Adaptive Synchronization of Chaotic Systems Based on Speed Gradient Method and Passification

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***Abstract*—A problem of synchronizing two nonlinear multidimensional systems with unknown parameters is considered. Two general procedures for adaptive synchronization law design based on speed-gradient method are proposed. Conditions ensuring the synchronization are given. The second procedure is based on passification of error system (making it passive by feedback). The results are illustrated by examples: synchronizing a pair of Chua's circuits and a pair of circuits with tunnel diodes. Computer simulation results confirming theoretical analysis are given.**

I. Introduction

During recent years the growing interest was observed in the problem of synchronizing chaotic systems [2], [13]. It was motivated not only by scientific interest in the problem, but also by practical applications in different fields, particularly in telecommunications [3], [10], [11]. However most design methods were suggested and justified under conditions that all the system parameters are known and states are available for measurement. Some methods apply only to low dimensional systems.

Of practical interest is the problem of synchronizing two or more systems when not only initial state but also values of some parameters are not available to the designer of synchronization device. This more complicated problem, which corresponds to the real situations, will be referred to as one of adaptive synchronization [4], [6], [7], [15].

This paper gives the brief exposition of the recent results on adaptive synchronization of chaotic systems obtained by the so called speed-gradient (SG) method [4], [8]. The speed-gradient method was used previously in nonlinear and adaptive control and successfully applied recently to control of chaotic systems [1], [4], [5], [16]. It has been shown in the paper how to formulate and solve the controlled synchronization problem in the speed-gradient framework. For systems with more specific structure (satisfying weakened matching conditions) the design of output feedback synchronization algorithm is suggested based on passification approach (rendering the system passive by feedback) [19]. The proposed methods are illustrated by examples extending the previous results of [5], [6], [9].

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II. Synchronization by Speed-Gradient Algorithms

A. Synchronization and Adaptive Synchronization Problems

To describe general procedures of synchronization algorithms design by speed-gradient methods consider two interconnected systems

$$\dot{x}_i = F_i(x_i, u, t) \tag{1}$$

where $i = 1, 2$, $x_i \in R^n$ are state vectors, $u \in R^m$ is interconnection (or coupling) signal.

The problem is to choose synchronization algorithm

$$u = U(x_1, x_2, u, t) \tag{2}$$

or adaptive synchronization algorithm

$$u = U(x_1, x_2, \theta, t) \tag{3}$$
$$\dot{\theta} = \Theta(x_1, x_2, \theta, t) \tag{4}$$

where $\theta \in R^N$ is vector of adjustable parameters ensuring the synchronization goal[1]

$$x_1(t) - x_2(t) \to 0 \quad \text{when } t \to \infty. \tag{5}$$

If we interpret coupling signal $u(t)$ as control input, then both synchronization and adaptive synchronization problems can be considered as special cases of the following control problem.

B. Description of the Speed-Gradient Method

Consider the controlled system equation in the state space form

$$\dot{x} = F(x, u, t) \tag{6}$$

where $x \in R^n$ is a state vector, $F(x, u, t)$ is vector-function, continuously differentiable in (x, u) and piecewise continuous in t, $\dot{x}$ stands for dx/dt and $u \in R^m$ is input vector.

Consider the problem of finding the control law $u(t)$, ensuring the control goal

$$Q(x(t), t) \to 0, \qquad \text{when } t \to \infty. \tag{7}$$

where $Q(x, t) \geq 0$ is a smooth objective function.

[1] More general synchronization problem statement can be found in [12].

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One can determine a function $\omega(x,u,t)$ as a speed of changing $Q(x,t)$ along the trajectory of system (6) as follows: $\omega(x,u,t) = (\nabla_x Q)^T F(x,u,t) + \partial Q/\partial t$.

Then the algorithm in the form:

$$\frac{d}{dt}(u + \psi(x,u,t)) = -\Gamma \nabla_u \omega(x,u,t), \tag{8}$$

where $\Gamma = \Gamma^T > 0$ is $m \times m$ gain matrix and $\psi(x,u,t)$ forms sharp angle with the speed-gradient $\nabla_u \omega(x,u,t)$ $(\psi^T \nabla_u \omega \geq 0)$ is called the *speed-gradient algorithm in combined form* [8].

The main special cases of (8) are speed-gradient algorithms in differential form

$$\frac{du}{dt} = -\Gamma \nabla_u \omega(x,u,t) \tag{9}$$

and in finite form:

$$u = -\psi(x,u,t). \tag{10}$$

The typical forms of algorithm (10) are linear $u = -\Gamma \nabla_u \omega(x,u,t)$ and relay ones $u = -\Gamma\ \mathrm{sign}(\nabla_u \omega(x,u,t))$, where components of vector $\mathrm{sign}(z)$ are signs of the corresponding components of vector z.

The stability theorems for speed-gradient systems (6), (8) can be found in [8] and [16], [17]. For our purposes the following two results are useful which can be derived from the results of [17].

Theorem 1: (for combined form of SG-algorithm)*:* Assume that the right-hand sides (RHS) of the system (6), (8) are bounded together with derivatives in any region where $Q(x,t)$ is bounded. Assume also that $\omega(x,u,t)$ is convex in u and the following conditions are valid:

1) Attainability condition: there exist $u_* \in R^m$ and scalar continuous function $\rho(Q)$, with property $\rho(Q) > 0$ when $Q > 0$ such that inequality $\omega(x,u_*,t) \leq -\rho(Q(x,t))$ holds.
2) Growth condition: boundedness of $Q(x,t)$ implies boundedness of x.

