to a narrow set of models, a high performance control system is potentially achieved, but poor performance will possibly occur until there is sufficient data to obtain an accurate estimate of the process model.

An alternative approach, based on a probability measure computed on-line was suggested recently. Campi *et al*. [29] has proposed a hierarchical switching scheme based on a probability measure of the likelihood of the different models, which is estimated on-line from the data is obtained by the control system. This probability measure is employed to select a controller that suitably compromises robustness versus performance, given the current level of uncertainty. As time goes by, the probability distribution becomes more sharply peaked around the model that best describes the system.

The main contribution of this work is a robust adaptive predictive controller based on receding horizon and multiple models, switching and tuning techniques. The control law is designed by a multi-objective optimization that employs a set of LTI models to characterize the system dynamic with a minimum error at every sample. This set is built from a bigger one, which is employed to describe the system in the operating domain; using switching techniques. The switching is performed directly in the objective function and constraints of the optimization problem. The proposed approach to adaptive control corresponds to having an infinite number of controllers in the multiple-model implementation, with the additional benefit of constraints handling.

The paper is organized as follow: Firstly, in Section 2 the formulation of the predictive feedback control is revisited. The meaning of the design parameters is discussed. The original formulation is modified by changing the constraint employed to guarantee the closed loop stability. Then, in Section 3 the multiple models, switching and tuning approach is introduced. This involves modifying the objective function and the constraints employed by the optimization problem to design the controller. At the end of this section, a robust switching strategy that takes account of the uncertainty during the controller design is introduced. Section 4 shows the results obtained from the application of the proposed algorithm to a nonlinear continuous stirred tank reactor. Finally, the conclusions are presented in Section 5.

2. ROBUST PREDICTIVE FEEDBACK CONTROL

MPC is an optimal control approach involving the direct use of a system model and on-line optimization technique to compute the control actions such that a measure of the closed-loop performance is minimized and all constraints are and will be fulfilled [1]. The basic formulation implies a control philosophy similar to an optimal open-loop control law which includes, in a simple and efficient way, constraints present in the system. However, as pointed out by Lee and Yu [2] this formulation can give poor closed-loop performance, especially when uncertainties are assumed to be time-invariant in the formulation. This is true even when the underlying system is time-invariant. When the uncertainty is allowed to vary from one time step to next in the prediction, the open loop formulation gives robust, but cautious, control.

A way to solve these problems is to introduce a feedback action in the predictive controller [5]. This idea implies the use of the closed-loop prediction error instead of the open-loop prediction error to compute the future control actions. For a stable *SISO* system described by a *FIR* model, the predictive control law is given by

$$u(k)=\sum\nolimits_{j=0}^{v} q_j \hat{e}^0(J,k-j)+\sum\nolimits_{j=1}^{w} q_{j+v}u(k-j), \quad (1)$$

where v and w are the error and the control horizons of the control law, q_j $j = 0,1,\ldots,v+w$ are the control law's parameters, $u(k–j)$ is the past control action at time $k–j$ and $\hat{e}^0(J,k–j)$ is the J step ahead predicted error based on measurements until time $k–j$

$$\hat{e}^0(J,k-j)=e(k-j)-\mathcal{P}\left(J,z^{-1}\right)u(k-j). \quad (2)$$

In this equation $e(k–j)=r(k–j)–y(k–j)$ is the measured error at time $k–j$ and $\mathcal{P}(J,z^{-1})$ is the transfer function of the open-loop predictor given by [5]

$$\mathcal{P}\left(J,z^{-1}\right)=\tilde{a}_J z^{-1}+\sum\nolimits_{j=J+1}^{N}\tilde{h}_j z^{-j}-\sum\nolimits_{j=1}^{N}\tilde{h}_j z^{-j}, \quad (3)$$

where N is the *FIR* model length, $\tilde{a}_j$ is the step response coefficient and $\tilde{h}_j$ is the Markov coefficient of the model of the system respectively.

The control law (1) has three set of parameters to be tuned: the error and the control horizons (v and w), the prediction time J and the parameters q_j j=0,…,$v+w$. It includes a feedback action-based on present and past errors $e(k–j)$ j=0,…,v– which improves the closed-loop system response [19]. This is the main reason why the control law (1) reduces the effect of non measurable disturbances more aggressively than a standard model predictive control.

The stability of the predictive feedback controller depends on the parameters q_j $j = 0,\ldots,v+w$ and the prediction time J simultaneously. This result is summarized in the following theorem.

Theorem 1: Given a system controlled by a predictive feedback controller (1), the closed-loop system will be robustly stable if and only if

$$\frac{1-\sum_{j=1}^{w}|q_{j+v}|}{\sum_{j=0}^{v}|q_j|}+\tilde{a}_J > \sum_{i=J+1}^{N}|\tilde{h}_i|+\sum_{i=1}^{N}|h_i-\tilde{h}_i|, \qquad (4)$$

where h_j is the Markov coefficient of the system.

Proof: See Appendix A. □

Assuming that a polytopic linear model (PLM) W of m linear *FIR* models characterizes the behavior of an uncertain, as well as a nonlinear, stable system up to a desired accuracy ε over a bounded region D, the robust stability problem becomes the problem of finding a set of parameters q_j and J such that (4) is satisfied for all models of W

$$\frac{1-\sum_{j=1}^{w}|q_{j+v}|}{\sum_{j=0}^{v}|q_j|}+\tilde{a}_J > \sum_{j=J+1}^{N}|\tilde{h}_j|+\max_{l=1,\ldots,m}\left(\left|\sum_{j=1}^{N}\hat{h}_{jl}-\tilde{h}_j\right|\right), \qquad (5)$$

where $\hat{h}_{jl}$ is the Markov coefficient of the lth model l=1,…,m of the PLM W.

If the prediction time J is given, the stability problem reduces to the problem of finding a set of parameters q_j j=0,…,v+w, so that (4) is satisfied for all models of W simultaneously. This condition can be written as a set of m linear inequalities

$$\frac{1-\sum_{j=1}^{w}|q_{j+v}|}{\sum_{j=0}^{v}|q_j|}+\tilde{a}_J > \sum_{j=J+1}^{N}|\tilde{h}_j|+\sum_{j=1}^{N}|\hat{h}_{jl}-\tilde{h}_j| \quad l=1,\ldots,m, \qquad (6)$$

to be satisfied simultaneously.

Equations (4)-(6) guarantee the robust *superstability* of the closed-loop system [19]. These stability conditions have being formulated as a condition on the models and controller parameters, rather than in terms of closed-loop eigenvalues position. Superstable systems are a narrower class than stable systems. In the parameters space, the stabilization problem becomes convex and numerous problems presenting serious difficulties within the framework of the standard theory, such as simultaneous stabilization of more than two models and robust stabilization under parametric uncertainty, are solved easily for this class of systems.

