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ADDENDUM TO OUR CHARACTERIZATION OF THE UNIT POLYDISC

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Abstract

In 2008, we obtained an intrinsic characterization of the unit polydisc Δ^n in $\mathbf{C}^n$ from the viewpoint of the holomorphic automorphism group. In connection with this, A. V. Isaev investigated the structure of a complex manifold M with the property that every isotropy subgroup of the holomorphic automorhism group of M is compact, and obtained the same characterization of Δ^n as ours among the class of all such manifolds. In this paper, we establish some extensions of these results. In particular, Isaev's characterization of the unit polydisc Δ^n is extended to that of any bounded symmetric domain in $\mathbf{C}^n$.

1. Introduction

This is a continuation of our previous paper [8], and we retain the terminology and notation there.

Let M be a connected complex manifold and $\mathrm{Aut}(M)$ the group of all biholomorphic automorphisms of M. Then, equipped with the compact-open topology, $\mathrm{Aut}(M)$ is a topological group acting continuously on M. It should be remarked here that $\mathrm{Aut}(M)$ does not have the structure of a Lie group, in general; this often causes difficulties in studying various problems related to $\mathrm{Aut}(M)$.

In 1907, it was shown by Poincaré [10] that the Riemann mapping theorem does not hold in the higher dimensional case. In fact, he proved that *there exists no biholomorphic mapping from the unit polydisc* Δ^2 *onto the unit ball* B^2 *in* $\mathbf{C}^2$ by comparing carefully the topological structures of the isotropy subgroups of $\mathrm{Aut}(\Delta^2)$ and $\mathrm{Aut}(B^2)$ at the origin o of $\mathbf{C}^2$. In view of this fact, for a given complex manifold M, it seems to be an interesting problem to bring out some complex analytic nature of M under some topological conditions on $\mathrm{Aut}(M)$. Taking this into account, we asked the following question in [8]: *Let M and N be connected complex manifolds and assume that their holomorphic automorphism*

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groups $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as topological groups. Then is M biholomorphically equivalent to N?* And, as our main result, we obtained the following intrinsic characterization of the unit polydisc Δ^n from the viewpoint of the holomorphic automorphism group:

Theorem A ([8, Theorem]). *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to* Δ^n.

Later, related to this theorem, Isaev [6] investigated the structure of a complex manifold M with the property that every isotropy subgroup of the $\mathrm{Aut}(M)$-action is compact, and showed the following:

Theorem B ([6, Theorem 1.2]). *Let M be a connected complex manifold of dimension n satisfying the following two conditions*:

(1) *The isotropy subgroup of* $\mathrm{Aut}(M)$ *at every point of M is compact.*

(2) $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\Delta^n)$ *as topological groups.*

Then M is biholomorphically equivalent to Δ^n.

The main purpose of this paper is to establish the following extensions of Theorems A and B, which were announced at the 17th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications in Ho Chi Minh City, Vietnam, August 2009:

Theorem 1. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that there exists a topological subgroup G of* $\mathrm{Aut}(M)$ *that is isomorphic to the identity component of* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to* Δ^n.

This theorem will be proved in Section 2 by modifying the proof of Theorem A.

Let W be an arbitrary domain in $\mathbf{C}^n$. Then it is well-known that W admits a smooth envelope of holomorphy (cf. [9]). Hence, as an immediate consequence of this theorem, we obtain the following:

Corollary 1. *Let M be a connected Stein manifold of dimension n or a domain in* $\mathbf{C}^n$. *Assume that there exists a topological subgroup G of* $\mathrm{Aut}(M)$ *that is isomorphic to the identity component of* $\mathrm{Aut}(\Delta^n)$ *as topological groups. Then M is biholomorphically equivalent to* Δ^n.

