

ADAPTIVITY AND GROUP INVARIANCE IN MATHEMATICAL MORPHOLOGY

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ABSTRACT

The standard morphological operators are (i) defined on Euclidean space, (ii) based on structuring elements, and (iii) invariant with respect to translation. There are several ways to generalise this. One way is to make the operators adaptive by letting the size or shape of structuring elements depend on image location or on image features. Another one is to extend translation invariance to more general invariance groups, where the shape of the structuring element spatially adapts in such a way that global group invariance is maintained. We review group-invariant morphology, discuss the relations with adaptive morphology, point out some pitfalls, and show that there is no inherent incompatibility between a spatially adaptive structuring element and global translation invariance of the corresponding morphological operators.

Index Terms— Group morphology, adaptive morphology, space-variant structuring elements.

1. INTRODUCTION

Mathematical morphology is an approach to image analysis that studies image transformations with a simple geometrical interpretation. Small subsets, called *structuring elements*, of various forms and sizes are translated over the image plane to perform shape extraction. The classical approach is characterised by the following two properties: (i) the structuring element is *fixed*, i.e., does not depend on the spatial location at which it is centered; (ii) the basic image operations are invariant under translation. This can be extended to grey value images, using a lattice formulation, see [1–3].

Generalisations fall into two major categories:

1. Translation invariance is replaced by various other forms of invariance, with their associated *group morphologies*.
2. Structuring elements become dependent on position or the input image itself, leading to *adaptive morphology*.

In both cases the size or shape of structuring element becomes dependent upon the spatial location, but for reasons which are entirely different.

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The literature on adaptive morphology shows a rather confusing picture. First, a large set of different terms for the location-dependence of structuring elements is in use, with sometimes subtle differences of meaning: “space-variant”, “spatially variant”, “adaptive”, “spatially adaptive”, “extrinsic”, “intrinsic”, “adaptive neighbourhood”, “adaptive-weighted”, “dynamic”. Another confusion concerns the statements in various papers which use “space-variant” as equivalent with “not translation invariant” [4, 5].

The goal of this paper is to shed some light on these issues. The main arguments put forward below can be summarised as follows:

- Care is required when deriving properties of morphological operators involving the word “adaptive”. A distinction between location adaptivity and input adaptivity is essential.
- There is no inherent incompatibility between a spatially-variant structuring element and global translation invariance (or other types of group invariance) of the associated morphological operators.

2. PREVIOUS WORK

2.1. Group morphology

We mention a few examples of binary image transformations on a set E with different types of symmetry. For surveys of the resulting *group morphology*, see Heijmans and Ronse [6, 7] for the case of *abelian* symmetry groups, Roerdink [8] for the case of arbitrary (abelian and non-abelian) symmetry groups, as well as the book by Heijmans [3].

In *Euclidean morphology*, $E = \mathbb{R}^2$ or $E = \mathbb{Z}^2$ and the image operations are invariant under the group of Euclidean translations. All translated structuring elements have the same size, shape, and orientation.

In *Circular morphology* [6, 9], $E = \mathbb{R}^2 \setminus \{0\}$, and the image transformations comprise the abelian group generated by rotations and scalar multiplication w.r.t. the origin. The size of the structuring element at a point x is proportional to the distance of x to the origin, and the orientation depends on the angle which the line from the origin to x makes with the horizontal axis.

In *Perspective morphology* one requires invariance of image operations under object translation parallel to the image plane [8]. In this case the structuring element has to be adapted according to the law of perspective.

Other group morphologies are generated by the similarity, affine, spherical or projective groups [8]. These are different from the three morphologies described above in the sense that the corresponding group is non-abelian. Another marked difference from the abelian case is that structural group-invariant openings and closings can in general no longer be decomposed into products of group-invariant dilations and erosions [8].

2.2. Adaptive morphology

Location-adaptive structuring elements. A first form of “adaptivity” is to make the structuring element, now called *structuring function*, dependent on the location in the image. The structuring function is fixed *a priori*, i.e., does not depend on the input image. In [10] this approach is called ‘extrinsic’.

