The Journal of Geometric Analysis
Volume 14, Number 2, 2004

# Addenda to "The Entropy Formula for Linear Heat Equation"

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*ABSTRACT.* *We add two sections to [8] and answer some questions asked there. In the first section we give another derivation of Theorem 1.1 of [8], which reveals the relation between the entropy formula, (1.4) of [8], and the well-known Li–Yau's gradient estimate. As a by-product we obtain the sharp estimates on 'Nash's entropy' for manifolds with nonnegative Ricci curvature. We also show that the equality holds in Li–Yau's gradient estimate, for some positive solution to the heat equation, at some positive time, implies that the complete Riemannian manifold with nonnegative Ricci curvature is isometric to $\mathbb{R}^n$. In the second section we derive a dual entropy formula which, to some degree, connects Hamilton's entropy with Perelman's entropy in the case of Riemann surfaces.*

## 1. The relation with Li–Yau's gradient estimates

In this section we provide another derivation of Theorem 1.1 of [8] and discuss its relation with Li–Yau's gradient estimates on positive solutions of heat equation. The formulation gives a sharp upper and lower bound estimates on Nash's 'entropy quantity' $-\int_M H \log H \, dv$ in the case $M$ has nonnegative Ricci curvature, where $H$ is the fundamental solution (heat kernel) of the heat equation. This section is following the ideas in the Section 5 of [9].

Let $u(x,t)$ be a positive solution to $\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0$ with $\int_M u \, dv = 1$. We define

$$N(u,t) = \int_M -(\log u)\, u \, dv$$

and

$$\tilde{N}(u,t) = N(u,t) - \frac{n}{2}\log(4\pi t) - \frac{n}{2} \, .$$

Simple calculation shows that

$$\frac{d\tilde{N}}{dt} = -\int_M \left(\Delta(\log u) + \frac{n}{2t}\right) u \, dv \, . \tag{1.1}$$

*Math Subject Classifications.* Primary: 58G11.
*Key Words and Phrases.* Entropy formula, heat equation, Li–Yau inequality.
*Acknowledgements and Notes.* Research partially supported by NSF grant DMS-0328624.


ISSN 1050-6926

Notice that the integrand is nothing but the expression in Li–Yau's gradient estimate, which states

$$\Delta \log u + \frac{n}{2t} \geq 0 \tag{1.2}$$

when $M$ has nonnegative Ricci curvature. Using (1.2) one can arrive at the following sharp estimate on $-\int_M H \log H \, dv$. The lower bound (not sharp) was proved in the case of $M$ being the Euclidean space, for example, in Nash's article [7, p. 936], via the so-called Nash inequality.

**Proposition 1.1.** *Let $M^n$ be a complete Riemannian manifold with nonnegative Ricci curvature. Let $H$ be a the heat kernel. Then $\tilde{N}(H, t)$ satisfies the following properties:*

i) $\frac{d\tilde{N}}{dt} < 0$, *unless $M$ is isometric to $\mathbb{R}^n$.*

ii) $\lim_{t\to 0} \tilde{N}(H, t) = 0$ *and* $\lim_{t\to\infty} \tilde{N}(H, t) = \log \theta_\infty$. *Here* $\theta_\infty = \lim_{r\to\infty} \frac{V_x(r)}{\omega_n r^n}$ *and* $\omega_n$ *is the volume of the unit ball in $\mathbb{R}^n$.*

*In particular, $\tilde{N}(H, t)$ is bounded from below if and only if $M$ is of maximum volume growth and*

$$\frac{n}{2}\log(4\pi e t) + \log \theta_\infty \leq -\int_M H \log H \, dv \leq \frac{n}{2}\log(4\pi e t) . \tag{1.3}$$

***Proof.*** The monotonicity is a simple consequence of Li–Yau's estimates (1.2). The proof of the equality case is similar to the proof of Theorem 1.4 of [8] (See [8, p. 93].) The argument is that $\frac{d\tilde{N}}{dt} \leq 0$ and equality holds implies that $\Delta \log H + \frac{n}{2t} = 0$. Letting $t \to 0$, one can have that $\Delta r^2(x, y) = 2n$. The rest of the proof to (i) is the same as in the proof of Theorem 1.4 of [8]. The proof of the second part follows from estimates given in Theorem 2.1 of Li–Tam–Wang [5]. We leave the details to the interested readers. □

The above result gives the sharp lower bound on 'Nash's entropy' $\tilde{N}(H, t) \geq \log(\theta_\infty)$. It is interesting to compare it with the sharp lower bound stated in [7, p. 936] for the uniformly parabolic linear operator of divergence form on Euclidean spaces, which says that $\tilde{N}(H, t) - \frac{n}{2} \geq \log(c_1^{\frac{n}{2}})$, where $c_1$ is the lower bound of the eigenvalues of the coefficient matrix of the considered divergence parabolic operator.

