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ADAPTIVE TENSOR PRODUCT GRIDS FOR SINGULAR PROBLEMS

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ABSTRACT

We consider piecewise polynomial approximation of order M in N-dimensions with a tensor product partition of the space. We assume that the partition is to be chosen to minimize the maximum error in the approximation. The optimal rate of convergence for piecewise polynomial approximation to a smooth function for unconstrained partitions is known to be order $\frac{1}{K^{M/N}}$ where K is the number of elements in the partition. This rate of convergence is achieved by a uniform grid which may be taken to be a tensor product. In 1979 de Boor and Rice gave an adaptive algorithm which achieves this same order of convergence for a wide variety of singular functions. We now study whether this optimal order of convergence can be achieved by partitions constrained to be tensor products. We show that the optimal order of convergence is achieved by tensor product grids (partitions) for functions with point or boundary layer singularities. For some other singularities, the tensor product constraint reduces the order of convergence substantially.

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1. STATEMENT OF THE PROBLEM

1.1 *Notation and Definitions*

We consider a domain D in N-dimensional space R^N and a function f defined on D. For simplicity of notation we assume that D is the unit cube

$$D = \{ x \mid x = (x_1, x_2, ..., x_N) \text{ and } -1 \le x_i \le 1 \}$$

This assumption does not limit the applicability of the results of this paper. We consider a partition

$$E = \{ e_i \mid i = 1, 2, ..., K \}$$

of D into elements and the set $P_{n,K}$ of piecewise polynomials defined on E with total order n. The *approximation problem* is to determine E and $p^* \in P_{n,k}$ so that

$$\| f - p^* \| = \max \mid f(x) - p^*(x) \mid$$

is minimized. A *rectangular tensor product grid* G is a partition of D which is the tensor product of partitions of the N coordinate axes in R^N. Thus $\pi_i = \{ -1 < t_1 < t_2 \cdots < t_{K_i} < 1 \}$ is a partition of $-1 \le x_i \le 1$ and $G = \pi_1 \times \pi_2 \times \cdots \times \pi_N$ has $K = \prod_{i=1}^{N} K_i$ rectangular elements. A *tensor product grid* G is a partition that is topologically rectangular. That is, G is the continuous image of a rectangular tensor product grid.

We consider a manifold $S \in D$ of dimension m to be the image of a map (also called S) $S : R^m \to R^N$ intersected with D. In this paper S is assumed to be smooth, the map S has as many continuous derivatives as required at any point of the discussion. We define the function *dist* to be

$$dist(S, x) = \min_{y \in S} \| x - y \|$$

We choose the L_∞ norm in R^N throughout this paper. The diameter of an element e of E is defined by

$$diam(e) = \max_{x, y \in e} \| x - y \|$$

1.2 *Known Results*

The following theorem is a version of well known results concerning piecewise polynomial approximations to smooth functions in R^N.

Theorem 1. If $f(x)$ has n continuous derivatives ($\| f^{(n)} \| < \infty$) then there is a $p^* \in P_{n,K}$ so that

$$\| f - p^* \| \le \text{Const} * \max_i (diam(e_i))^n$$

where $\{ e_i \}$ is the set of elements constituting E. If the partition E is essentially uniform, that is

$$diam(e_i) / diam(e_j) \le \text{constant}$$

independently of K, then

$$\| f - p^* \| = O(K^{-n/N})$$

There is an algorithm [de Boor and Rice, 1979] which extends the convergence result of Theorem 1 to a broad class of singular functions. A particular case of the results established by deBour and Rice is the following.

Theorem 2. Assume the $f^{(n)}$ is continuous in $D - S$ and that

$$|f^{(n)}| \leq \text{Const} * dist\,(S, x)^{\alpha - n}$$

with $\alpha > m\,n\,/N$. Then the adaptive algorithm produces a partition E of D and a $p^* \in P_{n,K}$ so that

$$||\, f - p^* \,|| = O\,(K^{-n/N})$$

The striking thing about this result is that the order of convergence achieved is the same for a broad class of singular functions as for smooth functions. This theorem illustrates the power of adaptive approximation where one adjusts the partition E to the singularities of f.

