Addenda to the Survey of Layout Problems

Jordi Petit
Departament de Llenguatges i Sistemes Informàtics
Universitat Politècnica de Catalunya

Abstract

In 2002, J. Díaz, M. Serna and the author published "A Survey of Graph Layout Problems", which then was a complete view of the current state of the art of layout problems from an algorithmic point of view. The current review expands the contents of the original survey with updated results from these latest ten years and contributes an extensive bibliography.

1 Introduction

Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such way that a certain objective cost is optimized. In [Díaz et al., 2002], Maria Serna, Josep Díaz and me presented a survey that considered their motivation, complexity, approximation properties, upper and lower bounds, heuristics and probabilistic analysis on random graphs. The result was a complete view of the current state of the art with respect to layout problems from an algorithmic point of view.

In the occasion of Professor Josep Díaz's 60th birthday, a conference on Trends in Theoretical Computer Science was organized [Àlvarez et al., 2011]. Besides celebrating his birthday, the purpose of the conference was to bring together researchers to discuss various aspects of Theoretical Computer Science. As invited speaker, I opted to take the opportunity to revisit layout problems —the topic of my thesis and one of the favorites themes of Professor Díaz.

Indeed, after ten years, I found worthwhile to perform an exhaustive search on the newer literature on the topic and, consequently, to update the results in the survey. I was also curious to know whether major results in heuristics for the Minimum Linear Arrangement problem had appeared. Furthermore, the current status of the open problems stated in the original survey intrigued me.

The result of this search is synthesised in these addenda to the survey. In particular, the following topics are covered: new **NP**-completeness results, new polynomial time solvable graph classes, new fixed parametrized complexity results,

Bulletin of the EATCS no 105, pp. 177–201, October 2011

new approximability results, and new practical results on the Minimum Linear Arrangement problem. It does not cover: new application domains, new results on random graphs, other variants of layout problems.

2 Layout problems

For the sake of completeness, we start with the definition of several graph layout problems and associated concepts. This enables us to treat different layout problems using a unique framework.

Linear layouts. A *linear layout*, or simply a *layout*, of an undirected graph $G = (V, E)$ with $n = |V|$ vertices is a bijective function $\varphi\colon V \to [n] = \{1, \dots, n\}$. A layout has also been called a *linear ordering* [Adolphson et al., 1973], a *linear arrangement* [Shiloach, 1979], a *numbering* [Chinn et al., 1982] or a *labeling* [Juvan et al., 1992] of the vertices of a graph. We denote by $\Phi(G)$ the set of all layouts of a graph G.

Given a layout φ of a graph $G = (V, E)$, its *reversed layout* is denoted φ^R and is defined by $\varphi^R(u) = |V| - \varphi(u) + 1$ for all $u \in V$.

Layout measures. Given a layout φ of a graph $G = (V, E)$ and an integer i, define the set $L(i, \varphi, G) = \{u \in V \mid \varphi(u) \leq i\}$ and the set $R(i, \varphi, G) = \{u \in V \mid \varphi(u) > i\}$. The *edge cut* at position i of φ is defined as

$$\theta(i, \varphi, G) = |\{uv \in E \mid u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|$$

and the *modified edge cut* at position i of φ as

$$\zeta(i, \varphi, G) = |\{uv \in E \mid u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G) \wedge \varphi(u) \neq i\}|.$$

The *vertex cut* or *separation* at position i of φ is defined as

$$\delta(i, \varphi, G) = |\{u \in L(i, \varphi, G) \mid \exists v \in R(i, \varphi, G) : uv \in E\}|.$$

Given a layout φ of G and an edge $uv \in E$, the *length* of uv on φ is

$$\lambda(uv, \varphi, G) = |\varphi(u) - \varphi(v)|.$$

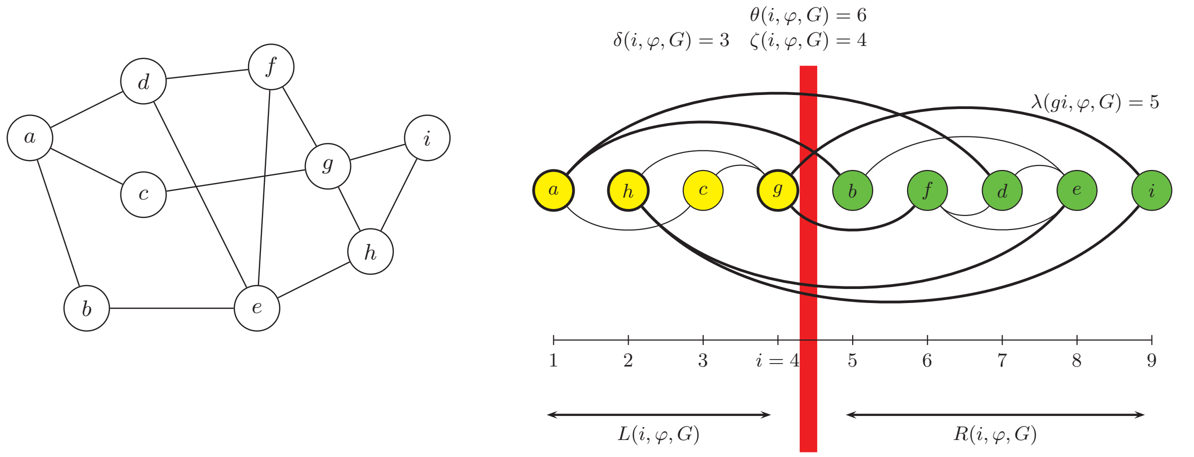
These measures are summarized in Table 1.

Table 1: Layout measures for a layout φ of a graph $G = (V, E)$.

$L(i, \varphi, G)$	$=$	$\{u \in V \mid \varphi(u) \leq i\}$.
$R(i, \varphi, G)$	$=$	$\{u \in V \mid \varphi(u) > i\}$.
$\theta(i, \varphi, G)$	$=$	$\lvert\{uv \in E \mid u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}\rvert$.
$\zeta(i, \varphi, G)$	$=$	$\lvert\{uv \in E \mid u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G) \wedge \varphi(u) \neq i\}\rvert$.
$\delta(i, \varphi, G)$	$=$	$\lvert\{u \in L(i, \varphi, G) \mid \exists v \in R(i, \varphi, G) : uv \in E\}\rvert$.
$\lambda(uv, \varphi, G)$	$=$	$\lvert\varphi(u) - \varphi(v)\rvert, \qquad uv \in E$.

Table 2: Layout problems and costs for a graph $G = (V, E)$ with $|V| = n$.

Problem	Name	Cost
Bandwidth	BANDWIDTH	$\text{BW}(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G)$.
Min. Lin. Arrangement	MINLA	$\text{LA}(\varphi, G) = \begin{cases} \sum_{uv \in E} \lambda(uv, \varphi, G), \\ \sum_{i=1}^{n} \theta(i, \varphi, G). \end{cases}$
Cutwidth	CUTWIDTH	$\text{CW}(\varphi, G) = \max_{i=1}^{n} \theta(i, \varphi, G)$.
Modified Cut	MODCUT	$\text{MC}(\varphi, G) = \sum_{i=1}^{n} \zeta(i, \varphi, G)$.
Vertex Separation	VERTSEP	$\text{VS}(\varphi, G) = \max_{i=1}^{n} \delta(i, \varphi, G)$.
Pathwidth	PATHWIDTH	$\text{PW}(\varphi, G) = \text{VS}(\varphi, G)$.
Sum Cut	SUMCUT	$\text{SC}(\varphi, G) = \sum_{i=1}^{n} \delta(i, \varphi, G)$.
Profile	PROFILE	$\text{PR}(\varphi, G) = \begin{cases} \sum_{u \in V} (\varphi(u) - \min_{v \in \Gamma^*(u)} \varphi(v)), \\ \text{SC}(\varphi^R, G). \end{cases}$
Edge Bisection	EDGEBIS	$\text{EB}(\varphi, G) = \theta(\lfloor n/2 \rfloor, \varphi, G)$.
Vertex Bisection	VERTBIS	$\text{VB}(\varphi, G) = \delta(\lfloor n/2 \rfloor, \varphi, G)$.