A bounded domain D in $\mathbf{C}^n$ is called *symmetric* if, for each point $p \in D$, there exists an element $s_p \in \mathrm{Aut}(D)$ such that $s_p \circ s_p = \mathrm{id}_D$, $s_p \neq \mathrm{id}_D$ and p is an isolated fixed point of s_p. Clearly, the unit polydisc Δ^n as well as the unit ball

B^n in $\mathbf{C}^n$ is a typical example of bounded symmetric domains. As a natural generalization of Theorem B, we can prove the following theorem in Section 3:

THEOREM 2. *Let M be a connected complex manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Assume that there exists a topological subgroup G of* $\mathrm{Aut}(M)$ *satisfying the following two conditions*:

(1) *The isotropy subgroup of G at every point of M is compact.*

(2) *G is isomorphic to the identity component of* $\mathrm{Aut}(D)$ *as topological groups. Then M is biholomorphically equivalent to D.*

Recall that the isotropy subgroup of $\mathrm{Aut}(M)$ at every point of M is compact, provided that M is hyperbolic in the sense of Kobayashi [7]. Hence we have the following:

COROLLARY 2. *Let M be a connected hyperbolic manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D)$ *as topological groups. Then M is biholomorphically equivalent to D.*

Finally, it should be remarked that, for a given connected complex manifold M, the following conditions (A) and (B) are mutually independent (for the detail, see Section 4):

(A) M is holomorphically separable and admits a smooth envelope of holomorphy.

(B) The isotropy subgroup of $\mathrm{Aut}(M)$ at every point of M is compact.

In this sense, our Theorems 1 and 2 may be considered as characterizations of model domains from different viewpoints.

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2. Proof of Theorem 1

Our proof is based on the argument developed in our previous paper [8]. Although there are some overlaps with that paper, we carry out the proof for the sake of completeness and self-containedness.

Let us start with fixing a coordinate system $z = (z_1, \dots, z_n)$ in $\mathbf{C}^n$ and setting

$$\Delta_j = \{z_j \in \mathbf{C} \mid |z_j| < 1\} \ (1 \le j \le n) \quad \text{and} \quad \Delta^n = \Delta_1 \times \cdots \times \Delta_n.$$

Recall that $\mathrm{Aut}(\Delta_j)$ is a connected, real simple Lie group of dimension 3 with trivial center. Let $\mathrm{Aut}^o(\Delta^n)$ be the identity component of $\mathrm{Aut}(\Delta^n)$. Then we know that $\mathrm{Aut}^o(\Delta^n)$ can be identified with the direct product of $\mathrm{Aut}(\Delta_j)$: $\mathrm{Aut}^o(\Delta^n) = \mathrm{Aut}(\Delta_1) \times \cdots \times \mathrm{Aut}(\Delta_n)$. Let $\mathfrak{g}(\Delta_j)$ and $\mathfrak{g}(\Delta^n)$, respectively, denote the real Lie algebras consisting of all complete holomorphic vector fields on Δ_j and on Δ^n. Then it is well-known that these Lie algebras are canonically identified with the Lie algebras of $\mathrm{Aut}(\Delta_j)$ and $\mathrm{Aut}(\Delta^n)$, respectively. Therefore we have

$$\mathfrak{g}(\Delta^n) = \mathfrak{g}(\Delta_1) \oplus \cdots \oplus \mathfrak{g}(\Delta_n), \quad [\mathfrak{g}(\Delta_i), \mathfrak{g}(\Delta_j)] = \{0\} \quad \text{for } 1 \leq i, j \leq n,\ i \neq j. \tag{2.1}$$

Moreover, we see that $\mathfrak{g}(\Delta_j)$ contains the holomorphic vector fields

$$H_j := \sqrt{-1} z_j \partial/\partial z_j \quad \text{and} \quad V_j := (1 - z_j^2)\partial/\partial z_j$$

induced by the one-parameter subgroups

$$z_j \mapsto (\exp \sqrt{-1} t) z_j \quad \text{and} \quad z_j \mapsto \frac{(\cosh t) z_j + \sinh t}{(\sinh t) z_j + \cosh t}$$

($t \in \mathbf{R}$) of $\mathrm{Aut}(\Delta_j)$, respectively. Then, putting $W_j = [H_j, V_j]$, we have

$$\mathfrak{g}(\Delta_j) = \mathbf{R}\{H_j, V_j, W_j\} \quad \text{and} \quad [H_j, [H_j, V_j]] = -V_j, \quad [W_j, V_j] = 4H_j \tag{2.2}$$

for $1 \leq j \leq n$. These bracket relations will be very important in our proof.