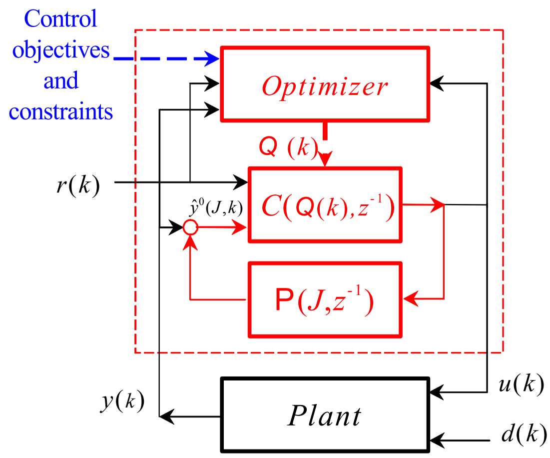

Fig. 1. Structure of the predictive controller.

Conditions for superstability have been introduced by several authors [20-22] and they have been applied to control problems [22,23]. Discrete superstable systems enjoy numerous important properties [22,24]. For this work the most relevant are:

1. Superstability guarantees the existences of a positively invariant set,
2. Superstability implies the existence of a non quadratic Lyapunov function, and
3. Superstability is retained in the time varying case, as well in the presence of time varying and nonlinear perturbations.

In the following sections we will employ these properties to guarantee the stability of the LTV system, used to approximate the original nonlinear system. The LTV system will be built from the set of LTI models W using a switching strategy. In the following paragraph we will introduce an optimization program to solve the controller design problem. The structure of the resulting controller is shown in Fig. 1.

Given a prediction horizon J and the structure of the control law, the parameters q_j j=0,…,v+w can be found solving the following nonlinear optimization problem [5]

$$\min_{q_j\ j=0,1,\ldots,v+w} F(r(k+i),y_l(i,k),u_l(i,k)) \qquad i\in[0,V] \qquad (7a)$$

$st.$

$$\hat{y}_l^0(J,k+i)=y_l(i,k)+P_l(J,q^{-1})u_l(i,k) \qquad i\in[0,V], \qquad (7b)$$

$$\hat{y}_l(i,k)=y(k)+P_l(J,q^{-1})u_l(k)+\sum_{j=0}^{i}\tilde{h}_{lj}u_l(k+j) \qquad l\in[1,m], \qquad (7c)$$

$$u_l(i,k)=\sum_{j=0}^{v}q_j\hat{e}_l^0(J,k+i-j)+\sum_{j=1}^{w}q_{v+j}u_l(k+i-j), \qquad (7d)$$

$$1>\sum_{j=1}^{w}|q_{j+v}|+\left(\sum_{i=J+1}^{N}|\hat{h}_{il}|-\hat{a}_{Jl}\right)\sum_{j=0}^{v}|q_j|, \qquad (7e)$$

where V is the overall number of sampling instants considered.

The objective function $F(\cdot)$ (equation (7a)) measures the future closed loop performance of the system. It should consider all models used to represent the controlled system, and it can be given by a general expression

$$F(\cdot)=\sum_{l=1}^{m}\gamma_l f_l\left(r(k+i),y_l(i,k),u_l(i,k)\right) \quad i\in[0,V], \quad (8)$$

where $\gamma_l \geq 0$ are arbitrary weights and f_l is the performance index for model l measured by any weighting norm

$$f_l(\cdot)=\left\|\hat{e}(i,k)\right\|_Q^p+\left\|u(i,k)\right\|_R^p \quad i\in[0,V], 1\leq p\leq\infty. \quad (9)$$

The first constraint (equation (7b)) is the corrected open loop prediction $\hat{y}_l^0(J,k+i)$ for each model, which is employed to compute the control action $u_l(i,k)$. It only uses the information available until time $k+i$. The second constraint (equation (7c)) is the closed loop prediction $\hat{y}_l(i,k)$ for each model, which is employed to evaluate the system performance and constraints and to compute $\hat{y}_l^0(J,k+i)$. It uses all the information available at time $k+i$. The third equation (equation (7d)) denotes the control law (1). Finally, the last constraint (equation (7e)) is the stability condition (6), which included ensuring superstability of the closed-loop system.

The optimization problem (7) contains a set of constraints for each model of $\mathcal{W}$, with control actions $u(k-j)$ $j=1,\ldots,w$ and past errors $e(k-j)$ $j=0,1,\ldots,v$ as common initial conditions and the parameters of the control law as common variables. The optimization problem readjusts the control law (1) until all design conditions are simultaneously satisfied, by a numerical search through a sequence of dynamic simulations. This reduces the computational burden in the minimization of the performance measure and fulfillment of constraints. Furthermore it replaces the open-loop prediction by a stable closed-loop prediction, thereby avoiding the ill-conditioning problems.

This formulation of predictive control allows for including constraints in the structure and parameters of the control law q_j $j=0,\ldots,v+w$ since they are the decision variables of the optimization problem (7). In this way, the resulting control law will combine features from predictive and variable structure control techniques [25,26]. This idea will help to improve the performance and robustness of the closed-loop system and to overcome the performance limitations imposed by the use of a time-invariant control law [25]. For example, the integral mode, which is employed to remove the steady-state error, can be included in the control law only when the system has reached its settling time. In this way, the closed-loop poles can be placed freely during the transient period, so that the performance is optimized and the integral mode is included after the transient has finished.

The stability of the closed-loop relies on the feasibility of the optimization problem (7). If there is no other constraint than the stability constraint (7e), the optimization problem is always feasible [22]. However, if there are constraints in the input ($g_u(\cdot)$) and output ($g_y(\cdot)$) variables

$$g_y\left(r(k+i),y_l(i,k),u_l(i,k)\right)\leq 0 \quad i\in[0,V], l\in[1,m],$$
$$g_u\left(y_l(i,k),u_l(i,k)\right)\leq 0,$$

the feasibility of the resulting optimization problem can be guaranteed by adding slack variables to constraints

$$g_y\left(r(k+i),y_l(i,k),u_l(i,k)\right)\leq \upsilon_y \quad \upsilon_y\geq 0,$$
$$g_u\left(y_l(i,k),u_l(i,k)\right)\leq \upsilon_u \quad \upsilon_u\geq 0,$$

and penalizing their deviation in the objective function F

$$F=\sum_{l=1}^{m}\gamma_l(k)f_l\left(r(k+i),y_l(i,k),u_l(i,k)\right)+\upsilon_y^2+\upsilon_u^2.$$

3. MULTIPLE MODELS, SWITCHING AND TUNING CONTROL

In many industrial applications it is frequently the case that during the design of a controller the plant is assumed approximately linear, with a given uncertainty. In practice, this assumption involves a simplification that is too strong. The resulting controller often leads to either intolerable constraint violations or over conservative control actions [2].