(a) $G = (V, E)$.

(b) Graphical representation of φ. Dividing the layout at $i = 4$, left vertices are shown in light yellow, right vertices in dark green, cut edges and separator vertices are bold.

Figure 1: A graph G together with some layout measures and a graphical representation of the layout $\varphi = \{(a, 1), (b, 5), (c, 3), (d, 7), (e, 8), (f, 6), (g, 4), (j, 9), (h, 2)\}$.

Graphical representation. A common way to represent a layout φ of a graph G is to align its vertices on a horizontal line, mapping each vertex u to position $\varphi(u)$, as shown in Figure 1. This graphical representation gives an easy understanding of the previously defined measures: By drawing a vertical line just after position i and before position $i + 1$, the vertices at the left of the line belong to $L(i, \varphi, G)$ and the vertices at the right of the line belong to $R(i, \varphi, G)$. It is easy to compute the cut $\theta(i, \varphi, G)$ by counting the number of edges that cross the vertical line. The modified cut $\zeta(i, \varphi, G)$ counts all the edges in $\theta(i, \varphi, G)$ except those that have vertex $\varphi^{-1}(i)$ as endpoint. It is also easy to compute the separation $\delta(i, \varphi, G)$ by counting the number of vertices at the left of the vertical line that are joined with some vertex at the right of the vertical line. Finally, the length $\lambda(uv, \varphi, G)$ of an edge uv corresponds to the natural distance between its endpoint images.

Layout costs. A *layout cost* is a function F that associates to each layout φ of a graph G an integer $F(\varphi, G)$. Let F be a layout cost; the optimization layout problem associated with F consists in determining some layout $\varphi^* \in \Phi(G)$ of an input graph G such that

$$F(\varphi^*, G) = \min_{\varphi \in \Phi(G)} F(\varphi, G).$$

For any F and G, we define $\textsc{min}F(G) = \min_{\varphi \in \Phi(G)} F(\varphi, G)$.

Layout problems. The particular costs we consider are listed below, together with the layout problems they give raise to:

- *Bandwidth* (Bandwidth): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{bw}(\varphi^*, G) = \textsc{minbw}(G)$ where

$$\textsc{bw}(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G).$$

- *Minimum Linear Arrangement* (MinLA): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{la}(\varphi^*, G) = \textsc{minla}(G)$ where

$$\textsc{la}(\varphi, G) = \sum_{uv \in E} \lambda(uv, \varphi, G) = \sum_{i=1}^{n} \theta(i, \varphi, G).$$

- *Cutwidth* (Cutwidth): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{cw}(\varphi^*, G) = \textsc{mincw}(G)$ where

$$\textsc{cw}(\varphi, G) = \max_{i \in [|V|]} \theta(i, \varphi, G).$$

- *Modified Cut* (ModCut): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{mc}(\varphi^*, G) = \textsc{minmc}(G)$ where

$$\textsc{mc}(\varphi, G) = \sum_{i \in [|V|]} \zeta(i, \varphi, G).$$

- *Vertex Separation* or *Pathwidth* (VertSep/Pathwidth): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{vs}(\varphi^*, G) = \textsc{minvs}(G)$ where

$$\textsc{vs}(\varphi, G) = \max_{i \in [|V|]} \delta(i, \varphi, G).$$

- *Sum Cut* (SumCut): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{sc}(\varphi^*, G) = \textsc{minsc}(G)$ where

$$\textsc{sc}(\varphi, G) = \sum_{i \in [|V|]} \delta(i, \varphi, G).$$

- *Profile* (Profile): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{pr}(\varphi^*, G) = \textsc{minpr}(G)$ where

$$\textsc{pr}(\varphi, G) = \sum_{u \in V} \left(\varphi(u) - \min_{v \in \Gamma^*(u)} \varphi(v) \right) \text{ and } \Gamma^*(u) = \{u\} \cup \{v \in V \mid uv \in E\}.$$

  The Profile and SumCut problems are equivalent.

- *Edge Bisection* (EdgeBis): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{eb}(\varphi^*, G) = \textsc{mineb}(G)$ where

$$\textsc{eb}(\varphi, G) = \theta(\left\lfloor \tfrac{1}{2}|V| \right\rfloor, \varphi, G).$$

- *Vertex Bisection* (VertBis): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\textsc{vb}(\varphi^*, G) = \textsc{minvb}(G)$ where

$$\textsc{vb}(\varphi, G) = \delta(\left\lfloor \tfrac{1}{2}|V| \right\rfloor, \varphi, G).$$

Although the Edge Bisection and Vertex Bisection problems are not strictly layout problems, they fit well in our framework for layout problems.

The definitions of the previous problems are summarized in Table 2.

3 Results

The results of addenda are mainly organized as entries in the following tables. In order to distinguish the new results, their respective entries are marked with a little star ($\star$) in the tables. Because some results have been published first in conference proceedings and then in journals, table entries are sorted topologically, rather than strictly chronologically.

Table 3: Complexity results

BW	in general	[Papadimitriou, 1976]
	for trees with maximum degree 3	[Garey et al., 1978]
	for caterpillars with hair-length ≤ 3	[Monien, 1986]
	for caterpillars with ≤ 1 hair per backbone vertex	[Monien, 1986]
	for cyclic caterpillars	[Muradyan, 1999]
	⋆ for cyclic caterpillars with hair length 1	[Muradian, 2003]
	for grid graphs and unit disk graphs	[Díaz et al., 2001]
LA	in general	[Garey et al., 1976]
	for bipartite graphs	[Even et al., 1975]
	⋆ for interval graphs	[Cohen et al., 2006]
	⋆ for permutation graphs	[Cohen et al., 2006]
CW	in general	[Gavril, 1977]
	for graphs with maximum degree 3	[Makedon et al., 1985]
	for planar graphs with maximum degree 3	[Monien et al., 1988]
	for grid graphs and unit disk graphs	[Díaz et al., 2001]
	⋆ for split graphs and chordal graphs	[Heggernes et al., 2008]
MC	for planar graphs with maximum degree 3	[Monien et al., 1988]
PW	in general	[Lengauer, 1981]
	for planar graphs with maximum degree 3	[Monien et al., 1988]
	for ⋆ starlike and chordal graphs	[Gustedt, 1993]
	for bipartite graphs	[Goldberg et al., 1995]
	for grid graphs and unit disk graphs	[Díaz et al., 2001]
	⋆ for split graphs	[Gustedt, 1993]
SC	in general	[Díaz et al., 1991]
		[Lin et al., 1994b]
		[Golovach, 1997]
	for cobipartite graphs	[Yuan et al., 1998]
	⋆ for split graphs	[Peng et al., 2006]
EB	in general	[Garey et al., 1976]
	for graphs with maximum degree 3	[MacGregor, 1978]
	for graphs with bounded maximum degree	[MacGregor, 1978]
	for d-regular graphs	[Bui et al., 1987]