1.3 *Examples*

Figure 1 shows a two dimensional tensor product grid G that might be suitable for a point singularity. It is clear that the tensor product grid can achieve small elements near the singularity. It is also clear that extraneous, long and thin elements are generated far away from the singularity.

Figure 2 shows a partition that might have been generated by an adaptive grid refinement algorithm such as in [de Boor and Rice, 1979], [Rheinboldt, 1980] or [Babuska and Rheinboldt, 1978]. Although it is not visually apparent, the partition of Figure 1 is a refinement of that of Figure 2. The grid in Figure 1 has 143 elements while the partition in Figure 2 has 58 elements.

Figure 3 shows a tensor product grid which has adapted to a curve of "near-singularity" in a function. One visualizes $f\,(x\,)$ as having a steep wave front along the curve seen in Figure 3.

1.4 *The Problem*

The advantage of the tensor product grid is that its data structure is much simpler than that of a general partition. One needs have only a mapping of R^N to R^N and a simple rectangular grid. The consequences of this are:

(a) Computer programs are much easier to write.
(b) Parallel algorithms are much easier to create.
(c) Vector algorithms are much more feasible.

The time to process the partition might or might not be less; it depends on the complexity of the mapping. However, experience shows that the time to process completely general partitions can be substantial.

The disadvantage of the tensor product grid is that more elements are used in the partition than needed to approximate f accurately. This leads to both more effort in computing the approximation and to a larger data structure once the approximation is found.

Thus we have two questions:

1. How damaging to the convergence is the constraint that the partition be a tensor product grid?

2. Are adaptive algorithms actually better in practice?

Note that adaptive algorithm here means one keeps the size of the partition fixed and adjusts its shape to approximate $f\,(x\,)$. This is a slightly different viewpoint than adaptive grid refinement where one continually refines an existing partition without otherwise changing its shape. On the other hand, one may attempt to adapt a fixed size partition by taking a much coarser partition and then refining it in some way. Thus, while we take the viewpoint of the moving finite element method [Miller and Miller, 1981], [Galinas et. al., 1982], the difference between it and grid refinement is more apparent than real.

The results established later in this paper show that the theoretical rate of convergence is not effected in some important cases by the constraint that the partition be a tensor product grid. The question as to whether adaptive algorithms are actually better in practice requires more experience and depends heavily on the particular context. For simple quadrature and approximation problems, it is well established that some adaptive methods greatly outperform non-adaptive ones. There is still too little experience, for example, in solving elliptic partial differential equations by adaptive methods. One must wait before drawing even tentative conclusions.

2. THREE LEMMAS

In this section we present three lemmas for the two dimensional case, $N=2$. The first lemma is a restatement of a result of [Rice, 1969].

Lemma 1. Give M, define the partition π of $[-1,1]$ by the points

$$t_0=0,\quad t_1=M^{-(n/\alpha)},\quad t_j=j^{n/\alpha}t_1,\quad t_{-j}=-t_j,\quad \pm 1$$

Then π has $2M$ points and there is a piecewise polynomial $p^*(x)\in P_{n,K}$ so that

$$\| x^\alpha - p^*(x)\| = O(M^{-n})$$

The next lemma relates the approximation of $f(x,y)=\operatorname{dist}((x,y),(0,0))^\alpha$ in R^2 to the approximation of x^α in R^1. We define G to be the tensor product of the partition π of Lemma 1 along the x and y axes, respectively. Let $e=e(i,j)$ be an element of G defined by

$$e(i,j)=[t_i,t_{i+1}]\times[t_j,t_{j+1}]$$

Without loss of generality, we may assume that $i\ge j\ge 0$. We denote by $p^*(x,y)$ the best approximation to f from $P_{n,K}$ on the element e and denote by $q^*(x)$ the best approximation to x^α on [0,1].