In order to guarantee constraint fulfillment for every possible realization of the system within a polytopic linear model (PLM) $\mathcal{W}$, it is clear that the control action has to be chosen safe enough to cope with the effect of the worst realization [27]. However, to improve the system's performance and robustness over a wider operational range and satisfy constraints, it is necessary to employ a better approximation to design the control law. A way of implementing this

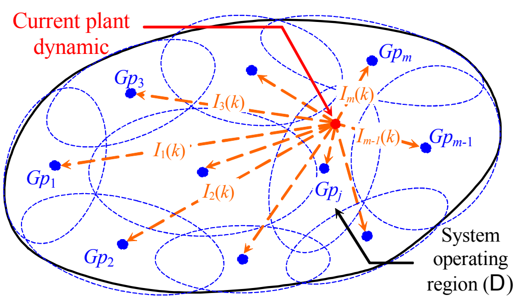

Fig. 2. Geometrical interpretation of index (11).

idea is to build a LTV ε-accurate model [28] from the PLM $\mathcal{W}$ using a switching rule. In this way, the closest model to the current system dynamic is identified (Fig. 2) and, then it is employed to design the control law. This idea implies the combination of *multiple models, switching and tuning control* (MMST) [15-17], with receding horizon techniques.

Generally, the switching algorithm is implemented by first computing performance indices $I_l(k)$ $l=1,\ldots,m$ based on the output prediction error

$$\varepsilon_l(k)=y(k)-\hat{y}_l(k) \quad l=1,\ldots,m,$$

where $\hat{y}_l(k)$ is the output of the l-th model. Then, based on performance indices $I_l(k)$, a supervisor selects the candidate controller that is better tuned to the currently estimated system model

$$I_{\min}(k)=\min_{l=1,\ldots,m}\{I_l(k)\}=I_j(k). \tag{10}$$

The compromise between robustness and performance is made off-line when the controller cover set is designed [16,17].

The switching criterion $I_l(k)$ plays a crucial role in the design of multiple MMST systems. The switching criterion depends upon the prior information assumed about the plant, and is chosen to ensure stability as well as to improve the performance [1]. The most common index employed in switching control can be presented as follows

$$I_l(k)=\alpha_l\varepsilon_l^2(k)+\beta_l\sum\nolimits_{i=N_0}^{k}\lambda_l^{k-i}\varepsilon_l^2(k) \quad l=1,\ldots,m, \tag{11}$$

where $\alpha_l \geq 0$, $\beta_l \geq 0$, $\lambda_l \in [0,1]$ and N_0 is the time instant when the change happens. Different performance indices can be obtained with different parameter values. For example, if $\beta_l=0$, the indices (11) become

$$I_l(k)=\alpha_l\varepsilon_l^2(k) \quad l=1,\ldots,m, \tag{12}$$

which ensures a fast adaptation response. However, the type of index defined in (12) is sensitive to uncertainties, disturbances and noise, deteriorating the performance of the adaptive systems. An index that overcomes these problems can be obtained by fixing $\alpha=0$, in this case the index (11) becomes

$$I_l(k)=\beta_l\sum\nolimits_{i=N_0}^{k}\lambda_l^{k-i}\varepsilon_l^2(k). \tag{13}$$

This index is less sensitive to noise and disturbance than index (12) because all prior data, including the most recent measurement, is exponentially weighted. The parameter λ determines the rate at which data enter into the calculation of index $I_l(k)$ and the depth of memory. It therefore introduces a lag in the selection of the model which is proportional to λ: as $\lambda\rightarrow 1$ all the data is equally considered and the lag is bigger, but when $\lambda=0$, the index (13) becomes (12).In the predictive feedback control framework analyzed in Section 2, the switching scheme can be implemented by calculating and comparing the above indices every sampling instant, generating the switching variables $S_l(k)$ from

$$S_l(k)=H\left(\Delta I_l(k)\right) \quad l=1,\ldots,m, \tag{14}$$

where

$$\Delta I_l(k)=I_{\min}(k)-I_l(k) \quad l=1,\ldots,m, \tag{15}$$

and $H(x)$ is the Heaviside unit step function given by

$$H_\sigma(x)=\begin{cases}1 & x\geq 0,\\ 0 & x<0.\end{cases} \tag{16}$$

Then, the objective function (8) -employed in problem (7)- is modified by replacing the weight γ_l with the switching variables $S_l(k)$ for each model and including them in the design constrains $g_y(\cdot)$ and $g_u(\cdot)$, which represent the output and input constraints respectively

$$\begin{aligned}
F&=\sum_{l=1}^{m}S_l(k)f_l\left(r(k+i),y_l(i,k),u_l(i,k)\right) \quad i\in[0,V], &\text{(17a)}\\
&g_y\left(S_l(k),y_l(i,k),u_l(i,k)\right)\leq 0 \qquad l\in[1,m], &\text{(17b)}\\
&g_u\left(S_l(k),y_l(i,k),u_l(i,k)\right)\leq 0.
\end{aligned}$$

Finally, the structure of predictive feedback controller must be modified by including the vector of switching variables

$$\mathcal{S}(k)=\left[S_1(k)\cdots S_m(k)\right]$$

as external inputs of optimizer (Fig. 3). In this way, the control law (1) is designed for problem (7) but with objective function and constraints (17). It will employ only the closest model to the current plant dynamic to measure the performance and evaluate the constraints while the stability constraint will be applied to all models. Then, a better closed-loop performance will be obtained because only the closest model to the current dynamic is used to evaluate the controller performance and constraints during the controller design.

The controller is computed such that all the closed-loop systems resulting from the elements of W are superstable at each sampling time. This is a sufficient condition to guarantee the stability of the switching sequence. This result is summarized in the following theorem.

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[1] A switching scheme based on a different criterion to measure the distance between two models, like gap-metric or a probability measure, can be employed. However, this topic is out of scope of this paper.

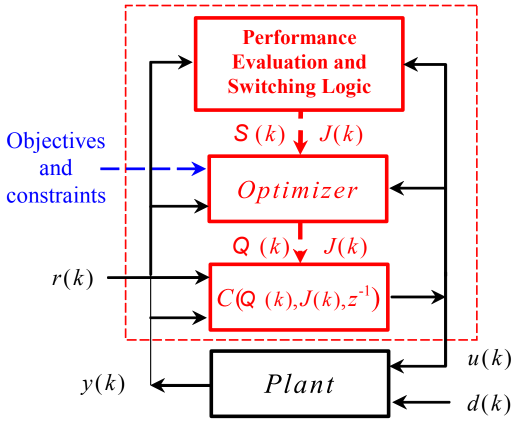

Fig. 3. Structure of the controller.

Theorem 2: Given a system represented by a PLM of m LTI models, $\mathcal{W}$, and controlled by the predictive controller (1) designed by problem (7) with objective function and constraints given by (17), the resulting closed loop system is exponentially stable for any switching sequence.

