**Finance and Stochastics manuscript No.**
(will be inserted by the editor)

# Addendum to: Multilevel dual approach for pricing American style derivatives

**Denis Belomestny · Mark Joshi · John Schoenmakers**

Received: 24 March 2015 / Accepted: 04 May 2015

**Abstract** In this note, we show how the dual approach in its particular form presented in [1] can be fitted into the framework of the recent work [2].

**Keywords** Optimal stopping, Dual approach, Multilevel Monte Carlo

**Mathematics Subject Classification (2010)** 91G60, 65C05, 60G40

**JEL Classification** G10, G12, G13

In the recent paper [2], a class of methodologies for developing upper bounds for Bermudan derivatives via Monte Carlo simulation is studied. This class, in particular, is designed to study methods that involve sub-simulations. Unfortunately, one of the most popular upper bound methodologies, that of Andersen and Broadie [1], does not lie within this class, since while the process used is an approximation to a martingale, the approximation itself is not a martingale with respect to some enlarged filtration as in Example 3.1 in [2].

We recall the dual approach, originally proposed by Rogers [4] and Haugh and Kogan [3], in the setup of [2]. Let

$$(Z_i : i = 0, 1, \ldots, T), \quad T \in \mathbb{N},$$

be a discrete-time, nonnegative stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and adapted to the filtration $\mathbb{F} := (\mathcal{F}_i : 0 \leq i \leq T)$. It is assumed that

$$\mathbb{E}[Z_i] < \infty \quad \text{for } 0 \leq i \leq T.$$

The problem is to find the optimal time to stop in order to maximize the value of $(Z_i)$. Let $\mathcal{T}_i$ denote the set of stopping times taking values in $\{i, i+1, \ldots, n\}$. A well-known

Duisburg-Essen University, Thea-Leymann-Str. 9, D–45127 Essen, Germany, and National Research University Higher School of Economics, Moscow, Russia, E-mail: denis.belomestny@uni-due.de · Centre for Actuarial Studies, Department of Economics, University of Melbourne VIC 3010, Australia, E-mail: mark.joshi@unimelb.edu.au · Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D–10117 Berlin, E-mail: schoenma@wias-berlin.de

fact is that the value of the optimal stopping problem is given by the Snell envelope

$$Y_i^* = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}_i[Z_\tau], \quad 0 \le i \le T,$$

at time $i = 0$, i.e., by $Y_0^*$. In the following, $\tau$ denotes a stopping time, $\mathbb{E}_i := \mathbb{E}_{\mathcal{F}_i}$ denotes the conditional expectation with respect to the $\sigma$-algebra $\mathcal{F}_i$, and sup (inf) is to be understood as *essential supremum* (*essential infimum*).

The dual approach is based on the following observation: for any martingale $(M_j)$ with $M_0 = 0$, we have

$$Y_0^* = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0[Z_\tau] = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0\left[Z_\tau - M_\tau\right] \le \mathbb{E}_0\left[\max_{0 \le j \le T}(Z_j - M_j)\right], \qquad (0.1)$$

so that the right-hand side provides an upper bound for $Y_0^*$. In [3] and [4], it is shown that (0.1) holds with equality (and the equality is almost sure) for the martingale part of the Doob decomposition of $Y^*$, that is,

$$Y_j^* = Y_0^* + M_j^* - A_j^*,$$

where

$$M_j^* = \sum_{i=1}^{j}(Y_i^* - \mathbb{E}_{i-1}[Y_i^*]), \quad A_j^* = \sum_{i=1}^{j}(Y_{i-1}^* - \mathbb{E}_{i-1}[Y_i^*]). \qquad (0.2)$$

In practice, there is a variety of ways to implement the dual method of [4] and [3]. A straightforward way is to approximate the Doob martingale of the Snell envelope by using sub-simulations to estimate continuation values, for example due to a given approximate value function or due to a given suboptimal exercise strategy. These approaches naturally lead to an upper-biased estimate and one objective in [2] was to treat them in a unified way. To this end, Belomestny, Schoenmakers and Dickmann [2] choose a setup where the *approximated martingales are martingales* themselves with respect to some enlarged filtration, thus allowing direct application of the results of [4] and ensuring upper-biased estimates. Although various sub-simulation-based algorithms fall into this setup, the particular Andersen and Broadie [1] algorithm (based on suboptimal stopping families) does not.

