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Inventiones mathematicae

Addendum to "On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space"

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1. Introduction

Let M be a compact symplectic manifold with symplectic form σ, and let T be a torus acting on M in a Hamiltonian way. The symbol X is used both for an element of the Lie algebra $\underline{t}$ of T and for the corresponding vector field on M. A Hamiltonian action means that there is a momentum mapping

$$J: M \to \underline{t}^* \tag{1.1}$$

having the properties

$$(\sigma|X) = -dJ_X, \qquad X \in \underline{t}, \tag{1.2}$$

$$\sigma(X, Y) = 0, \qquad X, Y \in \underline{t}. \tag{1.3}$$

Here we have used the notation $(\sigma|X)$ for the inner product of the vector field X with the form σ, and $J_X(m) = \langle X, J(m)\rangle$ for the X-component of J.

In an earlier paper [4] it was shown that the push forward $J_*(dm)$ of the Liouville measure dm on M under the momentum mapping J is a piecewise polynomial measure on $\underline{t}^*$. Moreover, in case X has isolated isolated zeros on M an explicit formula for the integral

$$\int_M e^{i\langle X, J(m)\rangle} dm \tag{1.4}$$

was obtained using the method of stationary phase. The goal of this note is to extend this formula to the case that the zeros of X are not necessarily isolated.

If we write N for the zero set of X and $i: N \to M$ for the inclusion, then $i^*(\sigma)$ is a symplectic form on N. The normal bundle E of N in M has the structure of a symplectic vector bundle. Denote this symplectic structure on E by τ. By linearization X induces a fiber preserving automorphism $LX: E \to E$ leaving τ invariant. Here invariant is meant in the infinitesimal sense. We can

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choose an automorphism $i: E \to E$ commuting with LX such that $i^2 = -\mathrm{id}$ and $\tau(i.,.)$ is a Riemannian structure on E. Conclusion is that E becomes a complex vector bundle over N, and LX a complex automorphism of E.

Choose an LX-invariant connection D on E, and let Ω be the curvature matrix. Now we can formulate the following

Theorem. *If $\lambda_1, \ldots, \lambda_r$ are the different weights for the action of LX on E, then*

$$\int_M e^{J_X} e^{\sigma} = \int_N \frac{e^{i^* J_X} e^{i^* \sigma}}{\det\left(\frac{LX + \Omega}{2\pi i}\right)} \tag{1.5}$$

for $X \in \underline{t}_{\mathbb{C}} = \underline{t} \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$\langle X, \lambda_j \rangle \neq 0, \quad j = 1, \ldots, r. \tag{1.6}$$

Here J_X and LX are extended linearly for $X \in \underline{t}_{\mathbb{C}}$.

Decompose $E = \bigoplus E_\lambda$ according to different weights λ, and let Ω_λ be the curvature matrix of E_λ. Clearly LX acts on E_λ by multiplication with $\langle X, \lambda \rangle$. Now

$$\begin{aligned}
\left\{\det\left(\frac{LX + \Omega}{2\pi i}\right)\right\}^{-1} &= \left\{\det\left(\frac{LX}{2\pi i}\right)\right\}^{-1} \{\det(1 + LX^{-1}\Omega)\}^{-1} \\
&= \left\{\det\left(\frac{LX}{2\pi i}\right)\right\}^{-1} \{\prod_\lambda \det(1 + LX^{-1}\Omega_\lambda)\}^{-1} \\
&= \left\{\det\left(\frac{LX}{2\pi i}\right)\right\}^{-1} \{\sum_{k \geq 0} \alpha_k(X)\}^{-1} \\
&= \left\{\det\left(\frac{LX}{2\pi i}\right)\right\}^{-1} \{\sum_{k \geq 0} \beta_k(X)\}.
\end{aligned}$$

Here $\alpha_k(X)$ and $\beta_k(X)$ are differential forms on N of degree $2k$ with $\alpha_0(X) = \beta_0(X) = 1$. Their cohomology classes in $H^{2k}(N, \mathbb{C})$ are polynomials in the Chern classes of the various E_λ. The coefficients of these polynomials are rational functions in X, and homogeneous of degree $-k$. Because $i^* J_X$ is locally constant on N we obtain that the integrand in the right hand side of (1.5) is a closed form on N whose cohomology class is a polynomial in the cohomology class of $i^* \sigma$ and the Chern classes of E_λ. The coefficients of this polynomial are meromorphic in X, and analytic for $X \in \underline{t}_{\mathbb{C}}$ satisfying (1.6). Because the left hand side of (1.5) is an analytic function in X on all of $\underline{t}_{\mathbb{C}}$ it is sufficient to prove (1.5) for $X \in \mathfrak{t}$ satisfying (1.6).