Then all solutions of the system (6), (8) are bounded and the goal (7) is achieved.

Theorem 2: (for finite form of SG-algorithm): Assume that function $Q(x,t)$ is smooth and the RHS of the system (6) are bounded together with derivatives in any region where $Q(x,t)$ is bounded. Assume that (10) is solvable for u for any $x \in R^n$ and the solution of the system (6), (10) exists locally for any initial $x(0) \in R^n$. Assume that $\omega(x,u,t)$ is convex in u and the following conditions are valid:

1) Attainability condition: There exist $u_*(x,t) \in R^m$ and scalar continuous function $\rho(Q)$, with property $\rho(Q) > 0$ when $Q > 0$ such that inequality $\omega(x,u_*(x,t),t) \leq -\rho(Q(x,t))$ holds.
2) Growth condition: boundedness of $Q(x,t)$ implies boundedness of x.

Then all solutions of system (6), (10) are bounded and the goal (7) is achieved if $\psi(x,u,t)^T \nabla_u \omega(x,u,t) \geq \gamma \|\nabla_u \omega(x,u,t)\|$, where $\gamma \geq \sup \|u_*(x,t)\|/\|\nabla_u \omega(x,u,t)\|$.

In adaptive synchronization problems when the generalized controlled system equation (6) consists of two subsystems the speed-gradient method can be used for design of synchronization (the main loop) and adaptation (the adaptation loop) algorithms as it is shown in the next section. Note that a number of existing control, adaptation and synchronization algorithms can be considered and analyzed in the speed-gradient framework, e.g., some algorithms proposed in [14], [15].

C. Synchronization and Adaptive Synchronization by the Speed-Gradient Algorithms

The speed-gradient method yields the following procedure of synchronization algorithms design.

Step 1. Choose the goal function $\bar{Q}(x) \geq 0,\ x \in R^n$, such that the synchronization goal (5) can be expressed as

$$\bar{Q}(x_1(t) - x_2(t)) \to 0 \quad \text{when } t \to \infty, \tag{11}$$

and growth conditions of Theorems 1 and 2 are fulfilled. For example the growth conditions hold if $\bar{Q}(x)$ is radially unbounded ($\bar{Q}(x) \to \infty$ when $\|x\| \to \infty$) and trajectory of one system, say x_2, is bounded. Then the state of the controlled system is introduced as $x = x_1$ and function $Q(x,t) = \bar{Q}(x - x_2(t))$. In many cases the good choice is to take quadratic form $\bar{Q}(x_1 - x_2) = (x_1 - x_2)^T P (x_1 - x_2)$, where $P = P^T > 0$.

Step 2. Calculate speed $\dot{Q} = \omega(x_1, x_2, u, t)$ and speed-gradient $\nabla_u \dot{Q} = \nabla_u \omega$ of the goal function.

Step 3. Check conditions of the Theorem 2 for finite form of speed-gradient algorithm (10). If speed-gradient $\nabla_u \omega$ depends only on measurable variables and known parameters then the problem is solved.

Step 4. If speed-gradient algorithm $\nabla_u \omega$ depends on vector of unknown parameters $\xi \in R^N$ then replace ξ by the vector of adjustable parameters $\theta \in R^N$ and consider the system (1) together with algorithm

$$u = -\psi(\nabla_u \omega(x_1, x_2, \theta, t)) \tag{12}$$

as new system of form (4). Choose adaptation algorithm (4) by speed-gradient method applied to the new system (1), (12) and the goal (11) in assumption that vector θ is new input. If conditions of Theorem 1 are fulfilled and adaptation algorithm depends only on measurable variables and known parameters then the problem is solved.

Examples illustrating how to ensure conditions of the Theorems 1 and 2 are given in Section II-C3. Sometimes extra conditions should be imposed for that purpose. For example, to provide growth condition two additional assumptions can be imposed

1) trajectories of both systems $x_i(t)$ are bounded for bounded input $u(t)$ (so-called bounded-input-bounded

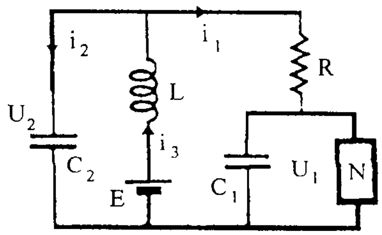


Fig. 1. Chaotic generator on tunnel diode.

state (BIBS) condition which is a sort of stability condition);

2) algorithm (10) (or (12) ensures boundedness of $u(t)$ (e.g., ψ is bounded or $\psi = \lambda(x_1 - x_2)\bar{\psi}$ where $\bar{\psi}$ is bounded and $|\lambda(x)| \to \infty$ for $x \to \infty$.

If the above procedure does not apply directly then some natural extensions can be made. For example if synchronization algorithm depends on nonavailable states then observer can be added to the system [18]. If conditions of Theorems 1 and 2 hold only in some region Ω of the overall system state space $\{x_1, x_2, \theta\}$ then the design parameters (gain matrix Γ, weighting matrix P, etc.) need to be chosen to ensure that the trajectories belong to Ω for all $t \geq 0$. The above procedure can be extended to solve synchronization problem for $k > 2$ interconnected subsystems in (1). Assuming that trajectory of one of the systems is bounded the goal function can be taken, e.g., in the form $Q(x_1, x_2, \ldots, x_k) = \|x_1 - x_2\|^2 + \cdots \|x_1 - x_k\|^2$.