As in Theorem 1 in the introduction, let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy and assume that there exists a topological group isomorphism $\Phi : \mathrm{Aut}^o(\Delta^n) \to G$, where G is the given topological subgroup of $\mathrm{Aut}(M)$. Since Δ^n is a Reinhardt domain in $\mathbf{C}^n$, the n-dimensional torus T^n acts naturally on Δ^n as a connected Lie transformation group, so that, via the isomorphism Φ, T^n now acts effectively and continuously on M by biholomorphic transformations. Hence this action is necessarily real analytic by a classical result of Bochner and Montgomery [3]. Therefore, by a well-known fact due to Barrett, Bedford and Dadok [1], we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and that there exists a topological group isomorphism $\Phi : \mathrm{Aut}^o(\Delta^n) \to G \subset \mathrm{Aut}(D)$ such that $\Phi(T(\Delta^n)) = T(D)$, where $T(\Delta^n)$ and $T(D)$, respectively, denote the subgroups of $\mathrm{Aut}(\Delta^n)$ and of $\mathrm{Aut}(D)$ induced by the restrictions of the standard T^n-action on $\mathbf{C}^n$ to Δ^n and to D.

Now, the group G can be turned into a Lie group by transferring the Lie group structure from $\mathrm{Aut}^o(\Delta^n)$ by means of Φ. Since the Lie group G endowed with the compact-open topology acts continuously on D by biholomorphic transformations, the action is real analytic with respect to the Lie group structure induced from $\mathrm{Aut}^o(\Delta^n)$ (cf. [3]). Thus G is now a Lie transformation group of D acting effectively on D by biholomorphic transformations; accordingly, the Lie algebra of G can be identified with the Lie algebra $\mathfrak{g}$ consisting of all holomorphic vector fields on D induced by one-parameter subgroups of G (so-called

G-vector fields on D). We thus obtain the Lie algebra isomorphism $d\Phi : \mathfrak{g}(\Delta^n) \to \mathfrak{g}$ induced by Φ. From now on, for the sake of simplicity, let us put

$$G_j = \Phi(\mathrm{Aut}(\Delta_j)), \quad \mathfrak{g}_j = d\Phi(\mathfrak{g}(\Delta_j)) \quad \text{and}$$
$$I_j = d\Phi(H_j), \quad X_j = d\Phi(V_j), \quad Y_j = d\Phi(W_j)$$

for $1 \le j \le n$. Then $G = G_1 \times \cdots \times G_n$ and, by (2.1) and (2.2), we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \quad \text{for } 1 \le i, j \le n,\ i \ne j; \tag{2.3}$$

$$\mathfrak{g}_j = \mathbf{R}\{I_j, X_j, Y_j\} \quad \text{and} \quad [I_j, [I_j, X_j]] = -X_j, \quad [Y_j, X_j] = 4I_j \tag{2.4}$$

for every $1 \le j \le n$.

Put $D^* = D \cap (\mathbf{C}^*)^n$ and, for a point $z \in D$, let $(\mathfrak{g}_j)_z$ denote the subspace of the tangent space to D at z that consist of the values of the elements of $\mathfrak{g}_j$ at z. Then, using the bracket relations (2.3) and (2.4), one can verify the following assertion:

1) *For every point $z_o \in D^*$, there exist a local holomorphic coordinate system $(U, w_1, \dots, w_n)$ on D^*, centered at z_o, and a nowhere dense real analytic subset $\mathscr{A}$ of U such that $(\mathfrak{g}_j)_p = \mathbf{C}\{(\partial/\partial w_j)_p\}$ for $p \in U \backslash \mathscr{A}$ and $1 \le j \le n$.*