The proof of the theorem follows with minor adaptions [2] where a similar formula is obtained for M a complex manifold, L a holomorphic line bundle on M with Chern class $[\sigma]$ and X a holomorphic vector field on M which acts on L.

That Bott's ideas could be extended from the holomorphic to the symplectic case was observed in [1], at least in the situation that X has isolated zeros.[1]

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1 After this note was written we learned that the case of non isolated zeros has also been treated in a somewhat more general context in: N. Berline et M. Vergne, Classes caractéristiques équivariantes. Formule de localisation et chomologie équivariante, to appear in the Comptes Rendus

The purpose of this paper is just to carry out the explicit calculation needed for this extension. The proof presented here is self-contained modulo the citation of formula (2.28).

2. Proof of the Theorem

Fix a T-invariant Riemannian metric g on M. Using the exponential mapping we obtain a diffeomorphism ψ from a neighborhood U of the zero section in E onto a neighborhood $\psi(U)$ of N in M. We can take U invariant under multiplication μ_ε in the fiber by positive constants $\varepsilon \leq 1$. The push forward under ψ of the linear vector field LX on U is equal to X. Suppose $L\theta$ is a 1-form on $E\setminus N$ having the properties

$$(L\theta | LX) = 1, \tag{2.1}$$

$$(dL\theta, LX) = 0, \tag{2.2}$$

$$\mu_\varepsilon^* L\theta = L\theta. \tag{2.3}$$

Given (2.1) the condition (2.2) is equivalent to $\mathscr{L}_{LX}(L\theta) = 0$. Using the 1-form $g(X,X)^{-1} g(X,.)$ on $M\setminus N$ and a partition of unity we obtain a 1-form θ on $M\setminus N$ satisfying

$$(\theta | X) = 1, \tag{2.4}$$

$$(d\theta | X) = 0 \tag{2.5}$$

and such that the pull back under ψ of the form θ on $\psi(U)$ is equal to $L\theta$.

Consider the $(2n-1)$-form

$$v = -\sum_{k=1}^{n} (-1)^k e^{Jx} \theta \wedge (d\theta)^{k-1} \wedge \frac{\sigma^{n-k}}{(n-k)!} \tag{2.6}$$

on $M\setminus N$. An easy computation shows that

$$(dv | X) = \left(e^{Jx} \frac{\sigma^n}{n!} \middle| X \right) \tag{2.7}$$

which in turn implies that

$$dv = e^{Jx} \frac{\sigma^n}{n!} \tag{2.8}$$

on $M\setminus N$. For positive ε denote by B_ε the ε-ball bundle in E, and by S_ε the boundary of B_ε. Clearly

$$\int_M e^{Jx} \frac{\sigma^n}{n!} = \lim_{\varepsilon \downarrow 0} \int_{M\setminus \psi(B_\varepsilon)} e^{Jx} \frac{\sigma^n}{n!} \tag{2.9}$$

which by Stokes' theorem is equal to

$$\lim_{\varepsilon \downarrow 0} \int_{\psi(S_\varepsilon)} \sum_{k=1}^{n} (-1)^k e^{Jx} \theta \wedge (d\theta)^{k-1} \wedge \frac{\sigma^{n-k}}{(n-k)!}. \tag{2.10}$$

The mapping $\psi\mu_\varepsilon$ is a diffeomorphism from the unit normal sphere bundle S to $\psi(S_\varepsilon)$. Applied to forms which are smooth on all of U $\lim_{\varepsilon\downarrow 0}(\psi\mu_\varepsilon)^*$ is equal to $(i\pi)^*=\pi^* i^*$. Here $\pi: E\to N$ is the natural projection. The 1-form $(\psi\mu_\varepsilon)^*\theta=L\theta$ does not depend on ε. Hence (2.10) is equal to

$$\int_S \sum_{k=1}^{n} (-1)^k e^{\pi^* i^* Jx} L\theta\wedge(dL\theta)^{k-1}\wedge\frac{(\pi^* i^*\sigma)^{n-k}}{(n-k)!}. \tag{2.11}$$

In order to construct the 1-form $L\theta$ on $E\setminus N$ we choose a connection

$$D: \Gamma(E)\to\Gamma(T^*N\otimes E) \tag{2.12}$$

on E. For an exposition of the theory of connections and characteristic classes see Chern's book [3]. From now on we will replace N by one of its connected components. Let q be the rank of E as a complex vector bundle.