D. Speed-Gradient Synchronization of Chaotic Generators Basedon Tunnel Diodes

Consider a chaotic generator (Fig. 1)—the reference model —based on tunnel diode [9] which is described by

$$\begin{cases} \dot{u}_{1m} = C_{21m}(u_{2m} - u_{1m} - i_m(u_{1m})R_m)/q_m, \\ \dot{u}_{2m} = (u_{1m} - u_{2m} + i_{3m}R_m)q_m, \\ \dot{i}_{3m} = q_m(E_m - u_{2m})/R_m, \end{cases} \tag{13}$$

where $C_{21m} = C_{2m}/C_{1m}$, $q_m = R_m/\omega_{2m}L_m$, $R_m > 0$, E_m are system parameters, u_{1m}, u_{2m}, i_{3m} are state variables and $i_m(u_{1m})$ is the nonlinear voltage versus current characteristics of the tunnel diode. In this paper we use a simple cubic approximation of the NDC (negative differential conductivity) region in the form

$$i_m(u_{1m}) = a_{1m}(u_{1m} - b_m)^3 - a_{2m}(u_{1m} - b_m) + a_{3m} \tag{14}$$

where b_m characterizes the initial shift on tunnel diode.

Denoting $G_m = 1/R_m$ rewrite system (13) in the form

$$\begin{cases} \dot{u}_{1m} = p_m((u_{2m} - u_{1m})G_m - i_m(u_{1m})) \\ \dot{u}_{2m} = ((u_{1m} - u_{2m})G_m + i_{3m})/d_m, \\ \dot{i}_{3m} = d_m(E_m - u_{2m}) \end{cases} \tag{15}$$

where $p_m = C_2\omega_2 L/C_1$, $d_m = 1/\omega_2 L$, E_m, $i_m(u_{1m})$ are independent from G_m.

The second system (controlled system) differs from the reference model by a field-effect transistor operating in the linear regime introduced instead of linear resistor R_m

$$G = \frac{1}{R} = \frac{1 - \sqrt{\frac{u - u_b}{u_0}}}{R_0} \tag{16}$$

where R_0 is an undepleted channel resistance, u_b is a built-in voltage, u_0 is a pinch off voltage, u is the control voltage. The system model is given by

$$\begin{cases} \dot{u}_1 = p((u_2 - u_1)G - i(u_1)) \\ \dot{u}_2 = ((u_1 - u_2)G + i_3)/d, \\ \dot{i}_3 = d(E - u_2), \end{cases} \tag{17}$$

where $i = i(u_1) = a_1(u_1 - b)^3 - a_2(u_1 - b) + a_3$.

Suppose that control voltage satisfies the inequality $u_b \leq u \leq u_0$ (we can achieve it by choosing the parameters of control algorithms, which will be obtained later) and that p_m, d_m, G_m, a_{1m}, a_{2m}, a_{3m}, b_m are unknown[2]. Choose the control goal as follows:

$$Q(u_1 - u_{1m}) = 0.5(u_1 - u_{1m})^2 \to 0, \quad \text{when } t \to \infty. \tag{18}$$

To design control algorithm calculate the derivative of $Q(\cdot)$ along the trajectory of the system (17), (15). Synchronization goal (18) can be achieved if the value of the control parameter $G = G_*$ satisfy inequality: $\omega(\cdot, G_*) < 0$. Assume[3] that $u_1 \neq u_2$ and take

$$G_* = (p_m((u_{2m} - u_{1m})G_m - i_m(u_{1m})) - \beta(u_1 - u_{1m}) + pi(u_1))/(p(u_2 - u_1)), \tag{19}$$

where $\beta > 0$.

To simplify the control algorithm design, linearize the nonlinear characteristics of the transistor (16) near the operation point $u_0/2$

$$G = \frac{1}{R} = \frac{1 - \alpha(u - u_b)/u_0}{R_0} \tag{20}$$

where α can be found from the condition: derivative $\partial G/\partial u$ should be equal before and after linearization.

Using (19) and (20) we obtain the ideal control law

$$u_* = (p_m((u_{1m} - u_{2m})G_m + i_m(u_{1m})) + \beta(u_1 - u_{1m}) - pi(u_1))\frac{u_0 R_0}{p(u_2 - u_1)\alpha} + \frac{u_0}{\alpha} + u_b \tag{21}$$

which satisfies inequality $\omega(\cdot, G(u_*)) < 0$. However this algorithm is inapplicable because of its dependence on unknown parameters p_m, G_m, i_m.

According to speed gradient method the real control, independent of unknown parameters and ensuring the synchronization aim (18), can be obtained in the two forms.

1) Nonadaptive Synchronization Algorithm: According to speed-gradient method [8] we obtain the following control algorithm:

$$u = -\gamma \nabla_u \omega(\cdot) = \gamma \frac{(u_1 - u_{1m})(u_2 - u_1)p\alpha}{u_0 R_0} \tag{22}$$

where $\gamma > 0$ is a gain coefficient.