Dilations and erosions without any invariance property were first considered by Serra [2, Ch.2]. Let $\mathcal{P}(E)$ denote the set of all subsets of a set E ordered by set-inclusion. A mapping $\delta : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is a dilation if and only if there exists a function $N : E \rightarrow \mathcal{P}(E)$, called structuring function, such that

$$\delta(X) = \bigcup_{x \in X} N(x). \quad (1)$$

This statement can be interpreted as follows. Attach to each point x of E a subset (“neighbourhood”) $N(x)$ of E . Then the dilation $\delta(X)$ is the union of all the subsets which are attached to points of X .

Recall that a pair of transformations (ε, δ) on $\mathcal{P}(E)$ is called an *adjunction*, if for all subsets X and Y of E the following equivalence holds: $\delta(X) \leq Y \iff X \leq \varepsilon(Y)$. It is easy to see that the erosion ε associated by adjunction to the dilation δ in (1) is given by

$$\varepsilon(X) = \{y \in E : N(y) \subseteq X\}. \quad (2)$$

The formulas (1)-(2) reduce to the classical case when the structuring function is chosen as $N(x) = A_x$ with A a fixed structuring element; here $A_x = \{a + x : a \in A\}$ denotes the translate of A along the vector x . Similar expressions for the grey-scale case exist.

A systematic analysis of morphological operators with location-adaptive structuring elements for both binary and grey scale images was made by Bouaynaya *et al.* [4, 5] (called “spatially-variant morphology” by them). A “locally adaptable” morphology for binary images was considered by Cuisenaire [11], who used disks with a position dependent radius as structuring elements.

Input-adaptive structuring elements. The second form of adaptivity is to make the structuring element depend on the

local features of the input image (thereby, the structuring element also becomes dependent on the location in the image).

Lerallut *et al.* [12] have introduced *morphological amoebas* as filter kernels which adapt themselves to the local content (such as contour variations) of the grey-scale input image. In this case, the structuring function depends on f ,

$$N^{(f)}(x) = \{y : d_\lambda(x, y) \leq r\}$$

where d_λ is the so-called amoeba distance which depends on the intensities of the input image f itself.

Another approach was followed by Braga Neto [13], who called it *adaptive neighbourhood morphology*. Here the structuring function is given by $N^{(f)}(x) = R_m^f(x)$, where $R_m^f(x)$ are input-adaptive regions defined in terms of m -th order connectivity. Later work along similar lines was presented by Debayle *et al.* [10], who called this type of adaptivity ‘intrinsic’.

3. INPUT-ADAPTIVE MORPHOLOGICAL OPERATORS

Let $\mathcal{L} = \text{Fun}(E, \mathcal{T})$ denote the complete lattice of grey scale functions with domain E , whose range is a complete lattice \mathcal{T} of grey values. Consider the mappings $\delta : \mathcal{L} \rightarrow \mathcal{L}$ and $\varepsilon : \mathcal{L} \rightarrow \mathcal{L}$ defined by

$$\begin{aligned} \delta(f)(x) &= \bigvee_{y \in \tilde{N}^{(f)}(x)} f(y), & x \in E \\ \varepsilon(f)(x) &= \bigwedge_{y \in N^{(f)}(x)} f(y), & x \in E \end{aligned} \quad (3)$$

where the reflected neighbourhood $\tilde{N}^{(f)}$ is defined by

$$y \in N^{(f)}(x) \iff x \in \tilde{N}^{(f)}(y).$$

Note that, since the neighbourhoods depend on the input f , the mappings $f \rightarrow \delta(f)$ and $f \rightarrow \varepsilon(f)$ are in general not a dilation and erosion, i.e., do not commute with suprema and infima, respectively (nor do they form an adjunction, hence products $\delta\varepsilon$ and $\varepsilon\delta$ are not guaranteed to satisfy the algebraic properties of opening and closing). So one should *not* call these operators (adaptive) “dilation” and “erosion”.

This fact seems to be often overlooked in the literature. For example, Debayle *et al.* simply state that the adjunction property for their input-adaptive operators is inferred from the lattice theory of increasing mappings [10, page 151]. Braga Neto, in his work on alternating sequential filters with input-adaptive structuring elements [13], refers for proofs of their algebraic properties to theorems which only hold for the case of location-adaptive structuring elements. Similarly, Bouaynaya *et al.* [4, 5], in their work on spatially variant morphology, mention several forms of input adaptive morphology, such as [10, 12], as examples covered by their general framework; in fact, their proofs only cover the location-adaptive case.