Table 4: Graphs optimally solvable in polynomial time I

BW	caterpillars with hair-length ≤ 2	$O(n \log n)$	[Assman et al., 1981]
	hypercubes	$O(n \log n)$	[Harper, 1966]
	⋆ hypercubes		[Wang et al., 2009]
	butterflies	$O(n \log n)$	[Lai, 1997]
	interval graphs	$O(n\Delta^2 \log \Delta)$	[Muradyan, 1986]
	interval graphs	$O(n \log n)$	[Mahesh et al., 1991]
	interval graphs	$O(n \log n)$	[Sprague, 1994]
	chain graphs	$O(n^2 \log n)$	[Kloks et al., 1998]
	complete k-level t-ary tree	$O(n)$	[Heckmann et al., 1998]
	⋆ rectangular grids	$O(1)$	[FitzGerald, 1974]
	⋆ cubic grids	$O(1)$	[FitzGerald, 1974]
	square grids	$O(n)$	[Mai et al., 1984]
	⋆ toroidal meshes	$O(1)$	[Paterson et al., 1993]
	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
	⋆ 3-dimensional grids	$O(1)$	*[Otachi et al., 2011]
LA	trees	$O(n^3)$	[Goldberg et al., 1976]
	rooted trees	$O(n \log n)$	[Adolphson et al., 1973]
	trees	$O(n^{2.2})$	[Shiloach, 1979]
	trees	$O(n^{\log 3/\log 2})$	[Chung, 1988]
	rectangular grids	$O(n)$	[Muradyan et al., 1980]
	square grids	$O(n)$	[Mitchison et al., 1986]
	2-dimensional cylinder	$O(n)$	[Muradyan, 1982]
	hypercubes	$O(n)$	[Harper, 1964]
	de Bruijn graph of order 4	$O(n)$	[Harper, 1970]
	d-dimensional c-ary cliques	$O(n)$	[Ellis et al., 1964]
	complete p-partite graphs	$O(n + p \log p)$	[Muradyan et al., 1988]
	⋆ proper interval graphs	$O(n)$	[Safro, 2002]
	⋆ certain Halin graphs	$O(n)$	[Easton et al., 1996]
	⋆ Outerplanar graphs	$O(\delta^2 n + n^2)$	[Frederickson et al., 1988]
	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
	⋆ chord graphs	$O(n \log n)$	[Rostami et al., 2008]
SC	trees	$O(n^{2.3})$	[Lepin, 1986]
	trees	$O(n)$	[Díaz et al., 1991]
	trees	$O(n^{1.722})$	[Kuo et al., 1994]
	interval graphs	?	[Lin et al., 1994a]
	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
	⋆ d-trapezoid and permutation graphs	$O(n^d)$	[Bodlaender et al., 1998]
	square grids	$O(n)$	[Díaz et al., 2000]
	⋆ cographs	$O(n)$	[Fomin et al., 2000]
	⋆ primitive starlike graphs	pol(n)	[Peng et al., 2006]
	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]

Table 5: Graphs optimally solvable in polynomial time II

CW	trees	$O(n \log^{\Delta-2} n)$	[Chung et al., 1982]
	trees	$O(n \log n)$	[Yannakakis, 1985]
	hypercubes	$O(n)$	[Harper, 1964]
	d-dimensional c-ary cliques	$O(n)$	[Nakano, 1994]
	max degree $\leq \Delta$ and treewidth $\leq k$	$O(n^{\Delta k^2})$	[Thilikos et al., 2001]
	ordinary 2- and 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal and cylindrical 2-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	complete p-partite graphs	$O(n + p \log p)$	[Muradyan et al., 1988]
	⋆ complete binary trees	$O(n)$	[Lengauer, 1982]
	⋆ abelian Cayley graphs	$O(1)$	[Berend et al., 2008]
	⋆ de Bruijn graphs	?	[Raspaud et al., 1995]
	⋆ bounded degree partial w-tree	$n^{O(wd)}$	[Thilikos et al., 2005b]
	⋆ threshols graphs	$O(n)$	[Heggernes et al., 2008]
	⋆ bipartite permutation graphs	$O(n)$	[Heggernes et al., 2010]
	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
PW	trees	$O(n \log n)$	[Ellis et al., 1979]
	trees	$O(n)$	[Skodinis, 2000]
	cographs	$O(n)$	[Bodlaender et al., 1993]
	permutation graphs	$O(n^2)$	[Bodlaender et al., 1995b]
	n-dimensional grids	$O(n^2)$	[Bollobás et al., 1991]
	⋆ bounded degree partial w-tree	$n^{O(wd)}$	[Thilikos et al., 2005b]
	⋆ unicyclic graphs	$O(n)$	[Ellis et al., 2004]
	⋆ 2-dimensional grids, cylinders, tori	$O(n)$	[Ellis et al., 2008]
	⋆ 3-dimensional grids	$O(1)$	[Otachi et al., 2011]*
	⋆ k-starlike, split, primitive starlike graphs		[Peng et al., 2000]
	⋆ trapezoid and permutation graphs	$O(n^2)$	[Bodlaender et al., 1998]
	⋆ biconvex bipartite graphs	$O(n)$	[Peng et al., 2007]
	⋆ circular-arc graphs	$O(n)$	[Suchan et al., 2007]
	⋆ cographs	$O(n)$	[Bodlaender et al., 1993]
	⋆ comparability graphs of interval orders	$O(n)$	[Garbe, 1995]

Table 6: Graphs optimally solvable in polynomial time III

MC	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
EB	trees	$O(n^3)$	[MacGregor, 1978]
	hypercubes	$O(n)$	[Nakano, 1994]
	d-dimensional c-ary arrays	$O(n)$	[Nakano, 1994]
	d-dimensional c-ary cliques	$O(n)$	[Nakano, 1994]
	ordinary 2- and 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal and cylindrical 2-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	grid graphs	$O(n^5)$	[Papadimitriou et al., 1996]
	treewidth $\leq k$	$O(n^2)$	[Soumyanath et al., 1990]
	cube-connected cycles graphs	$O(n)$	[Manabe et al., 1984]
	⋆ Abelian Cayley graphs	?	[Berend et al., 2008]

Table 7: Fixed parameterized complexity results

Bandwidth(2)	$O(n)$	[Garey et al., 1978]
Bandwidth(k)	$O(n^{k+1})$	[Saxe, 1980]
Bandwidth(k)	$O(n^k)$	[Gurari et al., 1984]
Bandwidth(k)	$\mathbf{W}[k]$	[Bodlaender et al., 1994]
Cutwidth(2)	$O(n)$	[Garey et al., 1978]
Cutwidth(k)	$O(n^k)$	[Gurari et al., 1984]
Cutwidth(k)	$O(n^{k-1})$	[Makedon et al., 1989]
Cutwidth(k)	$O(n^2)$	[Fellows et al., 1992]
⋆ Cutwidth(k)	$O(n)$	[Thilikos et al., 2005a]
ModCut(k)	$O(n^2)$	[Fellows et al., 1992]
⋆SumCut(k)	FPT	[Gutin et al., 2006b]
Pathwidth(k)	$O(n^2)$	[Fellows et al., 1988]
Pathwidth(k)	$O(n)$	[Bodlaender, 1996]
MinLA($n + k$)	$O(m + n + 5.88^k)$	[Gutin et al., 2006a]

Table 8: Approximability results I

BW	3-appr for δ-dense graphs	$n^{O(1/\delta)}$	[Karpinski et al., 1997]
	2-appr for AT-free graphs	$O(n^3)$	[Kloks et al., 1999]
	$O(\log n)$-appr for caterpillars	$O(n^2)$	[Haralambides et al., 1991]
	$O(\log n)$-appr for GHB-trees	$O(n^2)$	[Haralambides et al., 1997]
	rnd $O(\log^{4.5} n)$-appr	$O(m(\lg^4 n \lg\lg n)$	[Feige, 2000]
	rnd $O(\log^3 n \sqrt{\log\log n})$-appr	pol(n)	[Dunagan et al., 2001]
	rnd $O(\sqrt{n/\text{BW}(G)} \log n)$-appr	pol(n)	[Blum et al., 2000]
	rnd $O(\log^{2.5} n)$-approx. for trees and for chordal graphs	pol(n)	[Gupta, 2001]
	no **PTAS**	—	[Blache et al., 1998]
	no **PTAS** for trees	—	[Blache et al., 1998]
	no **APX**	—	[Unger, 1998]
	⋆ no **APX** for caterpillars	—	[Dubey et al., 2011]
PW	$O(\log^2 n)$-appr	pol(n)	[Bodlaender et al., 1995a]
	$O(\log n)$-appr for planar graphs	pol(n)	[Bodlaender et al., 1995a]
	⋆ $O(\log^{1.5} n)$-appr	?	[Feige et al., 2005]
	⋆ 3-appr for outerplanar graphs	$O(n)$	[Govindan et al., 1998]
	⋆ 3-appr for Halin graphs	$O(n)$	[Fomin et al., 2006]
	⋆ not appr within an additive constant	—	[Bodlaender et al., 1995a]