Therefore, if we choose a point $p \in U \backslash \mathscr{A}$ and consider the orbits

$$D_p := G \cdot p \quad \text{and} \quad S_j := G_j \cdot p \ \ (1 \le j \le n)$$

of G and of G_j passing through p, then the assertion 1) together with (2.3) guarantees us that every S_j is a complex submanifold of D and D_p is an open subset of D. Hence D_p is a Reinhardt domain in $\mathbf{C}^n$, because G is connected and contains the torus $T(D) = T^n$. More precisely, in exactly the same way as in the proof of [8, Theorem], it can be shown that

2) *every S_j is biholomorphically equivalent to the unit disc Δ_j*;

3) *D_p is biholomorphically equivalent to the unit polydisc Δ^n*; and

4) *D is a bounded domain in $\mathbf{C}^n$ and D_p is an open dense subset of D.*

Thus the proof of Theorem 1 is now reduced to showing that D_p is also closed in D. If G is a closed subgroup of $\mathrm{Aut}(D)$, then G acts properly on D, as seen in the proof of [8; Theorem]. Consequently, the orbit $D_p = G \cdot p$ has to be closed in D in this case. Here, whether or not G is closed in $\mathrm{Aut}(D)$, we want to verify the closedness of D_p in D. To this end, assume the contrary that there exists a boundary point $q \in \partial D_p$ in D. Let d_D denote the Kobayashi distance on D and let $K(x; r) = \{y \in D \mid d_D(x, y) < r\}$ be the Kobayashi ball of radius $r > 0$ with center $x \in D$. Since d_D induces the standard topology of D (cf. [2], [12]) and p is an interior point of D_p, one can pick a small $r > 0$ in such a way that $K(p; r) \subset D_p$. For such an $r > 0$, choose a point $x_o \in D_p \cap K(q; r)$ arbitrarily and let g_o be an element of G such that $x_o = g_o \cdot p$. Then, since d_D is invariant under the action of $G \subset \mathrm{Aut}(D)$, we have

$$d_D(g_o^{-1} \cdot q, p) = d_D(q, g_o \cdot p) = d_D(q, x_o) < r,$$

which means that $g_o^{-1} \cdot q \in K(p;r) \subset D_p$ and hence $q \in g_o \cdot D_p = D_p$, a contradiction to $q \in \partial D_p$. Therefore D_p is, in fact, closed in D and accordingly $D = D_p$ is biholomorphically equivalent to Δ^n; completing the proof of Theorem 1. □

3. Proof of Theorem 2

We shall use several fundamental facts on symmetric spaces without proofs. For the details, the reader may consult, for instance, Helgason's book [4].

Let M be a connected complex manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Let $\mathbf{G}$ be the identity component of $\mathrm{Aut}(D)$ and let $\mathfrak{G}$ be its Lie algebra. Fix a point $o \in D$ once and for all and let $\mathbf{K}$ be the isotropy subgroup of $\mathbf{G}$ at o. Then $\mathbf{G}$ is a semi-simple Lie group with trivial center that acts transitively on D and $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$. Note that, since a maximal compact subgroup of a connected Lie group is always connected, $\mathbf{K}$ is a connected Lie subgroup of $\mathbf{G}$. Moreover, D can now be represented as the coset space $D = \mathbf{G}/\mathbf{K}$. Consider here the involutive automorphism $\sigma : g \mapsto s_o g s_o$ of $\mathbf{G}$, where s_o denotes the symmetry of D with respect to o, and put $s = d\sigma$, the involutive automorphism of $\mathfrak{G}$ induced by σ. Let $\mathfrak{K}$ and $\mathfrak{P}$ be the eigenspaces of s for the eigenvalues $+1$ and -1, respectively. Then $\mathfrak{K}$ coincides with the Lie algebra of $\mathbf{K}$ and we have

$$\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}, \quad [\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}, \quad [\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P} \quad \text{and} \quad [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}. \tag{3.1}$$

As usual, we identify $\mathfrak{P}$ with the tangent space $T_o(D)$ to D at o; accordingly, $\mathfrak{P} = T_o(D)$ has the complex structure J_o^D induced by the standard complex structure tensor J^D on D. Thus $\mathfrak{P}$ can be regarded as a complex vector space. Moreover, under the identification $T_o(D) = \mathfrak{P}$, the linear isotropy group $\mathbf{K}^*$ of $\mathbf{G}$ at o is just the group $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K})$, where $\mathrm{Ad}_{\mathbf{G}}$ is the adjoint representation of $\mathbf{G}$. We will often use this fact in the proof.