Let us recap in more detail. First, a strategy is fixed. Since we have to consider values from forward starting points, this is a vector $(\tau_j)$ of stopping times with $\tau_j \ge j$ and

$$\tau_j > j \implies \tau_{j+1} = \tau_j.$$

Given these, we define a value process

$$Y_j = \mathbb{E}_j[Z_{\tau_j}].$$

[1] define a martingale analogously to (0.2) by

$$\begin{aligned} M_j^{AB} &= \sum_{i=1}^{j}(Y_i - \mathbb{E}_{i-1}[Y_i]) = \mathbb{E}_j[Y_{j+1}] - Y_0 + \sum_{i=0}^{j}(Y_i - \mathbb{E}_i[Y_{i+1}]) \\ &= \mathbb{E}_j[Y_{j+1}] - Y_0 + \sum_{i=0}^{j}(Z_i - \mathbb{E}_i[Y_{i+1}])\, 1_{\{\tau_i = i\}}. \end{aligned}$$

The expectations in this martingale are not immediate and have to be estimated via Monte Carlo simulation. The term $Y_0$ may be computed with high accuracy based on non-nested Monte Carlo simulations, and so $M_j^{AB}$ may be approximated by

$$M_j^{AB,k} = \frac{1}{k}\sum_{\ell=1}^{k}\xi_{j+1}^{(\ell)} - Y_0 + \sum_{i=0}^{j}\left(Z_i - \frac{1}{k}\sum_{\ell=1}^{k}\xi_{i+1}^{(\ell)}\right)1_{\{\tau_i=i\}}, \tag{0.3}$$

where, just as in [2], the conditionally on $\mathcal{F}_T$ independent random variables $\xi_j^{(\ell)}$ are characterized by

$$\mathbb{E}_T[\xi_j^{(\ell)}] = \mathbb{E}_{j-1}[\xi_j^{(\ell)}] = \mathbb{E}_{j-1}[Y_j] = \mathbb{E}_{j-1}[Z_{\tau_j}], \quad j = 1, \ldots, T,$$

and in particular it holds that

$$\mathbb{E}_j\left[M_j^{AB,k}\right] = M_j^{AB} \tag{0.4}$$

is a martingale with respect to the filtration $\mathbb{F}$.

The problem is that $(M_j^{AB,k})$ is not a martingale with respect to the canonically enlarged filtration $\mathbb{F}' := (\mathcal{F}_i' : 0 \le i \le T)$ with

$$\mathcal{F}_j' := \mathcal{F}_j \vee \sigma\left(\xi_p^{(\ell)},\ p = 1, \ldots, j, \ell = 1, \ldots, k\right),$$

unlike the martingale (3.2) of Example 3.1 in [2], since (0.3) is even not adapted to $\mathbb{F}'$. In fact, the adaptedness is destroyed by the presence of the indicator in (0.3), that is, by the fact that in general $1_{\{\tau_i=i\}} \neq 1$ with positive probability.

The lack of the martingale property for the "true" algorithm from [1] is not taken into account in [2]; but it turns out that the full martingale condition is not necessary to obtain an upper bound. Consider equation (0.1). There we assumed that $(M_j)$ is a martingale with respect to $\mathbb{F}$. We now show that the weaker condition that $(\mathbb{E}_j[M_j])$ is a martingale with respect to the filtration $\mathbb{F}$ is sufficient (cf. (0.4)). Indeed, if $(M_j)$, with $M_0 = 0$, is adapted to some extended filtration such that $(\widetilde{M}_j) := (\mathbb{E}_j[M_j])$ is a martingale, we have

$$Y_0^* := \sup_\tau \mathbb{E}[Z_\tau] = \sup_\tau \mathbb{E}[Z_\tau - \widetilde{M}_\tau] = \sup_\tau \mathbb{E}\sum_{j=0}^{T} 1_{\{\tau=j\}}(Z_j - \mathbb{E}_j[M_j])$$

$$= \sup_\tau \mathbb{E}\left[\sum_{j=0}^{T}\mathbb{E}_j[1_{\{\tau=j\}}(Z_j - M_j)]\right] = \sup_\tau \mathbb{E}[Z_\tau - M_\tau] \le \mathbb{E}\left[\max_{j=0,\ldots,T}(Z_j - M_j)\right].$$

Under this extension of the framework, we can set $M_j = M_j^{AB,k}$, and then the algorithm from [1] is encompassed by the proofs of [2] and all the results go over. In particular, it then follows that the rate of convergence of the bias caused by sub-simulations is as in that paper and a multi-level methodology can be implemented.

## References

1. L. Andersen and M. Broadie (2004). A primal-dual simulation algorithm for pricing multidimensional American options. *Management Science*, **50**, 1222-1234.
2. D. Belomestny, J. Schoenmakers, and F. Dickmann (2013). Multilevel dual approach for pricing American style derivatives. *Finance Stoch.*, **17**, 717–742.
3. M. Haugh, L. Kogan (2004). Pricing American options: a duality approach. *Operations Research*, **52**, 258–270.
4. L.C.G. Rogers (2002). Monte Carlo valuation of American options. *Mathematical Finance*, **12**, 271-286.