Table 9: Approximability results II

LA	**PTAS** for dense graphs	$n^{O(1/\epsilon^2)}$	[Arora et al., 1996]
	$O(\log^2 n)$-appr	pol(n)	[Hansen, 1989]
	$O(\log^2 n)$-appr	pol(n)	[Leighton et al., 1999]
	$O(\log n \log\log n)$-appr	pol(n)	[Even et al., 2000]
	$O(\log n)$-appr	pol(n)	[Rao et al., 1998]
	⋆ $O(\sqrt{\log n}\log\log n)$-appr	pol(n)	[Charikar et al., 2006]
			[Feige et al., 2007]
	$O(\log\log n)$-appr for planar graphs	pol(n)	[Rao et al., 1998]
	⋆ 4-appr for interval graphs	pol(n)	[Safro, 2002]
	⋆ 2-appr for interval graphs	pol(n)	[Cohen et al., 2006]
	⋆ no **APX** under Unique Games Conjecture	—	[Devanur et al., 2006]
	⋆ no **APX** if **NP-C** is not in rnd. sub-exp time	—	[Ambuhl et al., 2007]
CW	**PTAS** for dense graphs	$n^{O(1/\epsilon^2)}$	[Arora et al., 1996]
	$O(\log^2 n)$-appr	pol(n)	[Leighton et al., 1999]
SC	⋆ $O(\log^2 n)$-appr	pol(n)	[Ravi et al., 1991]
	$O(\log n \log\log n)$-appr	pol(n)	[Even et al., 2000]
	⋆ $O(\log n)$-appr for planar graphs	pol(n)	[Even et al., 2000]
	$O(\log n)$-appr	pol(n)	[Rao et al., 1998]
	$O(\log\log n)$-appr for planar graphs	pol(n)	[Rao et al., 1998]
EB	**PTAS** for dense graphs	$f(n,\epsilon)$	[Frieze et al., 1996]
	$O(\log^2 n)$-appr	pol(n)	[Feige et al., 2006]
	$O(\log^{1.5} n)$-appr	pol(n)	[Kao, 2008]
	$O(\log n)$-appr for planar graph	pol(n)	[Feige et al., 2006]
	⋆ no **APX** if **NP-C** not in rnd. sub-exp time	—	[Khot, 2006]

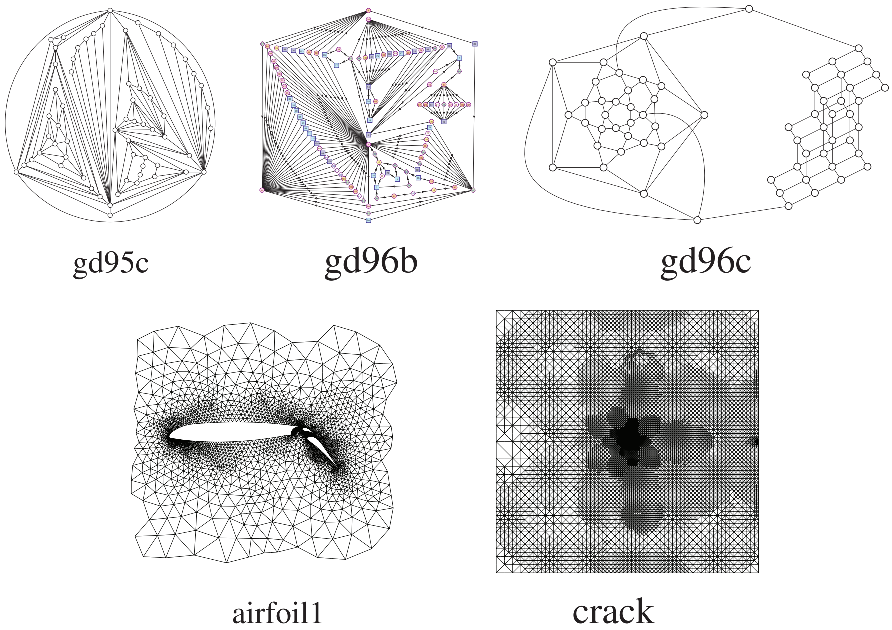


Figure 2: Sample graphs from the MinLA test suite

4 Practical results for MINLA

After having reviewed theoretical results on several layout problems, we turn now to practical results for the Minimum Linear Arrangement problem. We first discuss heuristic methods to get upper bounds, then we consider lower bounds.

Upper bounds. A first comparison of various heuristics for the MinLA problem was reported in [Petit, 1998, Petit, 2003]. This work stablished a test suite of graphs that has become an standard benchmark for this problem. Figure 2 illustrates some of these graphs. The algorithms considered included Successive Augmentation heuristics, Local Search heuristics and Spectral Sequencing. [Petit, 1998] concluded that the best approximations are usually obtained using Simulated Annealing, which involves a large amount of computation time wheras solutions found with Spectral Sequencing are close to the ones found with Simulated Annealing and can be obtained in significantly less time. Taken these considerations into account, [Petit, 2000] combined Spectral Sequencing and Simulated Annealing in a parallel setting to improve both time and solution quality.

The work of [Koren et al., 2002] presented a novel algorithm based on the multi-scale paradigm. Its results are on a par to those of Simulated Annealing, but the running time is significantly faster. Consequently it can be applied to larger graphs (ten thousands of vertices).

In [Poranen, 2005], a Genetic Hillclimbing algorithm is presented and analized on the same test suite. The solutions delivered by Genetic Hillclimbing improve those delivered by Simulated Annealing in quality and runtime in some

Heuristic	Reference	airfoil1	gd95c	gd96c	c1y
SS+SA	[Petit, 2003]	288977	506	519	63858
Genetic	[Poranen, 2005]	306005	506	519	64610
Multi-scale	[Koren et al., 2002]	291794	509	519	64934
Memetic	[Rodriguez-Tello et al., 2006]	285429	506	519	62333
Multi-level	[Safro et al., 2006]	272931	506	519	66262
2stage-SA	[Rodriguez-Tello et al., 2008]	276381	506	519	62230

Table 10: MinLA upper bounds computed by several heuristics on different graphs.

graphs, but not in others. Similarly, in [Rodriguez-Tello et al., 2006], a Memetic algorithm was presented. This algorithm uses a greedy algorithm to generate an initial population, a fine tuned Simulated Annealing algorithm, and a specialized crossover operator. The solutions obtained with this Memetic algorithm are better than those obtained with the Genetic Hillclimbing.

A major improvement on heuristics for MinLA was [Safro et al., 2006], which presented a Multi-level algorithm with weighted edge contraction. This method generates better results in less time and, again, even larger graphs can be considered.

The Two-Stage Simulated Annealing in [Rodriguez-Tello et al., 2008] integrates an heuristic to generate good quality initial solutions, a discriminating evaluation function, a special neighborhood space and an effective cooling schedule. This heuristic can be considered the current MinLA champion, improving several of the previous best results.

Table 10 tries to summarize the above results for a minimalistic benchmark made of four graphs.

Lower bounds. A comparison of various simple lower bounds was also provided in [Petit, 2003]. These included the Degree and Edge methods (both based on the local connectivity of each vertex), the Gomory-Hu tree method (based on fundamental cuts), and the Mesh method (aimed to get lower bounds for graphs arising from finite element methods). The results made evident the existence of a a big gap between the best obtained upper bounds and the best obtained lower bounds.

In [Caprara et al., 2011], so called Decorous lower bounds were presented. These are based on a linear-programming approach, with variables representing distances and an exponential number of constraints. For the smallest grahs in the test suite, the lower bounds stablish gaps of 5–20% with respect to the state-of-the-art heuristics.