Proof: To guarantee the stability of closed loop system for any switching sequence it is necessary to probe the stability of the PLM for any switching sequence, which implies the stability of the nonlinear system over the domain $\mathcal{D}$ [28].

Since the optimization problem (7) guarantee the superstability of all closed-loop systems resulting from of the PLM $\mathcal{W}$, it follows from Lemma 4 that the superstability of each individual model of $\mathcal{W}$, implies the superstability PLM itself. From Lemma 2 follows that the error trajectory will monotonically decrease in norm for all futures samples

$$\left\|e(k+j)\right\|_\infty \leq \sigma(k+j)\max\left(0,\left\|e(k)\right\|_\infty - \eta(k+j)\right) \quad \forall j>0,$$

where

$$\begin{aligned}\sigma(k+j) &= \prod_{i=1}^{j}\sum_{l=1}^{m} S_l(k+i)\sigma_l \quad j>0,\\ \eta(k+j) &= \sum_{l=1}^{m} S_l(k+j)\eta_l.\end{aligned} \tag{18}$$

Since the error converges monotonically to zero for any switching sequence, the resulting closed-loop for the PLM system is stable.

Finally, by construction of the PLM model [28], all trajectories of the nonlinear system are included in the set of the PLMs trajectories. Thus, the stability of the PLMs switching sequence over $\mathcal{D}$ implies the stability of the nonlinear system switching sequence over $\mathcal{D}$. □

The superstability of the PLM implies the existence of a common unbounded Lyapunov function[2]

$$V_l(k) = \max_{j=0,\ldots,N}\left|u(k-j)\right| \quad l=1,\ldots,m, \tag{19}$$

which is properly nested for at least N samples

$$V(k) \subset \cdots \subset V(k+N) \subseteq V(k+N+1) \subseteq \cdots.$$

The common Lyapunov function for the PLM $\mathcal{W}$ will exist as long as the optimization problem has a solution and it will be decreasing along all trajectories.

3.1. Robust adaptive switched control

Standard switching control schemes are based on the *certainty equivalence* philosophy [16,29]. At each switching time, the supervisor selects the candidate controller that is better tuned to the currently estimated system model. The compromise between robustness and performance is made off line when the controller cover is designed [16,17].

An alternative approach to robustness problem has been proposed by Campi *et al.* [29], who suggested a hierarchical switching scheme based on a probability measure of the likelihood of the different models. The probability measure is estimated on-line from the data obtained by the control system. It is employed to select a controller that suitably compromises robustness versus performance, given the current level of uncertainty. As time goes by, the probability distribution becomes more sharply peaked around the model that best describes the system.

In the switching and tuning framework presented in this work, the robustness can be obtained using a subset of models which lays into a distance δ(k) to the current system dynamic

$$\mathcal{M}(k) = \left\{Gp_l(z) / \Delta I_l(k) \leq \delta(k) \quad l=1,\ldots,m\right\} \tag{20}$$

instead of the closest model (Fig. 4) to design the control law. In this way, the control algorithm will build an ε-accurate model [28] every sample to approximate the nonlinear model, which will be used by the optimization problem (7) to design the control law. This idea will be implemented by modifying the switching function (16) in the following way

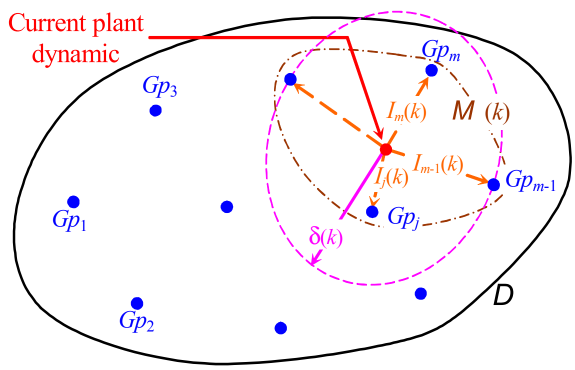

Fig. 4. Geometrical interpretation of index (20).

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[2] The FIR model is a non minimum state space realization of the system whose states are the previous control actions

$$H_\delta(x)=\begin{cases}1 & x\ge -\delta(k),\\ 0 & x<-\delta(k),\end{cases} \quad (21)$$

where

$$\delta(k)=\rho \min_{\substack{l=1,\dots,m\\ l\neq j}}\left(I_l(k)-I_j(k)\right) \quad \rho\ge 1,$$
$$I_j(k)=I_{\min}(k)=\min_{l=1,\dots,m}\left(I_l(k)\right), \quad (22)$$

and modifying the optimization problem (7) in two ways:

1. The switching variables $S(k)$ are included into stability constraints (7e)

$$S_l(k)\left(\sum_{j=1}^{w}\left|q_{j+v}\right|+\left(\sum_{j=J(k)+1}^{N}\left|\tilde{h}_j\right|-\left(\sum_{j=1}^{N}\left|\hat{h}_{jl}-\tilde{h}_j\right|-\tilde{a}_{J(k)}\right)\sum_{j=0}^{v}\left|q_j\right|\right)<1 \quad l=1,\dots,m. \quad (23)$$

2. The prediction time J will be computed at each sampling time using (6) such that only stabilizes the models involved in the control law design.

The modification of the switching rule implies the on-line building of an ε-accurate model[3], whose accuracy is defined by $\delta(k)$ and ρ

$$\varepsilon(k)=\rho\left(\varepsilon_{\min}(k)+\delta(k)\right),$$

and characterizes the region of the operational space D employed to design the control law (Fig. 4). In this way, these parameters define the number of models to be included in $M(k)$ at each sample.

In the first samples after a change, several models can have similar behavior and it is difficult to distinguish between them. Therefore, the set $M(k)$ will include more models than required to represent the system, providing closed-loop robustness according to the information available at each sampling time. As the time goes by, the indexes $I_l(k)$ $l=1,\dots,m$ will clearly differentiate and the set $M(k)$ will reduce the size (Fig. 5) until only the models required to represent the system with the desire accuracy will belong to $M(k)$. At least, it will include the two closest models to the current dynamic. In this way, the control law superstabilize the actual system's operating region, and their neighborhoods. As the time goes, the control law will change as the stabilization region is moving through the operating regions until the steady-state would be achieved. This is equivalent to a dynamical partition of the operating region during the controller design, which is carried out during the model selection trough the switching rule.

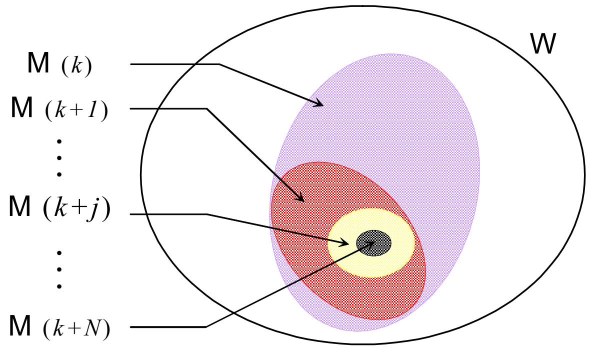

Fig. 5. Time evolution of model subset M(k).