[2] The more simple problem was considered in [9].

[3] It is possible because the trajectory of the reference model strongly influences the trajectory of the controlled system and can be chosen in such way that this situation would not appear at all.

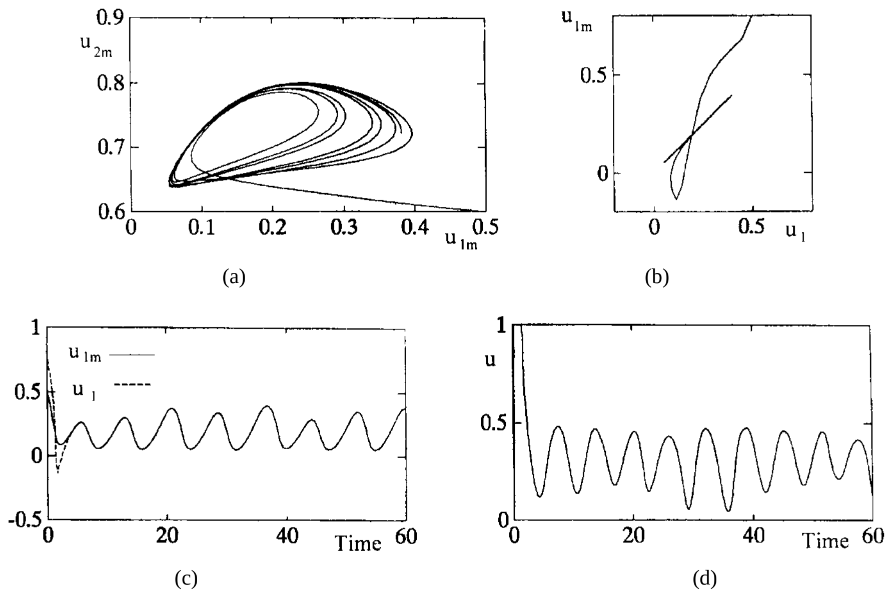


Fig. 2. (a) Phase plot (u_{1m}, u_{2m}). (b) Phase plot (u_1, u_{1m}). (c) Trajectories of outputs u_1 and u_{1m}. (d) Trajectory of control signal $u(t)$.

2) Synchronization Algorithm with Parametric Adaptation: According to adaptive control theory [8] the ideal control (21) also can be applied after some transformation. We replace the unknown parameters $p_m, d_m, G_m, a_{1m}, a_{2m}, a_{3m}, b_m$ in (21) and $i_m(u_{1m})$ by their estimates k_i

$$u = \frac{k_1(u_{1m} - u_{2m}) + k_2 u_{1m}^3 + k_2 u_{1m}^2 + k_4 u_{1m} + k_5}{u_2 - u_1} + \frac{(\beta(u_1 - u_{1m}) - pi)u_0 R_0}{p\alpha(u_2 - u_1)} + \frac{u_0}{\alpha} + u_b. \quad (23)$$

Noting that for implementation of the algorithm (23), u_{2m} must be measurable together with u_{1m}, which is undesirable, replace the expression $(u_{1m} - u_{2m})$ by its mean value $\delta = \mathrm{const}$ (it is supposed that the error may be canceled by tuning the parameter k_1) and join the two coefficients k_1, k_5 in one k_1.

$$u = \frac{k_1 + k_2 u_{1m}^3 + k_2 u_{1m}^2 + k_4 u_{1m}}{u_2 - u_1} + \frac{\beta(u_1 - u_{1m}) - pi)u_0 R_0}{p\alpha(u_2 - u_1)} + \frac{u_0}{\alpha} + u_b. \quad (24)$$

The adaptation algorithms for the tunable parameters k_i could be obtained using the differential form of speed-gradient algorithm [8] (we consider the generalized system (17), (15), (20), (24) with input signals $k_i(t)$ and use differential form of SG-algorithm because the ideal values of k_i are constant). Calculate the derivative of $\omega(\cdot)$ along k_i taking into account (20) and (24)

$$\begin{cases} \dot{k}_1 = \gamma_1 \alpha p(u_1 - u_{1m})/u_0 R_0 \\ \dot{k}_2 = \gamma_2 \alpha p(u_1 - u_{1m}) u_{1m}^3 / u_0 R_0 \\ \dot{k}_3 = \gamma_3 \alpha p(u_1 - u_{1m}) u_{1m}^2 / u_0 R_0 \\ \dot{k}_4 = \gamma_4 \alpha p(u_1 - u_{1m}) u_{1m} / u_0 R_0 \end{cases} \quad (25)$$

where $\gamma_i > 0$.

The achievement of the synchronization goal by the algorithms (22) and (24), (25) follows from stability theorems for speed-gradient algorithms. However stability and performance of simplified algorithm (24), (25) needs further analysis by means of computer simulation.