A further improvement was obtained in [Schwarz, 2010], which presents a novel approach based on a branch-and-cut algorithm using so called betweenness variables. Coupled with the upper bounds found by various heuristics, this method stablishes the opti-

Graph	best UB	[Petit, 2003]	[Caprara et al., 2011]	[Schwarz, 2010]
c1y	62230	14101	59971	
c2y	78757	17842	76253	
gd95c	**506**	292	443	**506**
gd96a	95263	5155	76253	
gd96b	**1416**	702	1281	**1416**
gd96c	**519**	241	402	**519**
gd96d	2393	595	2021	2391

Table 11: MinLA lower bounds computed by several methods on different graphs.

mality of some linear arrangements for the smallest graphs in the test suite (the have less than 120 vertices) and shows the high quality of the heuristics in medium sized graphs (arround 1000 vertices).

Table 11 gives a feeling on the lower bounds obtained with the above techniques and how they compare with the best upper bounds.

5 Conclusions

This paper extended the survey in [Díaz et al., 2002] with new results that have appeared in the literature up to the present.

Some of the new theoretical results show how the thin line that defines the hardness of some layout problems is being drawn more and more precisely. This can clearly be observed in the case of the Bandwidth problem: researchers have identified many particular classes of graphs where this problem remains **NP**-complete, or can be efficiently solved or cannot be approximated. Likewise, many similar results have been obtained in the latest ten years on the VertSep problem, mainly because of its equivalence with the Pathwidth problem, which has received a great interest due to its relation with the graph minors theory.

With regard to the experimental results for the MinLA problem, one can remark that the current research on lower bounding techniques based on linear programming is very promising, having stablished in some cases that heuristics yield solutions close to the optima. This may suggest that further development of heuristics for this problem is not worth, wheras the current lower bounding methods should be improved to be applied to larger graphs.

Regarding the open problems stated in the concluding remarks of the 2002 original survey, I could not find any reference tackling them.

References

[Adolphson et al., 1973] Adolphson, D. and Hu, T. C. (1973). Optimal linear ordering. *SIAM Journal on Applied Mathematics*, 25(3):403–423.

[Àlvarez et al., 2011] Àlvarez, C., Blesa, M., Duch, A., Gabarró, J., and Martínez, C. (2011). Report on trends in theoretical computer science: Celebrating Josep Díaz's 60th birthday. *Bulletin of the European Association for Theoretical Computer Science*, (104):144–148.

[Ambuhl et al., 2007] Ambuhl, Mastrolilli, and Svensson (2007). Inapproximability results for sparsest cut, optimal linear arrangement, and precedence constrained scheduling. In *FOCS: IEEE Symposium on Foundations of Computer Science (FOCS)*.

[Arora et al., 1996] Arora, S., Frieze, A., and Kaplan, H. (1996). A new rounding procedure for the assignment problem with applications to dense graphs arrangements. In *Proceedings of the 37th Annual Symposium on Foundations of Computer Science*, pages 21–30.

[Assman et al., 1981] Assman, S. F., Peck, G. W., Syslo, M. M., and Zak, J. (1981). The bandwidth of caterpillars with hair of lengths 1 and 2. *SIAM Journal on Algebraic and Discrete Methods*, 2:387–393.

[Berend et al., 2008] Berend, D., Korach, E., and Lipets, V. (2008). Minimal cutwidth linear arrangements of abelian cayley graphs. *Discrete Mathematics*, 308(20):4670–4695.

[Blache et al., 1998] Blache, G., Karpinski, M., and Wirtgen, J. (1998). On approximation intractability of the bandwidth problem. Technical report TR98-014, Electronic Colloquium on Computational Complexity.

[Blum et al., 2000] Blum, A., Konjevod, G., Ravi, R., and Vempala, S. (2000). Semidefinite relaxations for minimum bandwidth and other vertex-ordering problems. *Theoretical Computer Science*, 235(1):25–42.

[Bodlaender et al., 1994] Bodlaender, H., Fellows, M. R., and Hallet, M. T. (1994). Beyond NP-completeness for problems of bounded width: hardness for the W-hierarchy. In *Proceedings of the 26th Annual ACM Symposium on the Theory of Computing*, pages 449–458.

[Bodlaender et al., 1998] Bodlaender, H., Kloks, T., Kratsch, D., and Müller, H. (1998). Treewidth and minimum fill-in on d-trapezoid graphs. *J. Graph Algorithms Appl.*, 2:1–23.

[Bodlaender, 1996] Bodlaender, H. L. (1996). A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM Journal on Computing*, 25(6):1305–1317.

[Bodlaender et al., 1995a] Bodlaender, H. L., Gilbert, J. R., Hafsteinsson, H., and Kloks, T. (1995a). Approximating treewidth, pathwidth, and minimum elimination tree height. *Journal of Algorithms*, 18:238–255.

[Bodlaender et al., 1995b] Bodlaender, H. L., Kloks, T., and Kratsch, D. (1995b). Treewidth and pathwidth of permutation graphs. *SIAM Journal on Discrete Mathematics*, 8(4):606–616.

[Bodlaender et al., 1993] Bodlaender, H. L. and Möhring, R. H. (1993). The pathwidth and treewidth of cographs. *SIAM Journal on Discrete Mathematics*, 6(2):181–188.

[Bollobás et al., 1991] Bollobás, B. and Leader, I. (1991). Edge–isoperimetric inequalities in the grid. *Combinatorica*, 4:299–314.

[Bui et al., 1987] Bui, T., Chaudhuri, S., Leighton, T., and Sipser, M. (1987). Graph bisection algorithms with good average case behavior. *Combinatorica*, 7:171–191.

[Caprara et al., 2011] Caprara, A., Letchford, A., and Salazar, J. (2011). Decorous lower bounds for minimum linear arrangement. *INFORMS J. Comput.*, 23(1):??

[Charikar et al., 2006] Charikar, M., Hajiaghayi, M. T., Karloff, H., and Rao, S. (2006). l22 spreading metrics for vertex ordering problems. In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, SODA '06, pages 1018–1027, New York, NY, USA. ACM.

[Chinn et al., 1982] Chinn, P., Chvátalová, J., Dewdney, A., and Gibbs, N. (1982). The bandwidth problem for graphs and matrices—A survey. *Journal of Graph Theory*, 6:223–254.

[Chung, 1988] Chung, F. R. K. (1988). *Labelings of graphs*, pages 151–168. Academic Press, San Diego.

[Chung et al., 1982] Chung, M., Makedon, F., Sudborough, I. H., and Turner, J. (1982). Polynomial time algorithms for the min cut problem on degree restricted trees. In *Proceedings of the 23th Annual Symposium on Foundations of Computer Science*, pages 262–271.

[Cohen et al., 2006] Cohen, J., Fomin, F. V., Heggernes, P., Kratsch, D., and Kucherov, G. (2006). Optimal linear arrangement of interval graphs. In *In Proceedings of MFCS 2006, LNCS 4162*, pages 267–279. Springer.

[Devanur et al., 2006] Devanur, N., Khot, S., Saket, R., and Vishnoi, N. (2006). Integrality gaps for sparsest cut and minimum linear arrangement problems. In *STOC: ACM Symposium on Theory of Computing (STOC)*.

[Díaz et al., 1991] Díaz, J., Gibbons, A. M., Paterson, M. S., and Torán, J. (1991). The Minsumcut problem. In Dehen, F., Sack, R. J., and Santoro, N., editors, *Algorithms and Data Structures*, volume 519 of *Lecture Notes in Computer Science*, pages 65–79. Springer–Verlag.

[Díaz et al., 2000] Díaz, J., Penrose, M. D., Petit, J., and Serna, M. (2000). Convergence theorems for some layout measures on random lattice and random geometric graphs. *Combinatorics, Probability and Computing*, 9(6):489–511.

[Díaz et al., 2001] Díaz, J., Penrose, M. D., Petit, J., and Serna, M. (2001). Approximating layout problems on random geometric graphs. *Journal of Algorithms*, 39(1):78–116.