Using similar arguments to Theorem 2, but applied to ε-accurate model resulting from the switching rule (21)-(22), the set $M(k)$, the stability of the closed-loop system for any switching sequence can be guaranteed using the properties of superstable systems. This result is summarized in the following theorem.

Theorem 3: Given a nonlinear system represented by a set of models W and controlled by the control law (1) designed by problem (7) with the switching rule (21)-(22), the resulting closed loop system is exponentially stable for any switching sequence.

Proof: The superstability of the models included in $M(k+j)$ $j>0$ implies that the error trajectories of the resulting ε–accurate model will be monotonically decreasing in norm for all future samples

$$\left\|e(k+j)\right\|_\infty \le \sigma(k+j)\max\left(0,\left\|e(k)\right\|_\infty-\eta(k+j)\right) \quad \forall j>0,$$

where $\sigma(k+j)$ and $\eta(k+j)$ are give by (18). Since the error trajectories of the ε–accurate model converge monotonically to zero for any switching sequence, the resulting closed loop is exponentially stable.

Finally, for ε small enough, the stability of the ε–accurate implies the stability of the nonlinear dynamic over D [28]. □

Under this condition, a set of nested positively invariant sets and piece-wise Lyapunov functions

$$C(k+j+1)\subseteq C(k+j) \quad \forall j>0,$$
$$V(k+j+1)\subseteq V(k+j)$$

will exists, and the system trajectory will be confined to increasingly smaller regions, where the associated controller will be designed leading to control loops with response times that decrease, regulation proceeds much faster than if a single controller only were employed.

[3] The minimum accuracy of the adaptive scheme is given by the accuracy of the PLM, which depends on the number of models m. Assuming that the regimes at uniformly distributed over the operating space D, an upper bound of the number of models m_M required to have an accuracy ε was obtained by Angelis [28, chap. 4 pp. 36].

4. SIMULATIONS AND RESULTS

Consider the problem of controlling a continuously stirred tank reactor (*CSTR*) in which an irreversible exothermic reaction is carried out at constant volume. This is a nonlinear system originally used by Morningred *et al*. [13] for testing predictive control algorithms. The objective is to control the output concentration $Ca(t)$ using the coolant flow rate $q_C(t)$ as the manipulated variable. The reactor has a second output, the temperature of the reactor $T(t)$. The inlet coolant temperature $T_{CO}(t)$ (measurable) and the feed concentration $Ca_O(t)$ (non-measurable) represent the disturbances. The output concentration $Ca(t)$ has a measured time delay of t_d=0.5 *min*. The nonlinear nature of the system is shown in Fig. 6, where the open-loop response to changes in the manipulated variable is shown.

Fig. 6 shows the dynamic responses to the following sequence of changes in the manipulated variable $q_C(t)$: +10 $ltmin^{-1}$, –10 $ltmin^{-1}$, –10 $ltmin^{-1}$ the reactor control is quite difficult due to the change in the dynamics from one operational condition to another and the presence of zeros near the imaginary axis. Besides, the *CSTR* becomes uncontrollable when $q_C(t)$ goes to beyond 113 $ltmin^{-1}$.

We have chosen as operating space region the cube specified by

$$\left|Ca(t)-0.085\right| = 0.045\,mol\ lt^{-1},\ \left|T(t)-440\right| = 7.5\ {}^{\circ}K,$$
$$\left|q_C(t)-10\right| = 5\,lt\min^{-1}.$$

It is possible [28, chap. 4 pp. 36] to approximate the nonlinear model of the reactor within the specified working space, using m=4 linear models, leading to an estimate error ε=0.007. Using subspace identification techniques four discrete linear models for can be determined from the composition responses shown in Fig. 6. Notice that those changes imply three different operating points corresponding to the following stationary manipulated flow-rates: 100 $ltmin^{-1}$, 110 $ltmin^{-1}$, and 90 $ltmin^{-1}$. As in Morningred's work, the sampling time period was fixed at 0.1 *min*, which

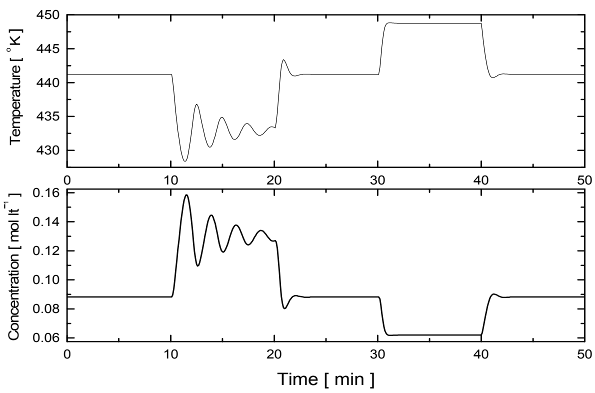

Fig. 6. Open-loop response of the CSTR.

Table 1. Vertices of polytopic model.

Operating conditions	Model
Model 1 $q_C = 100,\ \Delta q_C = 10$	$\frac{0.186\ \ 10^{-3} z^{-5}}{z^2 - 1.894z + 0.941}$
Model 2 $q_C = 110,\ \Delta q_C = -10$	$\frac{0.216\ \ 10^{-3} z^{-5}}{z^2 - 1.727z + 0.779}$
Model 3 $q_C = 100,\ \Delta q_C = -10$	$\frac{0.115\ \ 10^{-3} z^{-5}}{z^2 - 1.710z + 0.755}$
Model 4 $q_C = 90,\ \Delta q_C = 10$	$\frac{0.831\ \ 10^{-4} z^{-5}}{z^2 - 1.792z + 0.824}$

gives about four sampled-data points in the dominant time constant when the reactor is operating in the high concentration region.

Table 1 shows the four process transfer functions obtained. They define the polytopic linear model associated with the nonlinear behavior in the operating region being considered. They should be associated to the m vertex models in the above problem formulation.

The controller must be able to follow the reference and reject the disturbances present in this system having a settling time of 5 min for an error of 2%.

$$\begin{aligned} y(k) &\le 1.02 r_0 \quad \forall k > N_0, \\ \left|e(k)\right| &\le 0.02 r_0 \quad \forall k > N_0 + 50. \end{aligned} \tag{24}$$

where r_0 is the reference value and N_0 is the time of change. Besides, a zero-offset steady-state response is demanded for the steady-state controller

$$\sum\nolimits_{j=1}^{w} q_{j+v}(k) = 1 \quad \forall k > N_0 + 50. \tag{25}$$

To guarantee the system controllability over the whole operational region a hard constraint is physically used on the coolant flow rate at 110 $ltmin^{-1}$. Therefore, an additional restriction for the more sensitive model (Model 1 in Table 1) must be considered for the deviation variable $u(k)$

$$u_1(k) \le 10 \quad \forall k > N_0. \tag{26}$$

This assumes that the nominal absolute value for the manipulated variable is around 100 $ltmin^{-1}$ and the operation is kept inside the polytope whose vertices are defined by the linear models. The constraints (24)-(26) are then included in the optimization problem (7).