To make our point as clearly as possible, we will attempt a proof of the adjunction property of the operators in (3), and show at which point the proof is obstructed by the input-dependence of the structuring element. Subsequently, we give an explicit counterexample for the binary case.

Obstruction of the adjunction property. To form an adjunction, the operators $\delta(f)$ and $\varepsilon(f)$ defined by (3) would have to satisfy the following equivalence:

$$\delta(f) \leq g \iff f \leq \varepsilon(g) \quad (4)$$

Let us try to prove it along the usual lines.

$$\begin{aligned} & \delta(f) \leq g \\ \iff & \\ & \delta(f)(x) \leq g(x) \quad \forall x \in E \\ \iff & \{ \text{definition } \delta \} \\ & \bigvee_{y \in \tilde{N}(f)(x)} f(y) \leq g(x) \quad \forall x \in E \\ \iff & \\ & f(y) \leq g(x) \quad \forall y \in \tilde{N}(f)(x), \forall x \in E \\ \iff & \{ \text{definition reflected neighbourhood} \} \\ & f(y) \leq g(x) \quad \forall x \in N(f)(y), \forall y \in E \\ \iff & \\ & f(y) \leq \bigwedge_{x \in N(f)(y)} g(x) \quad \forall y \in E \end{aligned}$$

To complete the proof, the right hand side of the inequality in the last line above should equal $\varepsilon(g)(y)$, that is, $\bigwedge_{x \in N(g)(y)} g(x)$. However, this is not the case, as the infimum in the last line is over the neighbourhood $N(f)$ instead of $N(g)$. So the proof fails for the input-dependent case.

Counterexample. Consider a binary image f on \mathbb{Z}^2 with 4-connectivity and let B be the “cross” structuring element (center pixel with its 4-connected neighbours). Let B_x be the translate of B by x . A pixel x is called a *1-pixel* (foreground) when $f(x) = 1$ and a *0-pixel* (background) when $f(x) = 0$. A 1-pixel is called *isolated* when there are no 1-pixels that are 4-connected to it. Define the adaptive neighbourhood as

$$N^{(f)}(x) = \begin{cases} B_x & \text{if } x \text{ is a non-isolated 1-pixel of } f \\ \{x\} & \text{otherwise} \end{cases} \quad (5)$$

Then the operator δ in (3) is not a dilation, i.e., does not commute with supremum (union). Take the example in Figure 1, where cells with a black dot denote 1-pixels and empty cells denote 0-pixels. The images f and g both contain only isolated 1-pixels (note that 4-connectivity is used). Therefore, the operator δ does not change them, i.e., $\delta(f) = f$, $\delta(g) = g$. But the union $f \vee g$ contains non-isolated 1-pixels, so that its dilation results in the image on the lower right. Clearly, in this case $\delta(f \vee g) \neq \delta(f) \vee \delta(g)$, hence δ is not a dilation. For the same reason, ε in (3) is not an erosion, nor is (ε, δ) an adjunction (or $\delta\varepsilon$ an opening, or $\varepsilon\delta$ a closing).

In order to convert (3) to proper dilation and erosion we

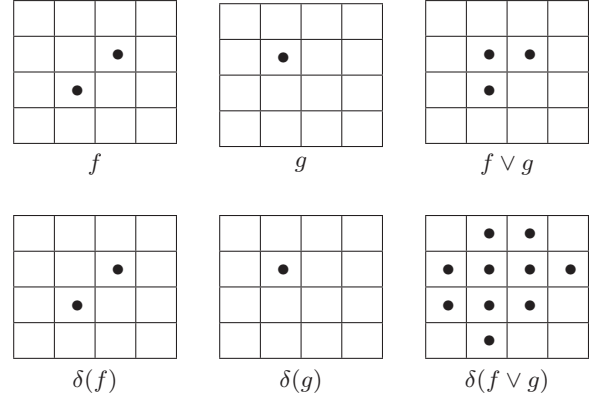


Fig. 1. Example showing that the operator δ defined in (3) is not a dilation when the structuring function is defined by (5).