To examine the efficiency of the proposed algorithms computer simulations were carried out. The overall system consisted of

1) Reference model (15) with parameters $G_m = 100/3$, $p_m = 0.068$, $d_m = 125$, $E_m = 0.72$, $u_{1m}(0) = 0.5$, $u_{2m}(0) = 0.6$, $i_{3m}(0) = 5$, $a_{1m} = 192.3$, $a_{2m} = -43.6$, $a_{3m} = 12$, $b_m = -0.37$;
2) Controlled system (17) with parameters $p = 0.08$, $d = 115$, $E = 0.73$, $G(0) = 100/3$, $u_1(0) = 0.8$, $u_2(0) = 0.67$, $i_3(0) = 7$, $a_1 = 190$, $a_2 = -40$, $a_3 = 10$, $b = -0.4$, $R_0 = 0.015$
3) Characteristic of the transistor channel resistance (16): $u_b = 0.001$, $u_0 = 14$;
4) Control algorithms, nonadaptive (22): $\alpha = 0.7$, $\gamma = 5$, and adaptive (24), (25): $\alpha = 0.7$, $\gamma_1 = 10$, $\gamma_{2,3,4} = 1$, $\beta = 100$, $k_i(0) = 0$.

Simulation results are shown in Figs. 2 and 3.

Fig. 2(a), phase plot (u_{1m}, u_{2m}), shows autonomous behavior of reference model after a transient process which is caused by location of initial conditions outside of its attractor.

Figs. 2(b) and 3(a), phase plots (u_1, u_{1m}), and Figs. 2(c) and 3(b), trajectories of outputs u_1 and u_{1m}, show that after transient process output trajectories of systems moves synchro.

Figs. 2(d) and 3(c) presents the trajectories of control signal $u(t)$.

The results may be summarized as follows.

1) The control signal u may be put into the region $u_b \leq u(t) \leq u_0$ by proper choice of parameters γ, β, R_0.
2) Algorithm (24), (25) provides better synchronization than the algorithm (22) but it is more complex and sensitive to the trajectory of the reference model.
3) Efficiency of both algorithms is independent from initial conditions.

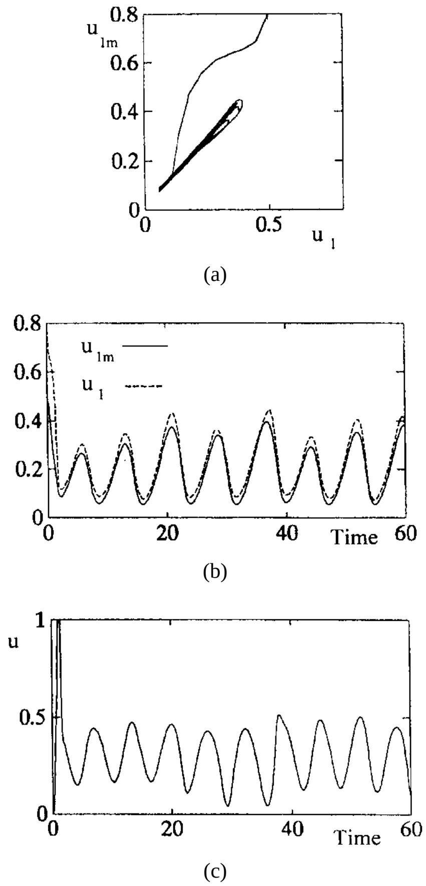


Fig. 3. (a) Phase plot (u_1, u_{1m}). (b) Trajectories of outputs u_1 and u_{1m}. (c) Trajectory of control signal $u(t)$.

III. Adaptive Synchronization of Passifiable Nonlinear Systems

A. Adaptive Synchronization: Output Feedback Design

In this section we consider the problem of synchronizing the two nonlinear systems by output feedback.

The system models are special case of those considered in Section II and looks as follows:

$$\dot{x}_i = f_i(x_1, x_2, t) + B_i u, \quad y_i = Cx_i, \tag{26}$$

where $i = 1, 2$, $x_i \in \mathcal{R}^n$ are state vectors, $y_i \in \mathcal{R}^l$ are measurable outputs, f_i are some functions consisting of linear and nonlinear parts, C is some constant matrix (we assume that outputs of systems are of similar type), B_i $(B_1 \neq B_2)$ are gain matrices, $u \in \mathcal{R}^m$ is a vector of control variables.

The synchronization goal is formulated as follows:

$$\lim_{t\to\infty} e(t) = 0 \tag{27}$$

where $e(t) = x_1(t) - x_2(t)$ is an error vector.

Suppose that some parameters of linear and nonlinear parts of (26) unknown to the designer of the synchronization algorithm. In other words, they depend on some vector of unknown parameters $\xi \in \Xi$, where Ξ is some known set.

The problem is to determine the control law using only measurable variables y_i and some information about nonlinearities such that the aim (27) is achieved for all $\xi \in \Xi$.

To solve the problem write down the error equation:

$$\dot{e} = Ae + \Phi(x_1, x_2, t) + Bu \tag{28}$$

where Φ is some function, consisting of linear and nonlinear parts, A is linear part of the error equation, $B = B_1 - B_2$. Now impose the main restriction on the class of the problems: suppose that the following representation is valid:

$$\Phi = \sum_{k=1}^{m} B_k \left[\xi_k^T z_k(x_1, x_2, t) + v_k(x_1, x_2, t)\right] \tag{29}$$

where B_k are the columns of matrix B, $\xi_k \in \mathcal{R}^N$ are vectors of unknown parameters and the values of vector-functions $z_k(\cdot) \in \mathcal{R}^N$ and scalar functions $v_k(\cdot)$ are measurable. Assumption (29) means that all the nonlinearities and uncertainties act in span of the control. It does mean however (unlike the standard matching conditions) neither that the unknown parameters appear linearly in the model, nor that all the uncertainties can be canceled by the proper choice of control (because the term with A in RHS of (28) may not be cancelable). Therefore (29) may be called *weakened matching condition*.