Now, define the parameters of the predictive feedback control law. The orders of the controller's polynomials are adopted arbitrarily such that the resulting controllers include the predictive version of popular *PID* controller (v=1 and w=2). The controller predictor $P(J, z^{-1})$ was built using the nonlinear model of the reactor. The prediction time J is chosen such that it guarantees the closed-loop stability for

Table 2. Prediction time for each region.

	Model 1	Model 2	Model 3	Model 4
J	12	11	9	10

each model. They were chosen such that they satisfy

$$a_J^l > \sum_{i=J+1}^{N} \left| h_i^l \right| \quad l = 1,\ldots,m. \tag{27}$$

The results are summarized in Table 2, where the prediction time for each model is shown.

The objective function employed to measure the closed-loop performance for each model of W is

$$f_l = \sum_{i=1}^{V} e_l^2(k+i,k) + \theta_l \Delta u_l^2(k+i) \quad l = 1,2,3,4, \tag{28}$$

where the time span is defined by V=150 and the control weight θ_l was fixed in a value such that the control energy has a similar effect than errors in the tuning process (θ_l=10^{-3} l=1, 2, 3, 4).

In these simulations the robust switching scheme with time-varying radius is employed. It uses the reactor temperature $T(k)$ as decision variable because it has no time delay. It is implemented using the index (11) with the parameters given by

$$\alpha_l = 0.6, \beta_l = 0.4 \quad l = 1,2,3,4.$$

To make more efficient the calculation of the indexes, the term corresponding to the moving average part of the index is implemented in a recursive way

$$Z_l(k) = \lambda_l(k)\varepsilon_l(k) + \left(1 - \lambda_l(k)\right) Z_l(k-1) \; Z_l(N_0) = 0.$$

Since the environment where the system operate is non stationary, a variable forgetting factor $\lambda_l(k)$ [30] is employed for each index, whose values are limited

$$0 \le \lambda_l \le 1 \quad l = 1,2,3,4.$$

The parameter ρ is set to 1.5 to increase the robustness of the system during the initial samples of the transitions between models and have a good transient response.

Finally, the optimization of the control law will be stopped when the system has achieved its settling time

$$\begin{aligned} &\left| e(i) \right| \le 0.02\, r_0 \qquad i = k, k-1, \ldots, k-5, \\ &\left| \Delta u(i) \right| \le 0.5. \end{aligned} \tag{29}$$

The multiple models and switching (MMS) controller was designed following the procedure proposed by Rantzer and Johansson [31]. The state space was divided in four regions or cells, with model 1 effective in cell 1 (Ca>0.09 and ΔCa>0), model 2 effective in cell 2 (Ca≥0.09 and ΔCa≤0), model 3 effective in cell 3 (Ca<0.09 and ΔCa<0) and model 4 effective in cell 4 (Ca<0.09 and ΔCa>0). A piecewise-quadratic Lyapunov function (PQLF) of form

$$V(x) = \left\{ x^T P_l x \text{ in cell } l \quad l = 1,2,3,4 \right.$$

is sought, where matrices P_i are parameterized so as to ensure that the function is continuous across the boundaries. Namely, following Johanson and Rantzer [32] the matrices P_i are parameterized as

$$P_l = F_l^T T F_l,$$

where the matrix T is to be determined and $F_l x$=$F_j x$ l≠j on the shared cells boundaries. Note that these matrices are not uniquely determined by the partition. This formulation relaxes the requirement of a common quadratic Lyapunov function in two ways. Firstly, we do not require a single positive-definite matrix P to simultaneously satisfy

$$A_l^T P + P A_l < 0 \quad l = 1,2,3,4.$$

Secondly, when implementing the search for such a function as a system of LMI, $x^T(A_l^T P + P A_l)x$ is not required to be negative for all non-zero x but only for those x in the cell i where the dynamics are given by the system matrix A_l. The problem of finding a PQLF for the system was formulated as a feasible problem for a system of LMIs and solved numerically [32]. The switching rule employed by this controller is similar to that one that is used by the adaptive predictive controller.

A robust MPC based on the worst-case minimization was developed to compare the closed-loop responses. The predictor was built using the model 2 assuming that the parameters are corrupted by some error ξ_i i=0, 1, …, p due to modelling error, i.e., a_i=a_{i2}+ξ_i i=0,1,…,p, such that a_i⊂[a_{i2}–ξ_i, a_{i2}+ξ_i]. The uncertainty bound ξ_i was calculated form the vertex of the polytopic models

$$\xi_i = \max_{j=1,3,4} \left(\left| a_{ji} - a_{2i} \right| \right) \quad i = 1,\ldots,p.$$

The error for the remaining parameters of the model has been computed in a similar way. Here it is assumed that the parameters' error is an independently identical uniform distributed variable. The remaining tuning parameters (the optimization horizon N, the control horizon N_U, control weight R, and the error weight Q) were set to

$$N = 200, N_U = 5, R = 510^{-3} I, Q = I.$$

The optimization problem was solved, at each step, using a min-max algorithm.

The simulation tests are similar to Morningred's work and consist of a sequence of step changes in the reference signal. The set point was changed in intervals of 10 min. from 0.09 $mollt^{-1}$ to 0.125, returns to 0.09, then steps to 0.055 and returns to 0.09 $mollt^{-1}$. Fig. 7 shows the results obtained when comparing the adaptive predictive controller with a standard MMS

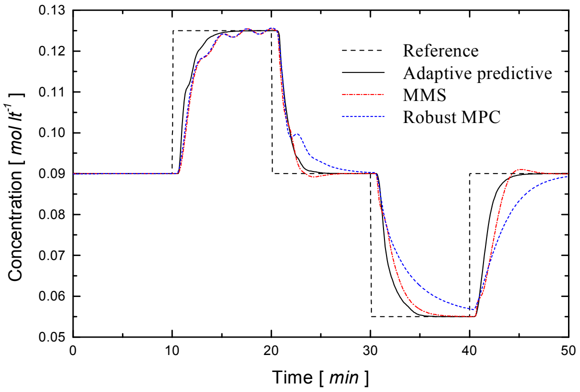

Fig. 7. Closed-loop responses to changes in setpoint.