introduce the following notation.

$$\begin{aligned} \delta^a(f, f^0)(x) &= \bigvee_{y \in \tilde{N}^{(f^0)}(x)} f(y) \\ \varepsilon^a(f, f^0)(x) &= \bigwedge_{y \in N^{(f^0)}(x)} f(y). \end{aligned} \quad (6)$$

Now the partial mappings $f \rightarrow \delta^a(f, f^0)$ and $f \rightarrow \varepsilon^a(f, f^0)$ (i.e., with f^0 fixed) are indeed a dilation and erosion with a space-variant structuring element.

Summarizing, to be able to talk about adaptive dilation and erosion [10], or about adaptive neighbourhood alternating sequential filters [13], one has to fix the adaptive neighbourhoods $N^{(f^0)}(x)$ once they have been derived from an initial input image f^0 . Then one can apply the operators in (6) to any input image f , and also use combinations of them to construct adaptive opening, closing, alternating sequential filters, etc., using the same adaptive neighbourhoods $N^{(f^0)}$ in all of them. One of the few papers we found which explicitly mentions this is the work by Lerallut *et al.* [12, section 2.2.2] (they call f^0 the “pilot image”).

4. ADAPTIVE NEIGHBOURHOOD MORPHOLOGY WITH GLOBAL INVARIANCE

Having clarified the notion of adaptive neighbourhood dilation and erosion by introducing the notation (6), we can now also address the question when such operators can be called translation-invariant.

Let f be an input image and define the translation f_h of f over the vector h by $f_h(x) = f(x - h)$ for all $x \in E$. Assume that the structuring function is invariant with respect to translation of f in the following sense:

$$N^{(f_h)}(x) = (N^{(f)}(x - h))_h. \quad (7)$$

In the cases of morphological amoebas or adaptive neighbourhood morphology mentioned above, it is easy to see that this formula holds: when the input image is translated, the corresponding amoebas or adaptive neighbourhoods will be translated accordingly.

Assuming (7) holds, the operators in (3) are easily shown to be translation invariant:

$$(\delta(f))_h = \delta(f_h), \quad (\varepsilon(f))_h = \varepsilon(f_h).$$

This translation invariance does not contradict the theorem that the only translation-invariant dilations or erosions have a fixed structuring element, since ε and δ do not form an adjunction on \mathcal{L} .

Under the same condition (7), the adaptive dilation and erosion in (6) are translation-invariant in the following sense. When we translate *both* input image *and* pilot image, and then carry out the adaptive dilation, the result is the same as when translating the output of the adaptive dilation (and similarly for the erosion). In other words,

$$(\delta^a(f, f^0))_h = \delta^a(f_h, f_h^0), \quad (\varepsilon^a(f, f^0))_h = \varepsilon^a(f_h, f_h^0)$$

This type of translation invariance is desirable by the same argument as for the classical morphological operators [1]. That is, when the image is obtained by a camera and the camera is slightly moved, the result of the image operation (adaptive or not) should move accordingly. Of course, in practice translation invariance can only be true modulo boundary effects, but this does not in the least diminish its fundamental importance.

One can now extend translation invariance to other types of group invariance for adaptive neighbourhood operators by extending the set of structuring elements attached to each pixel location, just as for the group morphologies with non-adaptive structuring elements. We will consider the details elsewhere.

5. CONCLUSIONS

We have recapitulated the various roles of spatial dependence of the structuring element in mathematical morphology. On the one hand, it allows to generalise translation invariance by letting the shape of the structuring element spatially adapt in such a way that global group invariance is maintained. On the other hand, morphological operators can be made adaptive by letting the size or shape of structuring elements depend on image location or local image features. We demonstrated that one has to be careful when speaking of dilation and erosion, or other types of morphological operators, in the input-adaptive case. Finally, we have shown that there is no inherent incompatibility between a spatially-variant structuring function and global translation invariance (or other types of group invariance) of the corresponding morphological operators. When interpreted in an appropriate way this type of invariance is perfectly sensible from a practical point of view.

6. REFERENCES

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