Lemma 2. There is an absolute constant C_n so that

$$\| f-p^*\|_e \le C_n \| x^\alpha - q^*\|$$

Proof. From [Morrey, 1966] we have, for $i>0$, that

$$\| f-p^*\|_e \le D_n'(t_{i+1}-t_i)^{n+1}\| f^{(n+1)}\|$$

where the norm of the multivariate derivative is taken to be

$$\| f^{(n+1)}\| = \max_{|\gamma|=1} | D^\gamma f |$$

and γ is a multi-index, D^γ is the usual multivariate differentiation operator. We observe for this particular f that $|D^\gamma f|$ is monotonic in each of the variables separately and thus

$$\| f^{(n+1)}\| \le t_i^{\alpha-n-1}(\alpha)(\alpha-1)\ldots(\alpha-n)$$

An examination of the proof of Lemma 1 (Theorem 2 of [Rice, 1969]) shows that

$$(t_{i+1}-t_i)^{\alpha} t_i^{\alpha-n-1}(\alpha)(\alpha-1)\ldots(\alpha-n) = O(K^{-n-1})$$

which is known to be $O(\|x^{\alpha}-q^*\|)$.

If i=0, then a simple direct analysis shows the bound on $\|f-p^*\|_e$ still holds. This concludes the proof.

We observe that the proof of Lemma 2 does not depend on the assumption of $N=2$ and state.

Corollary. Lemma 2 is valid for all N, with the constant C_n depending on N as well as n.

The third lemma allows us to incorporate smooth maps in the approximation process. Let M be a map from R^2 to R^2 which is smooth in the sense that with

$$M:(u,v)\rightarrow(x,y)$$

$$x = x(u,v)$$

$$y = y(u,v)$$

then all derivatives of x and y with respect to u and v are bounded and the Jacobian of the map is uniformly bounded from zero. Thus the map M is invertible.

Lemma 3. Let G be a tensor product grid and $p^*(u,v)\in P_{n,K}$ defined on G be an approximation to $f(u,v)$ with

$$\|f-p^*(u,v)\| = O(K^{-n-1})$$

Assume that the maximum element in G goes to zero as K goes to infinity. Then, for K sufficiently large, there is a piecewise polynomial $q^*(x,y)\in P_{n,K}$ defined on MG so that

$$\|M f(x,y)-q^*(x,y)\| = O(K^{-n-1})$$

Proof. The map M takes G into MG and f into Mf, but $M p^*$ is not a polynomial. Let q_K be the polynomial of order n which is the best approximation to Mp^*. As $K\rightarrow\infty$ the elements of G and MG become more and more rectangular. Since M is arbitrarily smooth, these elements converge to rectangles faster than $O(K^{-n-1})$. Thus the difference $\|M p^*-q_K\|$ on the elements becomes negligible compared to $\|f-p^*\|$; we may take $q^*(x,y)=q_K(x,y)$ and the lemma is established.

We observe that the proof of Lemma 3 also does not depend on the diversion N being 2 and state.

Corollary. Lemma 3 is valid for all N.

3. TWO DIMENSIONAL THEOREMS

In this section we establish the rates of convergence for piecewise polynomial approximation in two dimensions using tensor product grids. The essential elements of the results for higher dimensions are present in the case $N=2$ and it is easy to visualize them in this simpler case. For ordinary point singularities and boundary layer singularities we obtain the optimal rate of convergence; for a curve of singularities we obtain much less than the optimal rate of convergence for the more interesting cases.

Theorem 3. Let $f(x) = \text{dist}(x,0)^\alpha$ in R^2. Tensor product grids G exist so that there are piecewise polynomials p^* in $P_{n,K}$ defined on G with

$$\|f - p^*\| = O(K^{-n/2})$$

Proof. We use the grid G used in Lemma 2. The one dimensional partition π has $2M$ intervals and achieves an approximation error of $O(M^{-n})$. The tensor product of π with itself is G which has $4M^2$ elements and the best piecewise polynomial approximation on G achieves an error of $O(M^{-n})$. Set $K = 4M^2$ and we see that $O(M^{-n}) = O(K^{-n/2})$ which establishes the theorem.

The following corollary illustrates the applicability of this result to other functions with a point singularity that behaves like x^α.

Corollary. Let f_0 and f_1 be smooth functions and set

$$f(x) = f_0(x) + f_1(x)\,\text{dist}\,(x,0)^\alpha$$

Theorem 3 is satisfied for this $f(x)$ also.

A rectangular tensor product grid is not suitable for approximating a function with a curve S of singularities or even a boundary layer along S unless S is parallel to one of the coordinate axes. Consider, for example, a function f with a boundary layer along the line $x = y$. The grid G would have to have small elements along S and the rectangular, tensor product constraint would imply small elements everywhere.