[Díaz et al., 2002] Díaz, J., Petit, J., and Serna, M. (2002). A survey on graph layout problems. *ACM Computing Surveys*, 34(3):313–356.

[Dubey et al., 2011] Dubey, C., Feige, U., and Unger, W. (2011). Hardness results for approximating the bandwidth. *J. Comput. Syst. Sci.*, 77:62–90.

[Dunagan et al., 2001] Dunagan, J. and Vempala, S. (2001). On euclidean embeddings and bandwidth minimization. In Goemans, M., Jansen, K., Rolim, J. D. P., and Trevisan, L., editors, *Approximation, Randomization and Combinatorial Optimization: Algorithms and Techniques*, volume 2129 of *Lecture Notes in Computer Science*, pages 229–240. Springer-Verlag.

[Easton et al., 1996] Easton, T., Horton, S. B., and Parker, R. G. (1996). A solvable case of the optimal linear arrangement problem on halin graphs. *Congressus Numerantium (Utilitas mathematica)*, 119:3–17.

[Ellis et al., 2004] Ellis and Markov (2004). Computing the vertex separation of unicyclic graphs. *INFCTRL: Information and Computation (formerly Information and Control)*, 192.

[Ellis et al., 1979] Ellis, J., Sudborough, I. H., and Turner, J. (1979). The vertex separation and search number of a graph. *Information and Computation*, 113:50–79.

[Ellis et al., 1964] Ellis, J. and Warren, R. (1964). Assignments of numbers to vertices. *American Mathematical Monthly*, 71:508–516.

[Ellis et al., 2008] Ellis, J. and Warren, R. (2008). Lower bounds on the pathwidth of some grid-like graphs. *Discrete Applied Mathematics*, 156(5):545–555.

[Even et al., 2000] Even, G., Naor, J., Rao, S., and Schieber, B. (2000). Divide-and-conquer approximation algorithms via spreading metrics. *Journal of the ACM*, 47(4):585–616.

[Even et al., 1975] Even, S. and Shiloach, Y. (1975). NP-completeness of several arrangement problems. Technical report TR-43, Dept. of Computer Science, Technion, Haifa.

[Feige, 2000] Feige, U. (2000). Approximating the bandwidth via volume respecting embeddings. *Journal of Computer and System Sciences*, 60(3):510–539.

[Feige et al., 2005] Feige, U., Hajiaghayi, M., and Lee, J. R. (2005). Improved approximation algorithms for minimum-weight vertex separators. In *Proceedings of the thirty-seventh annual ACM symposium on Theory of computing*, STOC '05, pages 563–572, New York, NY, USA. ACM.

[Feige et al., 2006] Feige, U. and Krauthgamer, R. (2006). A polylogarithmic approximation of the minimum bisection. *SIAM Rev.*, 48:99–130.

[Feige et al., 2007] Feige, U. and Lee, J. R. (2007). An improved approximation ratio for the minimum linear arrangement problem. *Information Processing Letters*, 101(1):26–29.

[Fellows et al., 1988] Fellows, M. R. and Langston, M. A. (1988). Layout permutation problems and well-partially-ordered sets. In *Advanced research in VLSI*, pages 315–327. MIT Press, Cambridge, MA.

[Fellows et al., 1992] Fellows, M. R. and Langston, M. A. (1992). On well-partial-order theory and its application to combinatorial problems of VLSI design. *SIAM Journal on Discrete Mathematics*, 5(1):117–126.

[FitzGerald, 1974] FitzGerald, C. H. (1974). Optimal indexing of the vertices of graphs. *Mathematics of Computation*, 28(127):825–831.

[Fomin et al., 2000] Fomin, F. and Golovach, P. (2000). Graph searching and interval completion. *SIAM Journal on Discrete Mathematics*, 13(4):454–464.

[Fomin et al., 2006] Fomin, F. V. and Thilikos, D. M. (2006). A 3-approximation for the pathwidth of halin graphs. *J. of Discrete Algorithms*, 4:499–510.

[Frederickson et al., 1988] Frederickson and Hambrusch (1988). Planar linear arrangements of outerplanar graphs. *IEEETCS: IEEE Transactions on Circuits and Systems*, 35.

[Frieze et al., 1996] Frieze, A. and Kannan, R. (1996). The regularity lemma and approximation schemes for dense problems. In *Proceedings of the 37th Annual Symposium on Foundations of Computer Science*, pages 12–20.

[Garbe, 1995] Garbe, R. (1995). Tree-width and path-width of comparability graphs of interval orders. In Mayr, E., Schmidt, G., and Tinhofer, G., editors, *Graph-Theoretic Concepts in Computer Science*, volume 903 of *Lecture Notes in Computer Science*, pages 26–37. Springer Berlin / Heidelberg.

[Garey et al., 1978] Garey, M. R., Graham, R. L., Johnson, D. S., and Knuth, D. (1978). Complexity results for bandwidth minimization. *SIAM Journal on Applied Mathematics*, 34:477–495.

[Garey et al., 1976] Garey, M. R., Johnson, D. S., and Stockmeyer, L. (1976). Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1:237–267.

[Gavril, 1977] Gavril, F. (1977). Some NP-complete problems on graphs. In *11th Conference on Information Sciences and Systems*, pages 91–95. John Hopkins University, Baltimore.

[Goldberg et al., 1976] Goldberg, M. K. and Klipker, I. A. (1976). An algorithm for minimal numeration of tree vertices. *Sakharth. SSR Mecn. Akad. Moambe*, 81(3):553–556. In Russian.

[Goldberg et al., 1995] Goldberg, P. W., Golumbic, M. C., Kaplan, H., and Shamir, R. (1995). Four strikes against physical mapping od DNA. *Journal of Computational Biology*, 2(1):139–152.

[Golovach, 1997] Golovach, P. A. (1997). The total vertex separation number of a graph. *Diskretnaya Matematika*, 9(4):86–91. In Russian.

[Govindan et al., 1998] Govindan, R., Langston, M. A., and Yan, X. (1998). Approximating the pathwidth of outerplanar graphs. *Inform. Process. Lett*, 68:68–1.

[Gupta, 2001] Gupta, A. (2001). Improved bandwidth approximation for trees and chordal graphs. *Journal of Algorithms*, 40(1):24–36.

[Gurari et al., 1984] Gurari, E. and Sudborough, I. H. (1984). Improved dynamic programming algorithms for the bandwidth minimization and the mincut linear arrangement problem. *Journal of Algorithms*, 5:531–546.

[Gustedt, 1993] Gustedt, J. (1993). On the pathwidth of chordal graphs. *Discrete Applied Mathematics*, 45(3):233–248.

[Gutin et al., 2006a] Gutin, G., Rafiey, A., Szeider, S., and Yeo, A. (2006a). The linear arrangement problem parameterized above guaranteed value. In Calamoneri, T., Finocchi, I., and Italiano, G., editors, *Algorithms and Complexity*, volume 3998 of *Lecture Notes in Computer Science*, pages 356–367. Springer Berlin / Heidelberg.

[Gutin et al., 2006b] Gutin, G., Szeider, S., and Yeo, A. (2006b). Fixed-parameter complexity of minimum profile problems. *CoRR*, abs/cs/0604095.

[Hansen, 1989] Hansen, M. D. (1989). Approximation algorithms for geometric embeddings in the plane with applications to parallel processing problems. In *30th Annual Symposium on Foundations of Computer Science*, pages 604–609.

[Haralambides et al., 1997] Haralambides, J. and Makedon, F. (1997). Approximation algorithms for the bandwidth minimization problem for a large class of trees. *Theory of Computing Systems*, 30:67–90.

[Haralambides et al., 1991] Haralambides, J., Makedon, F., and Monien, B. (1991). Bandwidth minimization: an approximation algorithm for caterpillars. *Mathematical Systems Theory*, 24:169–177.