We denote $\tilde{F}(u, t) = \frac{d\tilde{N}}{dt}$. It is easy to check that 'Perelman's entropy' $\mathcal{W}(f, t)$ defined in (1.2) of [8], where $u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$, can be written as

$$\mathcal{W}(f, t) = \frac{d}{dt}\big(t\tilde{N}\big) = t\tilde{F}(u, t) + \tilde{N}(u, t) .$$

Proposition 1.1 in fact, is stronger than Theorem 1.4 of [8] since $\mathcal{W}(f, t) = 0$ for some $t > 0$ implies that both $\tilde{F}(u, t) = 0$ and $\tilde{N}(u, t) = 0$ since we observe that $\tilde{F}(u, t) \leq 0$ and $\tilde{N}(u, t) \leq 0$. The following lemma is Hamilton's formulation [4] of Li–Yau's gradient estimates in [6], which is just the Bochner formula, observing $\big(\frac{\partial}{\partial t} - \Delta\big)\Delta \log u = \Delta(\big(\frac{\partial}{\partial t} - \Delta\big)\log u) = \Delta(|\nabla \log u|^2)$.

**Lemma 1.2.** *Let $Q = \Delta \log u$. Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q = 2 < \nabla \log u, \nabla Q > +2\left|\nabla_i \nabla_j \log u\right|^2 + 2R_{ij}(\nabla \log u)_i(\nabla \log u)_j . \tag{1.4}$$

Integrating Lemma 1.2 one has that

$$\frac{dF}{dt} = -2\int_M \left(\left|\nabla_i \nabla_j \log u\right|^2 + R_{ij}(\nabla \log u)_i(\nabla \log u)_j\right) u \, dv . \tag{1.5}$$

Here $F(u,t) = \tilde{F}(u,t) + \frac{n}{2t}$. Now one can derive Theorem 1.1 of [8] from (1.5) as follows.

$$\begin{aligned}\frac{d\mathcal{W}}{dt} &= t\tilde{F}'(u,t) + 2F(t) - \frac{n}{t} \\ &= -2t\int_M \Big(|\nabla_i\nabla_j f|^2 + R_{ij} f_i f_j\Big) u\, dv + 2\int_M -(\Delta \log u) u\, dv - \frac{n}{2t}\,.\end{aligned}$$

Completing the square we have the entropy formula for the linear heat equation, namely (1.4) of [8].

With the above discussion, we can include a proof of Corollary 4.3 of [8].

***Proof of Corollary 4.3 of [8].*** From Proposition 1.1 we know that, under the assumption that $M$ has maximum volume growth, which is equivalent to $\mathcal{W}(f,t)$ being bounded, $|\tilde{N}(u,2t) - \tilde{N}(u,t)| \le \epsilon$ for $t >> 1$. This implies that there exists $t_i$ such that $t_i \tilde{F}(u,t_i) \to 0$ as $t_i \to \infty$. The monotonicity of $\mathcal{W}(f,t) = t\tilde{F}(u,t) + \tilde{N}(u,t)$ implies that $\lim_{t\to\infty} \mathcal{W}(f,t) = \lim_{t\to\infty} \tilde{N}(u,t)$. □

The discussion above can be summarized in the time derivative of the partition function ('Nash's entropy quantity') $-\int_M u \log u$ is the Li–Yau–Hamilton expression (in the Li–Yau–Hamilton inequalities), and the time derivative of the space integral of Li–Yau–Hamilton expression essentially gives the entropy formula (1.4) of [8]. In the next section we illustrate this principle for the Ricci flow on Riemann surfaces. We would like to thank Ben Chow for explaining to us how one can view Hamilton's differential Harnack as the gradient of Hamilton's entropy in the surfaces case. The corresponding formulation for Perelman's entropy formula is detailed in [2].