The general tensor product grid avoids this difficulty and we have for the boundary layer case the following. Indeed, the optimal grid for such a function will be nearly tensor product in any case, so we do not expect the tensor product constraint to affect the rate of convergence.

Theorem 4. Let S be a smooth curve in R and let $f(x) = \text{dist}(x,S)^\alpha$. Tensor product grids G exist so that there are piecewise polynomials p^* in $P_{n,K}$ defined on G with

$$\|f - p^*\| = O(K^{-n/2})$$

Proof. We use Lemma 3 to reduce the proof to the case that S is the line $x = 0$. Then we take G to be the tensor product of a uniform partition in y and the partition π of Lemma 1 in x. It is clear that the Mf, as a function of x, is smooth away from $x = 0$ and behaves like $|x|^\alpha$ as x approaches 0. Mf is smooth in y everywhere. Thus the best piecewise polynomial approximation $p^* \in P_{n,K}$ on G achieves an error of order $K^{-n/2}$. This concludes the proof.

It is appropriate to note that Theorems 3 and 4, as stated, cover only piecewise polynomial approximations. There is no smoothness assumptions about how the polynomial pieces join up. Thus these approximations are splines with multiple knots (the multiplicity is n which gives no continuity in the spline at all). In one dimension there is a standard technique of "pulling apart the knots" to extend order of convergence results of piecewise polynomial approximations to spline approximations. The application of this technique establishes the following.

Corollary. Theorem 2 and 3 are true with the set $P_{n,K}$ of piecewise polynomials replaced by the set of $S_{n,K}$ of splines (both defind on a tensor product grid).

Note that it is difficult to obtain both smooth and highly accurate piecewise polynomial approximations on arbitrary partitions, even if all the elements are rectangles. This topic is discussed further at the end of [de Boor and Rice, 1979] where the following conjecture is made.

"Blending function schemes are the only approximation methods with all of the following four desirable properties,

1. The approximations are smooth.
2. The order of convergence is optimal - or nearly so.
3. The approximation is locally determined.
4. The shape of the elements is specified a priori."

If $f(x,y)$ has a curve of singularities of more general type than boundary layer, then tensor product grids are not optimal. Suppose, for example, we use the hypothesis of Theorem 2. While this appears at first glance to be very similar to the hypothesis of Theorem 4, the latter implies that f is smooth "parallel" to the curve S independently of the distance from S. The hypothesis of Theorem 2 allows derivatives "parallel" to S to blow up just as fast as derivatives perpendicular to S. We have the following result.

Theorem 5. If f satisfies

$$f^{(n)}(x) \le \text{Const} * \text{dist}(S,x)^{\alpha-n}$$

then, with $N = 2$, there is a tensor product grid G with p^* from $P_{n,K}$ so that

$$\|f - p^*\| = O(K^{-\alpha/2})$$

Proof. One maps S so that MS is a straight line as in the proof of Theorem 4. There G is chosen to be uniform in y and the grid π of Lemma 1 in x. Consider a square element e with $x=0$ and size h, we see that

$$\|f - f(0,y_0)\| = O(h^\alpha)$$

where $(0,y_0)$ is any point on the y axis in e. This error estimate cannot be materially improved by using another polynomial approximant on e. If we set $M = \frac{1}{h}$ in the definition of π then we observe that the error of approximation away from $x=0$ remains $O(h^\alpha)$. The number K of elements in G is M^2 so that $O(h^\alpha)$ becomes $O(K^{-\alpha/2})$. This concludes the proof.

It is likely that this result is optimal for tensor product grids. In any case, the rate of convergence may be much reduced compared to that of Theorem 2 and the tensor product constraint produces a very significant decrease in the convergence rate for interesting values of α.

4. HIGHER DIMENSIONS

There are so many combinations of singularities that one can have in higher dimensions that it is not practical to attempt to explicitly cover them all. Thus, for $N=4$, one could have three point singularities, two curves of singularity (one of which is boundary layer) and a surface of singularities.