To solve the posed problem choose the following natural structure of adaptive controller:

$$u_k = \theta_{0k}^T (y_1 - y_2) + \theta_{1k}^T z_i(x_1, x_2, t) - v_k(x_1, x_2, t) \tag{30}$$

where $\theta_{0k} \in \mathcal{R}^l$, $\theta_{1k} \in \mathcal{R}^N$ are vectors of adjustable parameters. The adaptation algorithm is based on speed-gradient method and looks as follows:[4]

$$\theta_{jk}(t) = -\psi_{jk}(w_{jk}(t)) - \Gamma_{jk} \int_0^t w_{jk}(s)\, ds \tag{31}$$

where $j = 0, 1$; $w_{0k} = g_k^T(y_1 - y_2)(y_1 - y_2)$; $w_{1k} = g_k^T(y_1 - y_2) z_k$, $\Gamma_{jk} = \Gamma_{jk}^T \geq 0$ are gain matrices, $g_k \in \mathcal{R}^l$ are columns of some matrix G and $\psi_{jk}(w)^T w \geq 0$ for all w (for example, $\psi = w$ or $\psi = \mathrm{sign}(w)$).

The applicability conditions of the proposed algorithm and the value of matrix G can be obtained from the theorem below. Start with the following definition.

Definition [8]: System $\dot{x} = Ax + Bu$, $y = Cx$, where $u \in \mathcal{R}^m$, $y \in \mathcal{R}^l$ is called *hyper-minimum-phase* if it is *minimum-phase* (i.e., the polynomial $\varphi(\lambda) = \det(\lambda I - A) \det W(\lambda)$, where $W(\lambda) = C(\lambda I - A)^{-1} B$ is stable) and the matrix $CB = \lim_{\lambda\to\infty} \lambda W(\lambda)$ is symmetrical and positive definite.

Theorem 3: Choose $l \times m$-matrix G with columns $g_k, k = 1, \ldots, m$ such that the system with transfer function $W(\lambda) = G^T C(\lambda I - A)^{-1} B$ is hyper-minimum-phase for all $\xi \in \Xi$ and take the adaptation algorithm (31).

Then trajectories of the system (31) and error vector $e(t)$ are bounded and the following synchronization aim is achieved:

$$\int_0^{t_*} \|e(t)\|^2\, dt < \infty \tag{32}$$

where t_* is maximal time of existence of solution of (26), (30), (31).

[4] Special cases of the proposed algorithm—the differential and the finite form are also applicable.

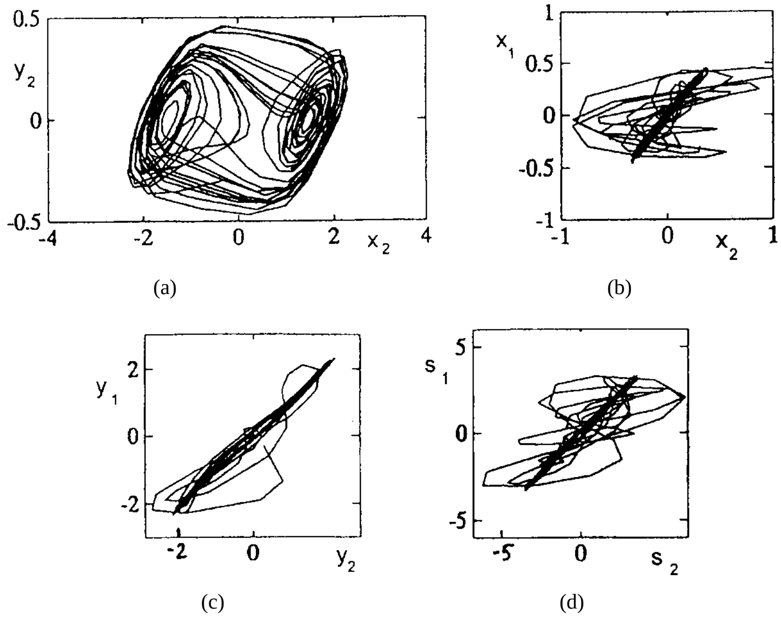


Fig. 4. (a) Phase portrait (x_2, y_2) of the system which is not influenced by control. (b), (c), (d) Phase plots (x_2, x_1), (y_2, y_1), (s_2, s_1).

Moreover, if function $z_k(x_1, x_2, t)$ is bounded in any region $\{(e,t) : ||e|| \le r, \, t \ge 0\}$ then trajectories of the system (28), (30) are also bounded and the aim (27) is achieved.

The proof of the Theorem 3 is based on Lyapunov function

$$V(x,\theta,t) = \frac{1}{2}e^T Pe + \frac{1}{2}\sum_{k=1}^{m}\left[\left\|\theta_{0k} + \psi_{0k} - \theta_{0k}^*\right\|^2_{\Gamma_0^{-1}} + ||\theta_{1k} + \psi_{1k} - \xi_k||^2_{\Gamma_1^{-1}}\right] \quad (33)$$

and the following lemma:

Lemma 1 [8]*:* Assume $\operatorname{rank}(B) = m$. Then there exist positive definite $n \times n$ matrix $P = P^T > 0$, the $l \times m$ matrix G and the $m \times l$ matrix θ_* such that $PA_* + A_*^T P < 0$, $PB = C^T G$, $A_* = A + B\theta_* C$ if and only if the system $\dot{x} = Ax + Bu$, $y = G^T Cx$ is hyper-minimum-phase.