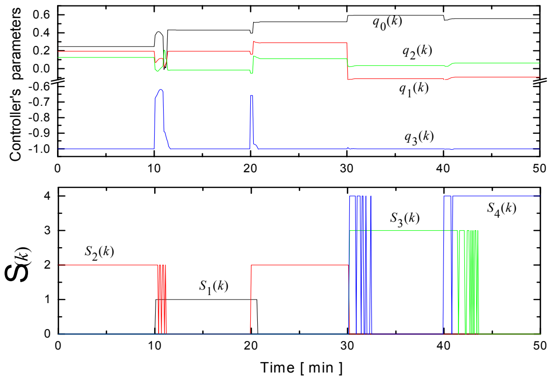

Fig. 9. Controller parameter sequence (upper figure) and switching-indices (lower figure) sequences.

and a robust MPC. The superior performance of the adaptive predictive controller proposed in this work is due to the combination of a switching scheme with the on-line design of the controller. In this way, the adaptive predictive controller is able to identify the local model and to optimize the closed-loop response whilst at the same time satisfying the constraints by modifying the controller's parameters (Fig. 9(a)).

The parameters of the adaptive controller are modified with changes in the reactor's operating region. They revealed an initial transient behavior, after each change, before achieving their steady state values (Fig. 9(a)). The major changes happen during the transitions from and to model 1 because the behavior of this model is different to others (see Fig. 6). This fact can be appreciated in the behavior of the switching variables, which show jitter during the first, third and fourth reference changes. These transitions correspond to changes to and from models 2, 3, and 4; which have similar dynamics and only differ in the gain.

The MMS shows a similar performance to the adaptive predictive controller; however the manipulated variable generated by the switching controller reveals bumps (Fig. 8) due to the switch between the different controllers. In this application, the bumps on the manipulated variable do not affect the output due to the high speed dynamics of the system.

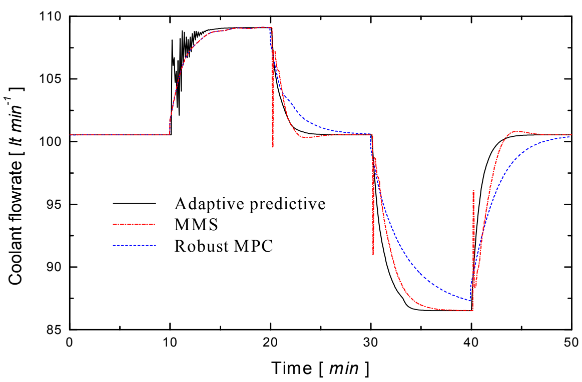

Fig. 8. Coolant flow rate inputs to setpoint changes.

5. CONCLUSIONS

A simple framework for the design of a robust predictive feedback controller with multiple models was presented. The approach was to relate the control law performance to the prediction of performance. The resulting controller identifies, at each sample, the closest linear model to the actual operational point of the controlled system, and reconfigures the control law so that it ensures robust stability of the closed-loop system. The reconfiguration of the controller is carried out by switching the function used to measure the closed-loop performance and the constraints.

The results obtained by simulating a continuously stirred tank reactor with significant non-linearities illustrate the effectiveness of the proposed controller.

APPENDIX A: ROBUST STABILITY CONDITION

Using the open-loop predictor (3) and the *FIR* representation of the system, the characteristic closed-loop equation with the predictive feedback controller (1) is given by

$$T(z^{-1}) = 1 + \sum_{n=1}^{w} q_{n+v} z^{-n} + \tilde{a}_J z^{-1} \sum_{n=0}^{v} q_n z^{-n} + \sum_{n=0}^{v} q_n z^{-n} \sum_{i=J+1}^{N} \tilde{h}_i z^{J-i}$$
$$+ \sum_{n=0}^{v} q_n z^{-n} \sum_{i=1}^{N} \left(h_i - \tilde{h}_i\right) z^{-i} + \sum_{n=0}^{v} q_n z^{-n} \sum_{i=N+1}^{\infty} h_i z^{-i}.$$

The stability of the closed-loop system depends on both: the prediction time J and the controller parameters (q_n n=0,…,v+w). It may be tested by any usual stability criteria. First, the following lemma is introduced.

Lemma 1: If the polynomial $T(z^{-1}) = \sum_{i=0}^{\infty} t_i z^{-i}$ has the property that

$$\inf_{|z|\geq 1} \left| T(z^{-1}) \right| > 0, \tag{30}$$

then the related closed-loop system will be asymptotically stable [21].

Applying Lemma 1 to characteristic closed-loop equation we have

$$\left|T(z^{-1})\right| \geq 1-\sum_{n=1}^{w}\left|q_{n+v}z^{-n}\right|-\tilde{a}_J\sum_{n=0}^{v}\left|q_n z^{-n-1}\right|-\sum_{n=0}^{v}\sum_{i=J+1}^{N}\left|q_n\tilde{h}_i z^{J-i-n}\right| -\sum_{n=0}^{v}\sum_{i=1}^{N}\left|q_n\left(h_i-\tilde{h}_i\right)z^{-i-n}\right|-\sum_{n=0}^{v}\sum_{i=N+1}^{\infty}\left|q_n h_i z^{-i-n}\right|.$$

The worst case happens when z=1, thus the closed-loop will be stable if

$$\begin{aligned}1-\sum_{n=1}^{w}\left|q_{n+v}\right|-\tilde{a}_J\sum_{n=0}^{v}\left|q_n\right|-\sum_{n=0}^{v}\left|q_n\right|\sum_{i=J+1}^{N}\left|\tilde{h}_i\right| \\ -\sum_{n=0}^{v}\left|q_n\right|\sum_{i=1}^{N}\left|h_i-\tilde{h}_i\right|-\sum_{n=0}^{v}\left|q_n\right|\sum_{i=N+1}^{\infty}\left|h_i\right|>0,\end{aligned} \tag{31}$$

which is equivalent to

$$\frac{1-\sum_{n=1}^{w}\left|q_{n+v}\right|}{\sum_{n=0}^{v}\left|q_n\right|}+\tilde{a}_J>\sum_{i=J+1}^{N}\left|\tilde{h}_i\right|+\sum_{i=1}^{N}\left|h_i-\tilde{h}_i\right|+\sum_{i=N+1}^{\infty}\left|h_i\right|. \tag{32}$$

The closed-loop stability depends on both parameters: the prediction time J and the controller parameters q_n n=0,1,…,v+w. So, for the controller design the prediction time J is fixed and then he parameters are tuned.

If the controller denominator verifies

$$\sum_{n=1}^{w}\left|q_{n+v}\right|=1 \tag{33}$$

the stability condition (32) becomes

$$\tilde{a}_J>\sum_{i=J+1}^{N}\left|\tilde{h}_i\right|+\sum_{i=1}^{N}\left|h_i-\tilde{h}_i\right|+\sum_{i=N+1}^{\infty}\left|h_i\right|. \tag{34}$$

This is condition was derived by Giovanini [19] for the predictive feedback controller. This equation mean that the prediction time J and controller parameters q_n n=0,1,…,v+w could be independently fixing. The same result is obtained if v=1 and q_{v+1}= –1.