Considering the equality case in the above estimates we have the following corollary.

**Corollary 1.3.** *Let $M^n$ be a complete Riemannian manifold with nonnegative Ricci curvature. Assume that $u$ is a positive solution to the heat equation. Then $M^n$ is isometric to $\mathbb{R}^n$ if (1.2) holds equality somewhere for some $t > 0$. One can draw the same conclusion if the equality holds in (1.5) of Theorem 1.2 in [8].*

***Proof.*** First, by the Li–Yau's proof of (1.2) and the strong maximum principle we conclude that (1.2) holds with equality everywhere. Now by (1.1) and the fact that $\mathcal{W}(f,t) = t\tilde{F} + \tilde{N}$ we know that $\mathcal{W}(f,t)$ is constant. By the entropy formula (1.4) of [8], we conclude that the right-hand side of (1.4) in [8] is zero. This implies that $f_{ij} = \frac{1}{2t} g_{ij}$. In particular, $f$ is a strictly convex function on $M$, which already implies that $M$ is diffeomorphic to $\mathbb{R}^n$. On the other hand, integrating along short geodesics showing that $2t(f(x) - f(x_0))$ equals $r^2(x_0, x)$, where $x_0$ is the minimum point of $f$. This shows that $\Delta r^2(x_0,x) = 2n$, which is enough to conclude that $M$ is isometric to $\mathbb{R}^n$. In our above proof we used implicitly that $u$ is integrable. But if one trace the proof of (1.2) in Li–Yau's article, especially Lemma 1.1 of [6], one can also apply the above argument. The similar proof also implies the second claim in the corollary. □

Due to the surprising similarity between the entropy formula for the Ricci flow and entropy formula for the linear heat equation, it is very natural to ask if the limiting value of the entropy for the Ricci flow has any geometric meaning or not. More precisely, the following questions may be interesting.

***Questions.*** Let $(M, g(t))$ be a $\kappa$-solution. (See [9, Section 11] for definition.) One may ask if

$$\lim_{\tau\to\infty} \mathcal{W}(g,f,\tau) = \lim_{\tau\to\infty} \tilde{N}(g,u,\tau) = \log\big(\theta_\infty(x_0,t_0)\big)?$$

Here $\theta_\infty(x_0, t_0) = \lim_{\tau\to\infty} \tilde{V}(\tau)$, the limit of the reduced volume (see [9, Section 7] for the definition) with respect to $(x_0, t_0)$, $\tau = t_0 - t$, $u(x, \tau) = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ is the fundamental solution to the conjugate heat equation originated from $(x_0, t_0)$.

One may also ask if the best $\kappa$ constant in the definition of the $\kappa$-solution satisfies

$$\kappa = \inf_{(x_0,t_0)} \theta_\infty(x_0, t_0)?$$

The observation that under the setting of Proposition 1.1,

$$\lim_{t\to\infty} \int_M \frac{e^{-\frac{r^2(x_0,y)}{4t}}}{(4\pi t)^{\frac{n}{2}}}\, dv = \log\theta_\infty$$

may be helpful to see the geometric meaning of the limit of the reduced volume in the above questions.

We want to thank Tom Ilmanen and Peng Lu for their interests and some discussions related to the above questions.

## 2. The dual entropy formula

In this section we present a dual entropy formula for the Ricci flow on Riemann sphere. The formulation indicates the connection between Hamilton's entropy and Perelman's entropy for this special case.

Let $(M, g(t))$ be a solution to the Ricci flow on $M \times [0, T]$. Let $u(x, t)$ be a positive solution to the heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x, t) = \mathcal{R}(x, t)u(x, t)\ . \tag{2.1}$$

We require that

$$\left(\int_M u\, dv\right)\Big|_{t=0} = 1\ .$$

It is easy to see that this is preserved under the flow. We define

$$\mathcal{N}(g, u, t) = \int_M -(\log u)\, u\, dv$$

and

$$\tilde{\mathcal{N}}(g, u, t) = \mathcal{N}(g, u, t) - \frac{n}{2}\log(4\pi t) - \frac{n}{2}\ .$$

Simple calculations show that

$$\frac{d\tilde{\mathcal{N}}}{dt} = -\int_M \left(\Delta(\log u) + \mathcal{R} + \frac{n}{2t}\right) u\, dv\ . \tag{2.2}$$