We state one result which typifies the general situations. Consider a manifold S of dimension $m < N$ which is, for simplicity of notation, the space of the first m coordinates, that $S = \{(x_1, ..., x_m, 0, ..., 0)$. Assume that f has boundary layer singularity in the first m_b dimensions of S, i.e.

$$f(x) = \text{dist}((x_1, ..., x_{m_b}, 0, ..., 0), (0, ..., 0))^\alpha + g(x)$$

and power type singularity in the next m_s dimensions, i.e.

$$|D^n g| \le \text{Const} * \text{dist}(x,0)^{\alpha/n}$$

where the D are differentiation operators in x_{m_b+1} through x_m. We state without proof.

Theorem 6. There is a tensor product grid G and p^* in $P_{n,K}$ so that

$$||f - p^*|| \leq O(K^{-1/(m_b/n + m_s/\alpha)})$$

Corollary. Tensor product grids provide optimal rates of convergence only if no singularities are of power type for manifolds of dimension greater than zero.

5. REMARKS ON APPLICATION TO ELLIPTIC PROBLEMS

We have obtained some optimal tensor product grids for solving elliptic problems by collocation with Hermite bicubics. Adaptive grids certainly pay off in some cases; for example consider the elliptic problem

$$u_{xx} + u_{yy} - 100u = 1200(\cosh(50y)/\cosh(50))$$

on the unit square with boundary conditions 1. With a 7 by 7 grid, the optimal adapted tensor product grid gives a maximum error in the computed solution of about .002 compared to an error of .2 for a 7 by 7 uniform grid. This factor of 100 in improvement in the error is very worthwhile. On the other hand, for a Poisson problem with true solution

$$\exp(-100[(x-.5)^2+(y-.117)^2]) * (x^2-x)(y^2-y)$$

adapting a 7 by 7 grid provides a improvement of only a factor of 11 (from .0045 to .0004) even though the solution appears to have a somewhat sharp, isolated bump.

This experience suggests that nearly optimal grids for elliptic problems are difficult to compute accurately and that optimal grids are rather different from what one initially expects.

Note that tensor product grids for singular problems produce very long and thin elements which have very large aspect ratios. The study of [Rice, 1985] indicates that these elements cause no numerical problems at all.

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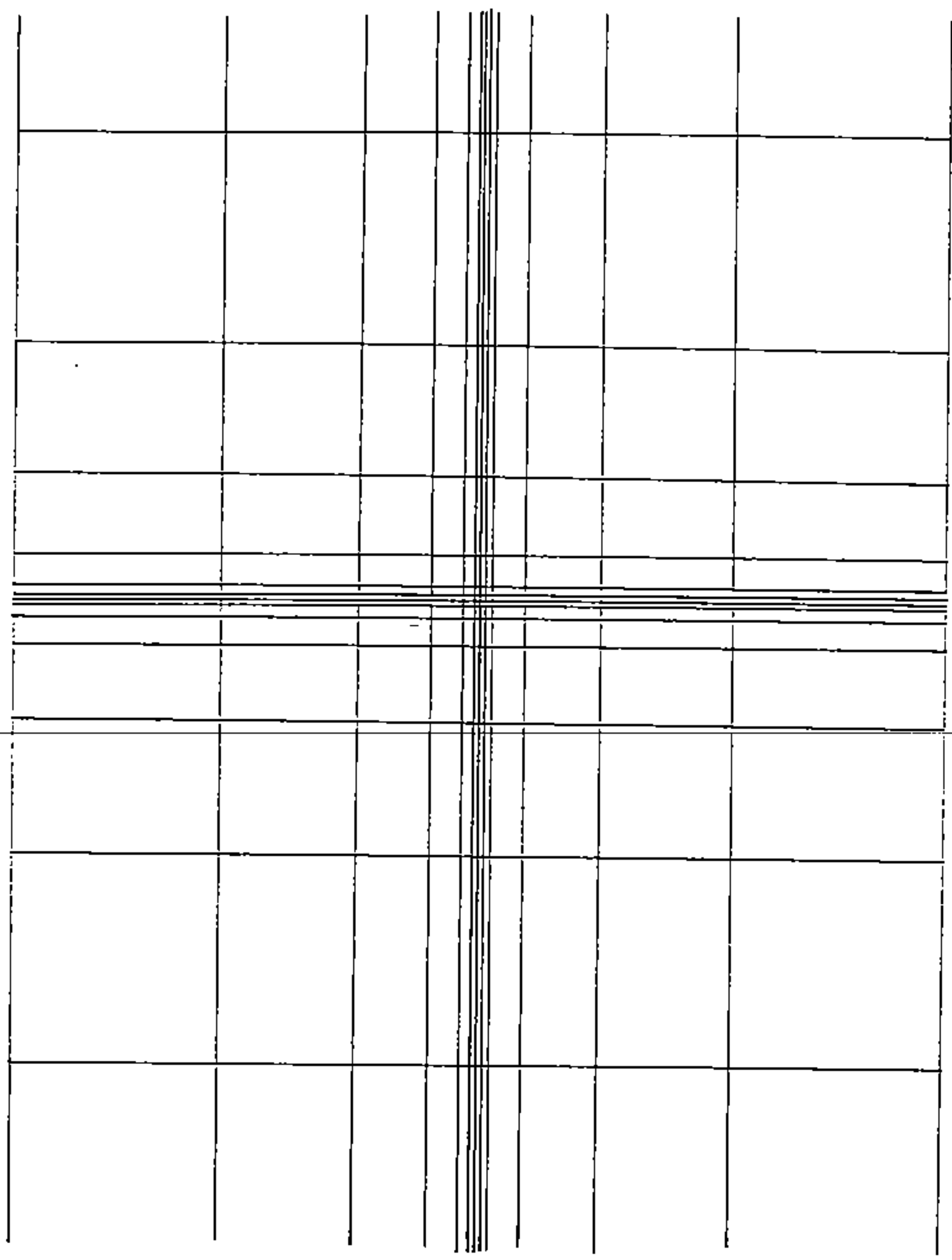