Assume now that there exists a topological group isomorphism $\Phi : \mathbf{G} \to G$, where G is the given topological subgroup of $\mathrm{Aut}(M)$ in Theorem 2. Since $\mathbf{G}$ is a Lie group, G has a unique Lie group structure with respect to which $\Phi : \mathbf{G} \to G$ is a Lie group isomorphism. Thus, by the same reasoning as in the proof of Theorem 1, G becomes a Lie transformation group of M acting effectively on M by biholomorphic transformations. We denote by $\mathfrak{g}$ the Lie algebra of G and by $d\Phi : \mathfrak{G} \to \mathfrak{g}$ the Lie algebra isomorphism induced by Φ.

Fix a point $p \in M$ arbitrarily and denote by K the isotropy subgroup of G at p. Then, by our assumption, K is a compact subgroup of G. Here, along the same line as in [6], we shall show that G acts transitively on M; accordingly, M can be written in the form $M = G/K$. To this end, choose a maximal compact subgroup $\hat{K}$ of G containing K. Then, since any two maximal compact subgroups of G are always conjugate under an inner automorphism of G, one can find an element $g_o \in G$ such that $\hat{K} = g_o \Phi(\mathbf{K}) g_o^{-1}$. Moreover, notice that the orbit $G \cdot p = G/K$ of G passing through p is a real analytic submanifold of M. Thus

$$2n \geq \dim G/K \geq \dim G/\hat{K} = \dim \mathbf{G}/\mathbf{K} = 2n,$$

from which we have $K = \hat{K}$, $\dim G/K = 2n$ and hence the orbit $G \cdot p = G/K$ is open in M. Since this is true for any point $q \in M$ with $q \neq p$ and since M is connected, we conclude that $M = G/K$, as desired. Therefore, by replacing Φ by $g_o \Phi(\cdot) g_o^{-1}$ if necessary, one may assume that $\hat{K} = \Phi(\mathbf{K})$; consequently, Φ induces a real analytic diffeomorphism, say again,

$$\Phi : D = \mathbf{G}/\mathbf{K} \to G/K = M. \tag{3.2}$$

Put $\mathfrak{k} = d\Phi(\mathfrak{K})$ and $\mathfrak{p} = d\Phi(\mathfrak{P})$. Then $\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{g}$ corresponding to K and we have the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the same properties as in (3.1). Let J^M be the G-invariant complex structure tensor on M and let J_p^M be the complex structure on $T_p(M) = \mathfrak{p}$ induced by J^M. Then, since J_p^M commutes with each element in the linear isotropy group K^* of G at p, so does with $\mathrm{Ad}_G(k)$ for all $k \in K$, where Ad_G is the adjoint representation of G.

In order to complete the proof of Theorem 2, we need to prove that, after a slight modification if necessary, the diffeomorphism Φ in (3.2) gives rise to a biholomorphic equivalence between D and M. For this purpose, by using the fact that $d\Phi$ gives a linear isomorphism from $\mathfrak{P}$ onto $\mathfrak{p}$, let us define the endomorphism J_o^* of $\mathfrak{P}$ by the formula

$$d\Phi(J_o^* X) = J_p^M(d\Phi(X)) \quad \text{for all } X \in \mathfrak{P}. \tag{3.3}$$