Lemma 1 establishes conditions of existence of feedback $u = \theta_* y + v$ making the closed loop system with input v and output Gy strictly passive. It is closely related to the Kalman-Yakubovich lemma and can be called "Feedback Kalman-Yakubovich lemma," see [8]. Theorem 3 shows that hyper-minimum-phaseness condition guarantees existence of the feedback making the closed loop system strictly passive with the storage function (33). Therefore the design of the synchronization system proposed in this section is based on making the system passive by feedback. Such an approach was called a passification [19].

Now write down the procedure of adaptive synchronization algorithm design.

Step 1. Write down the error equation (28).

Step 2. Determine function Φ in the form (29).

Step 3. Write control algorithm in the form (30) and matrices θ_{0k}, θ_{1k}.

Step 4. Choose the adaptation algorithm in general (31) (choose function ϕ_{jk}, see Section II.A), differential $(\psi_{jk} = 0)$ or finite $(\Gamma_{jk} = 0)$ forms.

Step 5. Using Theorem 3 obtain matrix G such that the system with transfer function $W = G^T C(\lambda I - A)^{-1} B$ is hyper-minimum phase.

Step 6. Check the other conditions of the Theorem 3 to determine the synchronization goal which can be achieved utilizing the algorithm obtained on steps 3 and 4.

Step 7. Choose matrices $\Gamma_{jk} = \Gamma_{jk}^T \ge 0$.

One can apply the proposed procedure to design synchronization law for systems with different structure as it was shown in [7]. Here we consider the more complicated problem: synchronizing Chua's circuits with unknown parameters and incomplete measurements.

B. Synchronization of Chua Circuits

Take a pair of Chua circuits which is given by [2]:

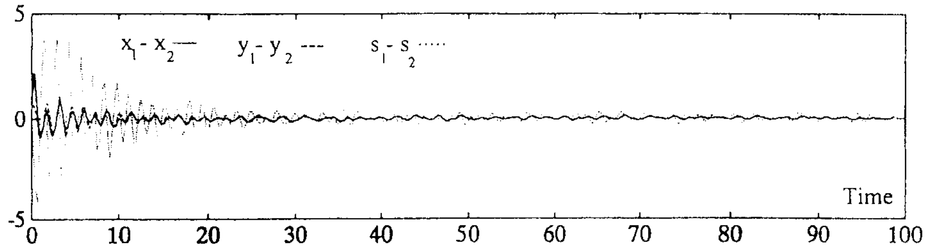
$$\begin{cases} \dot{w}_i = A_i w_i + D_i f_i(w_i) + B_i u, \\ v_i = C w_i \end{cases} \quad (34)$$

$$f_i(w_i) = M_{0i} x_i + 0.5(M_{1i} - M_{0i})(|x_i + 1| - |x_i - 1|),$$

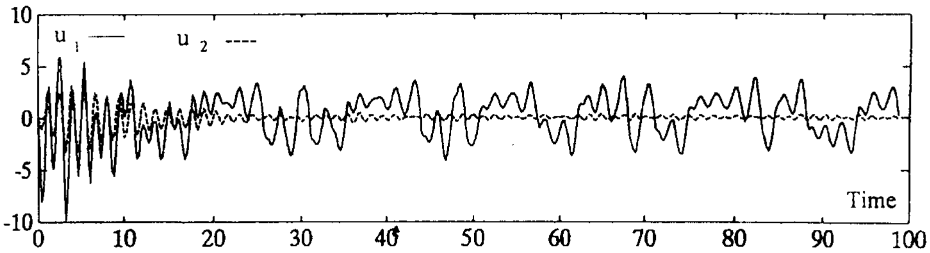
where

$$i = 1,2,\; C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\; A_i = \begin{pmatrix} 0 & p_i & 0 \\ 1 & -1 & 1 \\ 0 & -q_i & 0 \end{pmatrix}$$

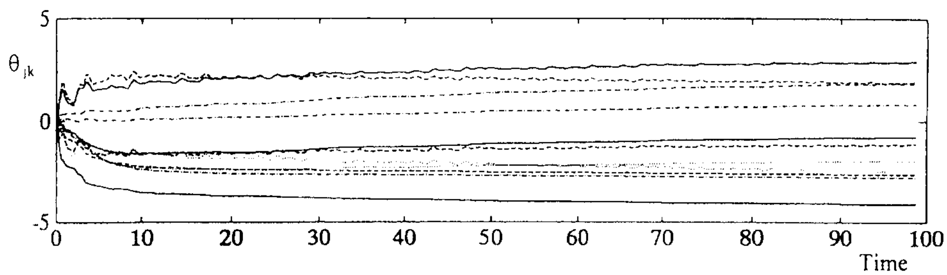
and $D_i = (-p_i \quad 0 \quad 0)^T$ are matrices of parameters, $w_i = (x_i \quad y_i \quad s_i)^T$ is a state vector, B_i are gain matrices, u is a control signal and v_i is a measurable output. Suppose that p_i, q_i, M_{0i}, M_{1i} are unknown.