[Harper, 1964] Harper, L. H. (1964). Optimal assignments of numbers to vertices. *Journal of SIAM*, 12(1):131–135.

[Harper, 1966] Harper, L. H. (1966). Optimal numberings and isoperimetric problems on graphs. *Journal of Combinatorial Theory*, 1(3):385–393.

[Harper, 1970] Harper, L. H. (1970). Chassis layout and isoperimetric problems. Technical report SPS 37–66, vol II, Jet Propulsion Laboratory.

[Heckmann et al., 1998] Heckmann, R., Klasing, R., Monien, B., and Unger, W. (1998). Optimal embedding of complete binary trees into lines and grids. *Journal of Parallel and Distributed Computing*, 49(1):40–56.

[Heggernes et al., 2008] Heggernes, P., Lokshtanov, D., Mihai, R., and Papadopoulos, C. (2008). Cutwidth of split graphs, threshold graphs, and proper interval graphs. In Broersma, H., Erlebach, T., Friedetzky, T., and Paulusma, D., editors, *Graph-Theoretic Concepts in Computer Science, 34th International Workshop, WG 2008, Durham, UK, June 30 - July 2, 2008. Revised Papers*, volume 5344 of *Lecture Notes in Computer Science*, pages 218–229.

[Heggernes et al., 2010] Heggernes, P., van 't Hof, P., Lokshtanov, D., and Nederlof, J. (2010). Computing the cutwidth of bipartite permutation graphs in linear time. In Thilikos, D., editor, *Graph Theoretic Concepts in Computer Science*, volume 6410 of *Lecture Notes in Computer Science*, pages 75–87. Springer Berlin / Heidelberg.

[Jinjiang et al., 1995] Jinjiang, Y. and Sanming, Z. (1995). Optimal labelling of unit interval graphs. *Applied Mathematics - A Journal of Chinese Universities*, 10:337–344. 10.1007/BF02662875.

[Juvan et al., 1992] Juvan, M. and Mohar, B. (1992). Optimal linear labelings and eigenvalues of graphs. *Discrete Applied Mathematics*, 36(2):153–168.

[Kao, 2008] Kao, M.-Y., editor (2008). *Encyclopedia of Algorithms*. Springer.

[Karpinski et al., 1997] Karpinski, M., Wirtgen, J., and Zelikovsky, A. (1997). An approximating algorithm for the bandwidth problem on dense graphs. Technical report TR 97-017, Electronic Colloquium on Computational Complexity, .

[Khot, 2006] Khot, S. (2006). Ruling out ptas for graph min-bisection, dense k-subgraph, and bipartite clique. *SIAM J. Comput.*, 36(4):1025–1071.

[Kloks et al., 1999] Kloks, T., Kratsch, D., and Müller, H. (1999). Approximating the bandwidth for asteroidal triple-free graphs. *Journal of Algorithms*, 32(1):41–57.

[Kloks et al., 1998] Kloks, T., Kratsch, D., and Müller, H. (1998). Bandwidth of chain graphs. *Information Processing Letters*, 68(6):313–315.

[Koren et al., 2002] Koren, Y. and Harel, D. (2002). A multi-scale algorithm for the linear arrangement problem. In *Revised Papers from the 28th International Workshop on Graph-Theoretic Concepts in Computer Science*, WG '02, pages 296–309, London, UK, UK. Springer-Verlag.

[Kuo et al., 1994] Kuo, D. and Chang, G. J. (1994). The profile minimization problem in trees. *SIAM Journal on Computing*, 23:71–81.

[Lai, 1997] Lai, Y.-L. (1997). *Bandwidth, edgesum and profile of graphs*. PhD Thesis, Dept. of Computer Science, Western Michigan Univ.

[Leighton et al., 1999] Leighton, T. and Rao, S. (1999). Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *Journal of the ACM*, 46(6):787–832.

[Lengauer, 1982] Lengauer (1982). Upper and lower bounds on the complexity of the min-cut linear arrangement problem on trees. *SIJADM: SIAM Journal on Algebraic and Discrete Methods*, 3.

[Lengauer, 1981] Lengauer, T. (1981). Black-white pebbles and graph separation. *Acta Informatica*, 16:465–475.

[Lepin, 1986] Lepin, V. V. (1986). Profile minimization problem of matrices and graphs. *Academy of Sciences of Belarus. Institute of Mathematics.*, 33(269). In Russian.

[Lin et al., 1994a] Lin, Y. and Yuan, J. (1994a). Profile minimization problem for matrices and graphs. *Acta Mathematicae Applicatae Sinica. English Series*, 10(1):107–112.

[Lin et al., 1994b] Lin, Y. X. and Yuan, J. J. (1994b). Profile minimization problem for matrices and graphs. *Acta Mathematicae Applicatae Sinica. English Series. Yingyong Shuxue Xuebao*, 10(1):107–112.

[MacGregor, 1978] MacGregor, R. M. (1978). *On partitioning a graph: a theoretical and empirical study*. PhD Thesis, University of California, Berkeley.

[Mahesh et al., 1991] Mahesh, R., Rangan, C. P., and Srinivasan, A. (1991). On finding the minimum bandwidth of interval graphs. *Information and Computation*, 95(2):218–224.

[Mai et al., 1984] Mai, J. H. and Luo, H. P. (1984). Some theorems on the bandwidth of a graph. *Acta Mathematicae Applicatae Sinica. Yingyong Shuxue Xuebao*, 7(1):86–95.

[Makedon et al., 1985] Makedon, F., Papadimitriou, C. H., and Sudborough, I. H. (1985). Topological bandwidth. *SIAM Journal on Algebraic and Discrete Methods*, 6(3):418–444.

[Makedon et al., 1989] Makedon, F. and Sudborough, I. H. (1989). On minimizing width in linear layouts. *Discrete Applied Mathematics*, 23(3):243–265.

[Manabe et al., 1984] Manabe, Y., Hagihara, K., and Tokura, N. (1984). The minimum bisection widths of the cube-connected cycles graph and cube graph. *Systems-Computers-Controls. The Transactions of the Institute of Electronics and Communication Engineers of Japan*, 15(6):9–18.

[Mitchison et al., 1986] Mitchison, G. and Durbin, R. (1986). Optimal numberings of an $n \times n$ array. *SIAM Journal on Algebraic and Discrete Methods*, 7(4):571–582.

[Monien, 1986] Monien, B. (1986). The bandwidth minimization problem for caterpillars with hair length 3 is NP-complete. *SIAM Journal on Algebraic and Discrete Methods*, 7(4):505–512.

[Monien et al., 1988] Monien, B. and Sudborough, I. H. (1988). Min Cut is NP-complete for edge weighted trees. *Theoretical Computer Science*, 58(1-3):209–229.

[Muradian, 2003] Muradian, D. (2003). The bandwidth minimization problem for cyclic caterpillars with hair length 1 is np-complete. *Theor. Comput. Sci.*, 307:567–572.

[Muradyan, 1999] Muradyan, D. (1999). Computer Science & Information Technologies Conference.

[Muradyan, 1982] Muradyan, D. O. (1982). Minimal numberings of a two-dimensional cylinder. *Akademiya Nauk Armyanskoĭ SSR. Doklady*, 75(3):114–119. In Russian.

[Muradyan, 1986] Muradyan, D. O. (1986). A polynomial algorithm for finding the bandwidth of interval graphs. *Akademiya Nauk Armyanskoĭ SSR. Doklady*, 82(2):64–66. In Russian.

[Muradyan et al., 1980] Muradyan, D. O. and Piliposyan, T. E. (1980). Minimal numberings of vertices of a rectangular lattice. *Akad. Nauk. Armjan. SRR*, 1(70):21–27. In Russian.

[Muradyan et al., 1988] Muradyan, D. O. and Piliposyan, T. E. (1988). The problem of finding the length and width of the complete p-partite graph. *Uchen. Zapiski Erevan. Gosunivers.*, 2:18–26. In Russian.