Then $J_o^* \circ J_o^* = -I$ and moreover, since

$$d\Phi(\mathrm{Ad}_{\mathbf{G}}(k)X) = \mathrm{Ad}_G(\Phi(k))\, d\Phi(X) \quad \text{for all } k \in \mathbf{K} \text{ and all } X \in \mathfrak{P},$$

it can be easily seen that J_o^* commutes with $\mathrm{Ad}_{\mathbf{G}}(k)$ for all $k \in \mathbf{K}$. Therefore $D = \mathbf{G}/\mathbf{K}$ admits a unique almost complex structure tensor J^* which coincides with J_o^* at o and is invariant under the action of $\mathbf{G}$. The proof is now divided into two cases as follows:

Case 1. *D is irreducible.* In this case, $\mathbf{G}$ is a simple Lie group and $\mathbf{K}$ is a maximal compact subgroup of $\mathbf{G}$ with one-dimensional center isomorphic to the circle group S^1. By definition of the irreducibility, $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K})$ now acts irreducibly on $\mathfrak{P}$. Hence, Schur's lemma implies that $J_o^* = cI$ with some constant $c \in \mathbf{C}$; accordingly $J_o^* = \pm\sqrt{-1}I = \pm J_o^D$ and $J^* = \pm J^D$, because $(J_o^*)^2 = -I$. Moreover, we would like to assert here the following: one may assume, without loss of generality, that D is invariant under the complex conjugation $\psi : z \to \bar{z}$ of $\mathbf{C}^n$ with respect to $\mathbf{R}^n$. Indeed, in the case where D is one of the four classical domains, it is well-known that D can be realized as a subdomain $\tilde{D}$ in some complex matrix space (cf. [5]). Then, a glance at $\tilde{D}$ tells us that it is invariant under the complex conjugation ψ. On the other hand, in the case where D is an exceptional bounded symmetric domain, it is shown in Roos [11; Section 3] that its Harish-Chandra realization $\tilde{D}$ has an explicit algebraic and geometric description using exceptional Jordan triple systems; from which it follows at once that $\tilde{D}$ is invariant under the complex conjugation ψ, as asserted. Thus, taking the diffeomorphism $\Phi \circ \psi$ instead of Φ in (3.2) if necessary, we may assume that $J^* = J^D$. This combined with (3.3) yields that $\Phi : D \to M$ is holomorphic;

consequently, it gives a biholomorphic equivalence between D and M, as required.

Case 2. *D is reducible*. In this case, D can be uniquely (up to an order) decomposed into the direct product

$$D = D_1 \times \cdots \times D_r, \tag{3.4}$$

where the factors D_i are irreducible bounded symmetric domains in $\mathbf{C}^{n_i}$ with $n_1 + \cdots + n_r = n$. Here, as in Case 1, one may assume that each D_i is invariant under the complex conjugation. Let $\mathbf{G}$ and $\mathbf{G}_i$ be the identity components of $\mathrm{Aut}(D)$ and of $\mathrm{Aut}(D_i)$. And, writing $o = (o_1, \dots, o_r)$ with $o_i \in D_i$ according to the decomposition (3.4), we denote by $\mathbf{K}$ and $\mathbf{K}_i$ the isotropy subgroups of $\mathbf{G}$ and of $\mathbf{G}_i$ at o and at o_i, respectively. Then, as mentioned in Case 1, each $\mathbf{G}_i$ is a simple Lie group with $\mathbf{K}_i$ as a maximal compact subgroup of it and D_i is a homogeneous space of $\mathbf{G}_i$. Moreover, we have $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ and $\mathbf{K} = \mathbf{K}_1 \times \cdots \times \mathbf{K}_r$, so that D can be expressed as

$$D = \mathbf{G}/\mathbf{K} = \mathbf{G}_1/\mathbf{K}_1 \times \cdots \times \mathbf{G}_r/\mathbf{K}_r. \tag{3.5}$$