Figure 1. Example of how a tensor product grid might adapt to a point singularity in a function of two variables.

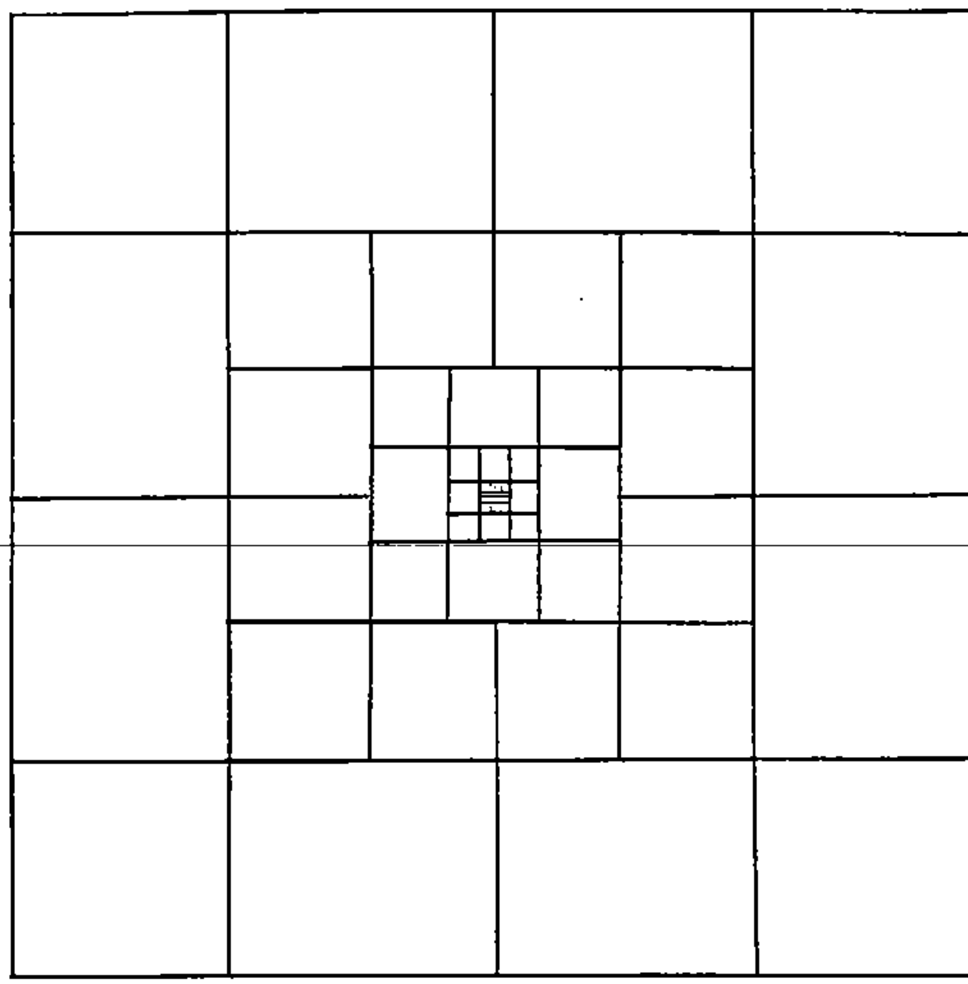

Figure 2. Example of how adaptive grid refinement might adapt to a point singularity in a