(e)



(f)



(g)

Fig. 4. (*Continued.*) (e) Trajectory of error vector $e(t)$. (f) Trajectory of control signal $u(t)$. (g) Trajectories of tunable parameters $\Theta(t)$.

Write down the error equation

$$\dot{e} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} p_1(y_1 - f_1(x_1)) - p_2(y_2 - f_2(x_2)) \\ 0 \\ q_2 y_2 - q_1 y_1 \end{pmatrix} + Bu, \quad (35)$$

where $B = B_1 - B_2$, $e(t) = w_1(t) - w_2(t)$ is an error vector.

One can obtain function Φ in the form (29) (take $B_1 = \left(\begin{smallmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \end{smallmatrix}\right)^T$, $B_2 = 0$) and write down the control law:

$$\begin{aligned} u_1 &= \theta_{01}^T (v_1 - v_2) + \theta_{11}^T z_1, \\ u_2 &= \theta_{02}^T (v_1 - v_2) + \theta_{12}^T z_2 \end{aligned} \quad (36)$$

where

$$\begin{aligned} \theta_{01}^T &= (\theta_{01}^{(1)}\ \theta_{01}^{(2)}), \\ \theta_{02}^T &= (\theta_{02}^{(1)}\ \theta_{02}^{(2)}), \\ \theta_{11}^T &= (\theta_{11}^{(1)}\ \theta_{11}^{(2)}\ \theta_{11}^{(3)}\ \theta_{11}^{(4)}\ \theta_{11}^{(5)}\ \theta_{11}^{(6)}), \\ \theta_{12}^T &= (\theta_{12}^{(1)}\ \theta_{12}^{(2)}) \end{aligned}$$

are tunable parameters, $z_1 = (y_1 \quad x_1 \quad |x_1 + 1| - |x_1 - 1| \quad y_2 \quad x_2 \quad |x_2 + 1| - |x_2 - 1|)^T, z_2 = (y_1\ y_2)^T$.

Take the adaptation algorithm in the differential form

$$\begin{cases} \dot{\theta}_{01} = -\Gamma_{01} g_1^T (v_1 - v_2)(v_1 - v_2), \\ \dot{\theta}_{02} = -\Gamma_{02} g_2^T (v_1 - v_2)(v_1 - v_2), \\ \dot{\theta}_{11} = -\Gamma_{11} g_1^T (v_1 - v_2) z_1, \\ \dot{\theta}_{12} = -\Gamma_{12} g_2^T (v_1 - v_2) z_2, \end{cases} \quad (37)$$

where $\Gamma_{jk} = \Gamma_{jk}^T > 0$ $(k, j = 1, 2)$, g_k are columns of matrix $G = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)$. The Theorem 3 implies that the system with transfer function $W = G^T C(\lambda I - A)^{-1} B$ is hyper-minimum phase if $b_1 > 0$, $b_2 > 0$, $g_{11} g_{22} - g_{12} g_{21} > 0$.

Now check the other conditions of the theorem: boundedness of z_k follows from boundedness of w_2 (w_2 is a chaotic trajectory of Chua circuit) and $e(t)$ ($\int_0^\infty \|e(t)\|^2\,dt < \infty$, see Theorem 3).

Therefore the trajectories of the system (35), (36) are bounded and the following synchronization goal is achieved: $\lim_{t\to\infty} e(t) = 0$.

Simulation results for the following values of parameters: $q_1 = 16.286$, $q_2 = 14.286$, $p_1 = 9$, $p_2 = 10.6$, $M_{01} = 4/7$, $M_{11} = -1/10$, $M_{02} = 2/7$, $M_{12} = -1/7$, $x_1(0) = y_1(0) = s_1(0) = 0.3$, $x_2(0) = y_2(0) = s_2(0) = -0.3$, $b_1 = b_2 = 1$, $\Gamma_{01} = \Gamma_{02} = \Gamma_{12} = I$, $\Gamma_{11} = I$, $G = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, elements of matrices $\theta_{jk}(0) = 0$, are shown in Fig. 4.

Fig. 4(a) presents phase portrait (x_2, y_2) of the system which is not influenced by control.

Fig. 4(b)–(d), phase plot (x_2, x_1), (y_2, y_1), (s_2, s_1), shows that under action of control $u(t)$ after some transient process systems synchronize [Fig. 4(f)], synchronization error $e(t)$ tends to zero [Fig. 4(e)] and tunable parameters $\Theta_{jk}(t)$ achieves some constant values[5] [Fig. 4(g)].

The system shows a good ability of getting synchronized and synchronization process is independent from initial conditions.

IV. Conclusion

The proposed speed-gradient based general adaptive synchronization methods can be applied for synchronization purposes in various fields. The methods provide powerful mechanism for synchronizing different nonlinear (including chaotic) systems with unknown parameters as well as the conditions of the synchronization goal achievement. Speed-gradient method as well as the procedures of control law design proposed in the paper does not use specific features of chaos and can be applied to synchronize both chaotic and periodic motions.

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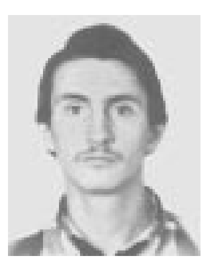

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