APPENDIX B: PROPERTIES OF SUPER-STABLE SYSTEMS

Consider the local approximation to a system model is given by the discrete model

$$\begin{aligned}x(k+1)&=Ax(k)+Bu(k),\quad x(0)=x_0,\\ y(k)&=Cx(k),\end{aligned} \tag{35}$$

where $x(k)\in R^n$, $A\in R^{n\times n}$, $B\in R^{n\times m}$ and the ∞ norm for the vector and matrices are given by

$$\left\|x(k)\right\|_\infty=\max_{i\in[1,n]}\left(\left|x_i(k)\right|\right),\left\|A\right\|_\infty=\max_{i\in[1,n]}\sum_{j=1}^{n}\left|a_{ij}\right|.$$

Definition 1: The system (35) is *superstable* if $\left\|A\right\|_\infty<1$.

This is a sufficient condition introduced by some [20-22] for different linear models (polynomials, FIR and state space) and it was later applied to control problems by several authors [5,22-24].

Superstable systems enjoy numerous important properties [24], for this work the most relevant are:

Lemma 2: Superstability implies the existence of a positively invariant set.

Proof: Given a sequence of bounded amplitude inputs, the system states are given by

$$x(k+j)\leq A^j x(k)+\sum_{i=1}^{j}A^{i-1}Bu(k+j-i). \tag{36}$$

If the input sequence is bounded, $\left\|u(k)\right\|_\infty\leq 1\,\forall j>0$, the norm of the states is given by

$$\begin{aligned}\left\|x(k+j)\right\|_\infty&\leq\left\|A\right\|_\infty^j\left\|x(k)\right\|_\infty+\sum_{i=1}^{j}\left\|A\right\|_\infty^{i-1}\left\|B\right\|_\infty\quad\forall j>0\\ &\leq\eta+\sigma^j\max\left(0,\left\|x(k)\right\|_\infty-\eta\right),\end{aligned} \tag{37}$$

where $\sigma=\left\|A\right\|_\infty$ and $\eta=\left\|B\right\|_\infty/1-\sigma$.

This estimate (37) implies that the norm of the system states decreases monotonically[4]

$$\left\|\Delta x(k+j)\right\|_\infty\leq\sigma^j\max\left(0,\left\|x(k)\right\|_\infty-\eta\right)\forall j\geq 0, \tag{38}$$

and for any state that satisfied $\left\|x(k)\right\|_\infty\leq\eta$, the system states will be bounded by

$$\left\|x(k+j)\right\|_\infty\leq\eta\quad\forall j>0, \tag{39}$$

consequently, the cube

$$C=\left\{x\in R^n:\left\|x\right\|_\infty\leq\eta\right\}$$

is a positively invariant set. □

Lemma 3: Superstability implies the existence of a non quadratic Lyapunov function.

Proof: From (38) is clear that the system states monotonically decrease at the rate

$$\left\|\Delta x(k+j)\right\|_\infty\leq\left\|A\right\|_\infty^j\left(\left\|x(k)\right\|_\infty-\eta\right)\forall j>0, \tag{40}$$

when the states are outside of the invariant set $C,\forall x(k):\left\|x(k)\right\|_\infty>\eta$.

For the particular case of a system without inputs, $u(k)\equiv 0$, the states will be bounded by

$$\left\|\Delta x(k+j)\right\|_\infty\leq\left\|A\right\|_\infty^j\left\|x(k)\right\|_\infty. \tag{41}$$

It follows from this equation that superstable systems has the nonquadratic Lyapunov function

$$V\left(x(k)\right)=\left\|x(k)\right\|_\infty. \tag{42}$$

[4] This property can be extended to inputs with outliers $\left\|x(k)\right\|_\infty<1\ \ \forall k\neq N,\ \left\|x(N)\right\|_\infty>1$.

This function grows linearly in any direction, $V(\lambda x)=\lambda V(x) \quad \forall x \in R^n, \forall \lambda \geq 0,$ and it is piecewise-linear and non differentiable, however it has lateral derivatives. At the same time, it has also the properties of the conventional Lyapunov function: $V(x)\geq 0$, $V(x) = 0$ for $x= 0$, it is convex and grows to infinity.

These results can be easily extended to time varying systems, as well in the presence of time varying and nonlinear perturbations. In this case, the following condition must satisfied for all k

$$\begin{aligned} &\|A(k)\|_\infty \leq r<1, \\ &\|f(x(k),k)\| \leq \mu+\nu\|x(k)\|_\infty, \quad 0 \leq \nu<1-r, \end{aligned} \tag{43}$$

which lead to the results showing in (37)-(42) with parameter η and σ given by

$$\sigma=r+\nu, \quad \eta=\frac{\mu}{1-\sigma}. \tag{44}$$

The proof follows literally the same lines as of Lemma 2 and 3. □

Lemma 4: The superstability of systems employed to build a polytopic linear model implies the superstability of time varying system and the existence of a common Lyapunov function.

Proof: Given a polytopic linear model [28]

$$\begin{aligned} x(k+1) &= \sum_{l=1}^{m} \omega_l(z)\left(A_l x(k)+B_l u(k)\right), \\ y(k) &= Cx(k), \end{aligned}$$

where $\omega_l(z(k))$ $l=1,\ldots,m$ are scheduling functions

$$\begin{aligned} &\omega_l(z(k)) \geq 0 \quad \forall (u,z)\in \mathcal{D}, l=1,\ldots,m, \\ &\sum_{l=1}^{M} \omega_l(z(k))=1 \quad \forall k \end{aligned}$$

and $z(k)$ is a general scheduling variable [28]. The superstability of individual systems

$$\|A_l\|_\infty <1 \quad l=1,\ldots,m$$

implies the superstability of the time varying system

$$\|A(k)\|_\infty=\left\|\sum_{l=1}^{m}\omega_l(z)A_l\right\|_\infty \leq \sum_{l=1}^{m}\left\|\omega_l(z)\right\|_1\left\|A_l\right\|_\infty<1.$$

Since each system (A_l,B_l) $l=1,\ldots,m$ is superstable they admit a Lyapunov function $V_l(x)=\|x\|_\infty$ $l=1,\ldots,m$, such that

$$\begin{aligned} V(k+1)&=\left\|\sum_{l=1}^{m}\omega_l(z)A_l x(k)\right\|_\infty \leq \sum_{l=1}^{m}\omega_l(z)\|A_l\|_\infty\|x(k)\|_\infty \\ &\leq \tilde{\omega}\|x(k)\|_\infty \qquad \tilde{\omega}=\sum_{l=1}^{m}\omega_l(z)\|A_l\|_\infty. \end{aligned}$$

and a positively invariant set

$$\mathcal{C}=\left\{x\in R^n : \|x\|_\infty \leq \eta(k)\right\} \quad \eta(k)=\sum_{l=1}^{m}\omega_l(z)\eta_l. \quad □$$

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