Now we similarly define

$$\tilde{\mathcal{F}}(g, u, t) = -\int_M \left(\Delta(\log u) + \mathcal{R} + \frac{n}{2t}\right) u\, dv$$

and $\mathcal{F}(g,u,t) = \tilde{\mathcal{F}}(g,u,t) + \frac{n}{2t}$. Then $\mathcal{N}'(g,u,t) = \mathcal{F}(g,u,t)$ and $\tilde{\mathcal{N}}'(g,u,t) = \tilde{\mathcal{F}}(g,u,t)$. For the case of the Riemann surfaces with $\mathcal{R}(x,t) > 0$ we let $u(x,t) = \frac{1}{4\pi}\mathcal{R}(x,t)$. Then $u(x,t)$ will satisfy all the above assumptions. The following result follows from Hamilton's proof of the differential Harnack inequality in the surface case.

**Lemma 2.1.** *For the above special chosen $u(x,t)$ one has that*

$$\frac{d\mathcal{F}}{dt} = -2\int_M \left|\nabla_i\nabla_j \log u + R_{ij}\right|^2 u\, dv\,. \tag{2.3}$$

***Proof.*** Let

$$\mathcal{Q} = \Delta \log u + \mathcal{R}\,. \tag{2.4}$$

The calculation in [4] shows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{Q} = 2 < \nabla \log u, \nabla \mathcal{Q} > +2\left|\nabla_i\nabla_j \log u + R_{ij}\right|^2 .$$

This implies that

$$\left(\frac{\partial}{\partial t} - \Delta - \mathcal{R}\right)(\mathcal{Q}u) = 2\left|\nabla_i\nabla_j \log u + R_{ij}\right|^2 u$$

which proves (2.3) after the integration. □

**Corollary 2.2.**

$$\mathcal{F}(g,u,t) \le \frac{n}{2t}\,. \tag{2.5}$$

***Proof.*** This is a consequence of Hamilton's differential Harnack in [4]. (See also [9, Proposition 1.2].) □

***Remark.*** $\mathcal{N}(g,u,t) - \log A(t)$ is Hamilton's entropy quantity, which is non-decreasing. Here $A(t)$ is the area of the evolving metric. While in [9], $\tilde{\mathcal{N}}(g,u,t)$ is called the partition function and the entropy was used for a different quantity which we shall illustrate further below.

As in [9] we define the entropy to be

$$\mathcal{W}(g,f,t) = t\tilde{\mathcal{F}}(g,u,t) + \tilde{\mathcal{N}}(g,u,t)\,.$$

Here we write $u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$. We also have the similar expression as [9].

$$\mathcal{W}(g,f,t) = \int_M \left(t\left(|\nabla f|^2 - \mathcal{R}\right) + f - n\right) u\, dv\,.$$

The dual entropy formula states as following.

**Proposition 2.3.**

$$\frac{d\mathcal{W}}{dt} = -2t\int_M \left|\nabla_i\nabla_j \log u + R_{ij} + \frac{1}{2t}g_{ij}\right|^2 u\, dv\,. \tag{2.6}$$

***Remark.*** Unlike Perelman's entropy, which is non-decreasing along the flow, the entropy define above is non increasing.

***Proof.*** Direct calculation shows that

$$\frac{d\mathcal{W}}{dt} = t\tilde{\mathcal{F}}'(g, u, t) + 2\mathcal{F}(g, u, t) - \frac{n}{t} .$$

Applying Lemma 2.1 we have that

$$\frac{d\mathcal{W}}{dt} = -2t \int_M \left|\nabla_i \nabla_j \log u + R_{ij}\right|^2 u\, dv + 2 \int_M -(\Delta \log u + \mathcal{R}) u\, dv - \frac{n}{2t} .$$

Completing the square finishes the proof. □

Besides that common expressions appear in both Perelman's entropy formula and the formula above, we call (2.6) the dual entropy formula since Perelman's entropy formula is considering backward Ricci flow and (2.6) is on the forward Ricci flow. The right-hand side of the Perelman's entropy formula is the shrinking gradient soliton equation while the right-hand side above is the expanding gradient soliton equation. In [3], the authors discovered another dual entropy formula which holds for all dimensions.

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Received November 12, 2003
Revision received February 20, 2004

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