Let $V\subset N$ be an open set, and ${}^t s=(s_1,\ldots,s_q)$ a frame field over V. This gives a trivialization of E over V by

$$\begin{gathered} V\times\mathbb{C}^q\to\pi^{-1}(V) \\ (n,z)\to\sum z_i s_i(n),\qquad {}^t z=(z_1,\ldots,z_q). \end{gathered} \tag{2.13}$$

If $s'=gs$ is a new frame field over V, then

$$z={}^t g z',\qquad dz={}^t dg z'+{}^t g dz'. \tag{2.14}$$

The connection matrix ω of D relative to the frame field s is defined by

$$Ds=\omega s. \tag{2.15}$$

The transformation formula for a change of frame field is

$$\omega' g=dg+g\omega. \tag{2.16}$$

Consider the vector valued 1-form Dz on E given by

$$Dz=dz+{}^t\omega z. \tag{2.17}$$

The form Dz transforms under a change of frame field according to

$$Dz={}^t g Dz'. \tag{2.18}$$

Moreover, an easy calculation shows that

$$d(Dz)={}^t\Omega z-{}^t\omega\wedge Dz \tag{2.19}$$

where

$$\Omega=d\omega-\omega\wedge\omega \tag{2.20}$$

is the curvature matrix relative to s.

Let H be a Hermitean structure on E. Assume that the connection D is admissable with respect to H. Hence, for a unitary frame field s over V we have

$$\omega + {}^t\bar{\omega} = 0, \qquad \Omega + {}^t\bar{\Omega} = 0. \tag{2.21}$$

From now on s and $s' = gs$ will be unitary frame fields over V. Because X generates a compact group we may assume that LX is leaving both the Hermitean structure H and the connection D invariant, i.e.

$$LX + {}^t\overline{LX} = 0, \tag{2.22}$$

$$[LX, \omega] = 0, \qquad [LX, \Omega] = 0. \tag{2.23}$$

Moreover we will choose the frame fields s and $s' = gs$ in such a way that $d(LX) = 0$.

Consider the 1-form

$$L\theta = (z, z)^{-1}(z, {}^tLX^{-1}Dz) \tag{2.24}$$

on $E \backslash N$. Clearly $L\theta$ is independent of the choice of the frame field s, and satisfies the conditions (2.1), (2.2) and (2.3). Using (2.19), (2.21) and (2.23) one has

$$L\theta \wedge (dL\theta)^{k-1} = (z, z)^{-k}(z, {}^tLX^{-1}Dz) \wedge \{(Dz, {}^tLX^{-1}Dz) + (z, {}^tLX^{-1}\,{}^t\Omega z)\}^{k-1}.$$

Hence (2.11) can be rewritten as

$$\int_N e^{i^*Jx} e^{i^*\sigma} \sum_{k=q}^{n} \sum_{j=1}^{k} (-1)^k \binom{k-1}{j-1}$$
$$\times \int_{S^{2q-1}} (z, {}^tLX^{-1}Dz) \wedge (Dz, {}^tLX^{-1}Dz)^{j-1} \wedge (z, {}^tLX^{-1}\,{}^t\Omega z)^{k-j}. \tag{2.26}$$

Here S^{2q-1} is the unit sphere in $\mathbb{C}^q$. The second integrand has to be integrated point wise for $n \in N$. The outcome of this integral is independent of the choice of the frame field s. But for fixed $n \in V$ one can always choose a frame field s, which is horizontal at n. Therefore, in the summation over j in (2.26) all summands vanish except the one with $j = q$. Hence (2.26) becomes

$$\int_N e^{i^*Jx} e^{i^*\sigma} (-1)^q \int_{S^{2q-1}} \frac{(z, {}^tLX^{-1}dz) \wedge (dz, {}^tLX^{-1}dz)^{q-1}}{\{(z, z) + (z, {}^tLX^{-1}\,{}^t\Omega z)\}^q}. \tag{2.27}$$

Using a formula of Bott ([2], p. 327)

$$\int_{S^{2q-1}} \frac{(z, dz) \wedge (dz, dz)^{q-1}}{\{(z, z) + t(z, Sz)\}^q} = \frac{1}{\det(1 + tS)} \int_{S^{2q-1}} (z, dz) \wedge (dz, dz)^{q-1} \tag{2.28}$$

for any $q \times q$ Hermitean matrix S, we can rewrite (2.27) as

$$\int_N \frac{e^{i^*Jx} e^{i^*\sigma} (-1)^q \det({}^t\overline{LX}^{-1})}{\det(1 + {}^tLX^{-1}\,{}^t\Omega)} \int_{S^{2q-1}} (z, dz) \wedge (dz, dz)^{q-1}. \tag{2.29}$$

With our orientation of $\mathbb{C}^q$ we get

$$\int_{S^{2q-1}} (z, dz) \wedge (dz, dz)^{q-1} = \int_{B^{2q}} (dz, dz)^q = (2\pi i)^q \tag{2.30}$$

and because LX^{-1} is skew-Hermitean (2.29) can be written as

$$\int_N \frac{e^{i^* J x}\, e^{i^* \sigma}}{\det\left(\frac{LX + \Omega}{2\pi i}\right)}. \tag{2.31}$$

This finishes the proof of the theorem.

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