function of two variables. The grid in Figure 1 is a refinement of this partition.

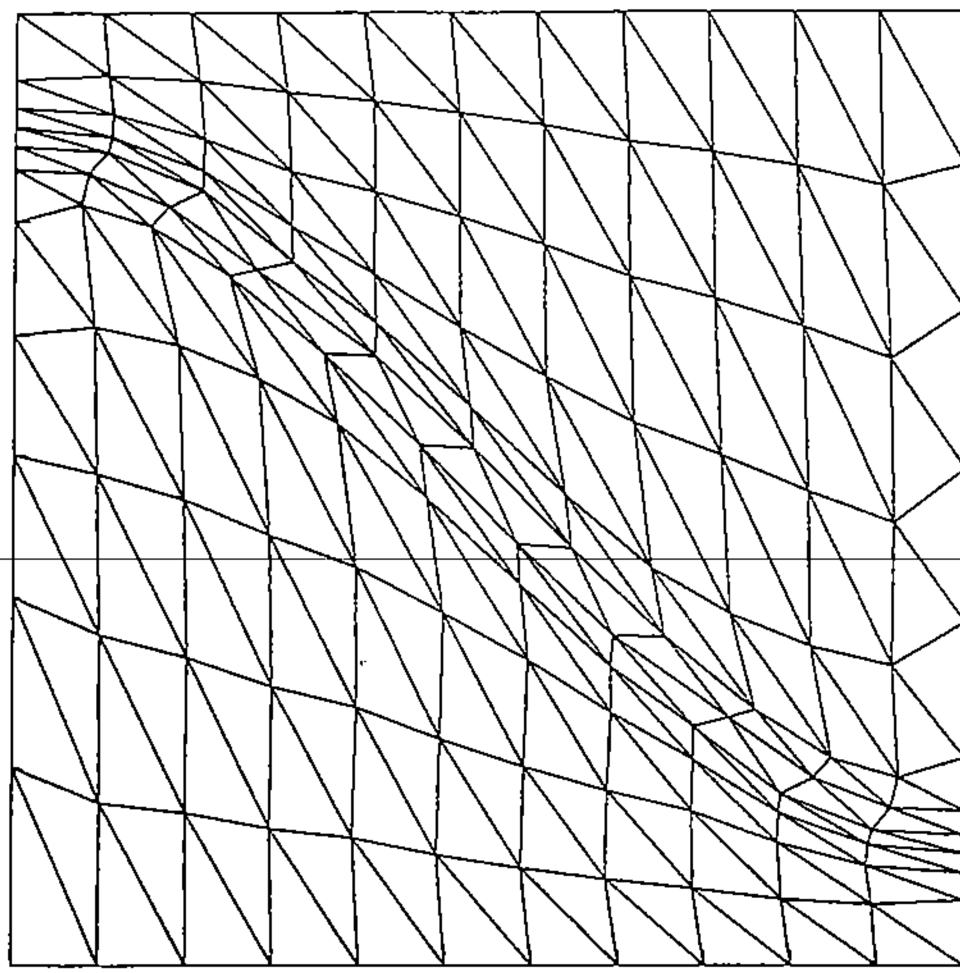

Figure 3. Example of how a tensor product grid might adapt to a steep wave front behavior in a function of two variables.