[Nakano, 1994] Nakano, K. (1994). Linear layouts of generalized hypercubes. In van Leeuwen, J., editor, *Graph-Theoretic Concepts in Computer Science*, volume 790 of *Lecture Notes in Computer Science*, pages 364–375. Springer–Verlag.

[Otachi et al., 2011] Otachi, Y. and Suda, R. (2011). Bandwidth and pathwidth of three-dimensional grids. Technical report, Cornell University Library, arXiv.org > cs > arXiv:1101.0964v1.

[Papadimitriou, 1976] Papadimitriou, C. (1976). The NP-completeness of the bandwidth minimization problem. *Computing*, 16:263–270.

[Papadimitriou et al., 1996] Papadimitriou, C. H. and Sideri, M. (1996). The bisection width of grid graphs. *Mathematical Systems Theory*, 29(2):97–110.

[Paterson et al., 1993] Paterson, M. S., Schröder, H., Sýkora, O., and Vrt'o, I. (1993). A short proof of the dilation of a toroidal mesh in a path. *Information Processing Letters*, 48(4):197–199.

[Peng et al., 2000] Peng, L., Tang, C. Y., Ko, M.-T., Ho, C.-W., and sheng Hsu, T. (2000). Graph searching on some subclasses of chordal graphs. *Algorithmica*, 33:395–426.

[Peng et al., 2006] Peng, S.-L. and Chen, C.-K. (2006). On the interval completion of chordal graphs. *Discrete Appl. Math.*, 154:1003–1010.

[Peng et al., 2007] Peng, S.-L. and Yang, Y.-C. (2007). On the treewidth and pathwidth of biconvex bipartite graphs. In *Proceedings of the 4th international conference on Theory and applications of models of computation*, TAMC'07, pages 244–255, Berlin, Heidelberg. Springer-Verlag.

[Petit, 1998] Petit, J. (1998). Approximation heuristics and benchmarkings for the MinLA problem. In Battiti, R. and Bertossi, A., editors, *Alex '98 — Building bridges between theory and applications*, pages 112–128. .

[Petit, 2000] Petit, J. (2000). Combining spectral sequencing and parallel simulated annealing for the minLA problem. Technical report LSI-01-13-R, Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya.

[Petit, 2003] Petit, J. (2003). Experiments on the minimim linear arrangement problem. *ACM Journal of Experimental Algorithmics*, 8.

[Poranen, 2005] Poranen, T. (2005). A genetic hillclimbing algorithm for the optimal linear arrangement problem. *Fundam. Inf.*, 68:333–356.

[Rao et al., 1998] Rao, S. and Richa, A. W. (1998). New approximation techniques for some ordering problems. In *Proceedings of the 9th ACM-SIAM Symposium on Discrete Algorithms*, pages 211–218.

[Raspaud et al., 1995] Raspaud, A., Sýkora, O., and Vrto, I. (1995). Cutwidth of the bruijn graph. *ITA*, 29(6):509–514.

[Ravi et al., 1991] Ravi, R., Agrawal, A., and Klein, P. (1991). Ordering problems approximated: single-processor scheduling and interval graph completition. In Leach, J., Monien, B., and Rodriguez, M., editors, *Automata, Languages and Programming*, volume 510 of *Lecture Notes in Computer Science*, pages 751–762. Springer–Verlag.

[Rodriguez-Tello et al., 2006] Rodriguez-Tello, E., Hao, J.-K., and Torres-Jimenez, J. (2006). Memetic algorithms for the minla problem. In Talbi, E.-G., Liardet, P., Collet, P., Lutton, E., and Schoenauer, M., editors, *Artificial Evolution*, volume 3871 of *Lecture Notes in Computer Science*, pages 73–84. Springer.

[Rodriguez-Tello et al., 2008] Rodriguez-Tello, E., Hao, J.-K., and Torres-Jimenez, J. (2008). An effective two-stage simulated annealing algorithm for the minimum linear arrangement problem. *Comput. Oper. Res.*, 35:3331–3346.

[Rolim et al., 1995] Rolim, J., Sýkora, O., and Vrt'o, I. (1995). Optimal cutwidths and bisection widths of 2- and 3-dimensional meshes. In *Graph theoretic concepts in computer science*, volume 1017 of *Lecture Notes in Computer Science*, pages 252–264. Springer.

[Rostami et al., 2008] Rostami, H. and Habibia, J. (2008). Minimum linear arrangement of chord graphs. *Applied Mathematics and Computation*, 203(1):358–367.

[Safro, 2002] Safro, I. (2002). *The minimum linear arrangement problem*. PhD Thesis, Weizmann Institute of Science.

[Safro et al., 2006] Safro, I., Ron, D., and Brandt, A. (2006). Graph minimum linear arrangement by multilevel weighted edge contractions. *J. Algorithms*, 60:24–41.

[Saxe, 1980] Saxe, J. B. (1980). Dynamic-programming algorithms for recognizing small-bandwidth graphs in polynomial time. *SIAM Journal on Algebraic Discrete Methods*, 1(4):363–369.

[Schwarz, 2010] Schwarz, R. (2010). *A branch-and-cut algorithm with betweenness variables for the Linear Arrangement Problem*. Diplomarbeit, Universität Heidelberg.

[Shiloach, 1979] Shiloach, Y. (1979). A minimum linear arrangement algorithm for undirected trees. *SIAM Journal on Computing*, 8(1):15–32.

[Skodinis, 2000] Skodinis, K. (2000). Computing optimal linear layouts of trees in linear time. In Paterson, M., editor, *Algorithms — ESA 2000*, volume 1879 of *Lecture Notes in Computer Science*, pages 403–414. Springer–Verlag.

[Soumyanath et al., 1990] Soumyanath, K. and Deogun, J. S. (1990). On the bisection width of partial k-trees. In *Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing*, volume 74 of *Congressus Numerantium*, pages 25–37. Utilitas Math.

[Sprague, 1994] Sprague, A. P. (1994). An $O(n \log n)$ algorithm for bandwidth of interval graphs. *SIAM Journal on Discrete Mathematics*, 7(2):213–220.

[Suchan et al., 2007] Suchan, K. and Todinca, I. (2007). Pathwidth of circular-arc graphs. In *Proceedings of WG 2007*, number 4769 in Lecture Notes in Computer Science, pages 258–269. Lecture Notes in Computer Science.

[Thilikos et al., 2005a] Thilikos, Serna, and Bodlaender (2005a). Cutwidth I: A linear time fixed parameter algorithm. *ALGORITHMS: Journal of Algorithms*, 56.

[Thilikos et al., 2005b] Thilikos, Serna, and Bodlaender (2005b). Cutwidth II: Algorithms for partial w-trees of bounded degree. *ALGORITHMS: Journal of Algorithms*, 56.

[Thilikos et al., 2001] Thilikos, D. M., Serna, M. J., and Bodlaender, H. L. (2001). A polynomial time algorithm for the cutwidth of bounded degree graphs with small treewidth. In Meyer auf der Heide, F., editor, *Algorithms, ESA 2001*, volume 2161 of *Lecture Notes in Computer Science*, pages 380–390. Springer–Verlag.

[Unger, 1998] Unger, W. (1998). The complexity of the approximation of the bandwidth problem. In *37th Annual Symposium on Foundations of Computer Science*, pages 82–91.

[Wang et al., 2009] Wang, X., Wu, X., and Dumitrescu, S. (2009). Note: On explicit formulas for bandwidth and antibandwidth of hypercubes. *Discrete Appl. Math.*, 157:1947–1952.

[Yannakakis, 1985] Yannakakis, M. (1985). A polynomial algorithm for the min-cut linear arrangement of trees. *Journal of the ACM*, 32(4):950–988.

[Yuan et al., 1998] Yuan, J., Lin, Y., Liu, Y., and Wang, S. (1998). NP-completeness of the profile problem and the fill-in problem on cobipartite graphs. *Journal of Mathematical Study*, 31(3):239–243.