Let $\mathfrak{G}_i$ be the Lie algebra of $\mathbf{G}_i$. Let σ_i be the involutive automorphism $g \mapsto s_{o_i} g s_{o_i}$ of $\mathbf{G}_i$ and put $s_i = d\sigma_i$. Then, denoting by $\mathfrak{K}_i$ and $\mathfrak{P}_i$, respectively, the eigenspaces of s_i for the eigenvalues $+1$ and -1, we obtain the direct sum decomposition $\mathfrak{G}_i = \mathfrak{K}_i \oplus \mathfrak{P}_i$ as in (3.1). As before, we identify $\mathfrak{P}_i = T_{o_i}(D_i)$ and we denote also by J^{D_i} the standard complex structure tensor on D_i. Let J_o^* be the complex structure on $\mathfrak{P} = \mathfrak{P}_1 \oplus \cdots \oplus \mathfrak{P}_r$ defined by (3.3). Then, since J_o^* commutes with $\mathrm{Ad}_{\mathbf{G}}(k)$ for all $k \in \mathbf{K}$ and since $\mathrm{Ad}_{\mathbf{G}}(\mathbf{K}_i)$ acts irreducibly on $\mathfrak{P}_i$ and trivially on $\mathfrak{P}_j$ for $j \neq i$, it follows that each $\mathfrak{P}_i$ is invariant under J_o^*. Thus J_o^* is decomposed $J_o^* = J_{o_1}^* \times \cdots \times J_{o_r}^*$, where each $J_{o_i}^*$ is the restriction of J_o^* to $\mathfrak{P}_i$. Therefore, letting J_i^* be the unique $\mathbf{G}_i$-invariant almost complex structure tensor on D_i which coincides with $J_{o_i}^*$ at o_i, we have $J^* = J_1^* \times \cdots \times J_r^*$. Moreover, since $\mathrm{Ad}_{\mathbf{G}_i}(\mathbf{K}_i)$ acts now irreducibly on $\mathfrak{P}_i$, Schur's lemma again implies that $J_i^* = \pm J^{D_i}$ for each $1 \leq i \leq r$. Finally, consider a real analytic diffeomorphism $\hat{\Phi} : D = D_1 \times \cdots \times D_r \to M$ given by

$$\hat{\Phi}(u) = \Phi(\gamma_1(u_1), \dots, \gamma_r(u_r)) \quad \text{for } u = (u_1, \dots, u_r) \in D_1 \times \cdots \times D_r = D,$$

where $\gamma_i(u_i) = u_i$ or $\gamma_i(u_i) = \bar{u}_i$, the complex conjugation in $\mathbf{C}^{n_i}$, for $1 \leq i \leq r$ and Φ is the diffeomorphism appearing in (3.2). Then, replacing Φ by a suitable $\hat{\Phi}$ if necessary, we have $J^* = J^D$. This means that $\Phi : D \to M$ is holomorphic. Therefore, we have shown that Φ gives a biholomorphic equivalence between D and M; thereby completing the proof of Theorem 2. □

4. A concluding remark

In this section, we would like to illustrate that the conditions (A) and (B) stated in the introduction are mutually independent, in general, with concrete examples as follows:

Example 1. Consider the two-dimensional complex Euclidean space $\mathbf{C}^2$, for instance. Then, the condition (A) is trivially satisfied for $\mathbf{C}^2$. On the other hand, notice that the isotropy subgroup $\mathrm{Aut}_o(\mathbf{C}^2)$ of $\mathrm{Aut}(\mathbf{C}^2)$ at the origin o of $\mathbf{C}^2$ contains the biholomorphic mappings $\varphi_\nu : \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$\varphi_\nu(z, w) = (z, w \exp(\nu z)), \quad (z, w) \in \mathbf{C}^2 \text{ for } \nu = 1, 2, \ldots.$$

Clearly this says that $\mathrm{Aut}_o(\mathbf{C}^2)$ is not to be compact; hence, the condition (B) is not satisfied for $\mathbf{C}^2$.

Example 2. Take an arbitrary compact connected hyperbolic manifold X of dimension $\geqq 2$ and consider the manifold M obtained from X by deletion of one point, say $M = X \backslash \{p\}$ $(p \in X)$. Then, being a complex submanifold of the hyperbolic manifold X, M is also hyperbolic. Accordingly, the condition (B) is automatically satisfied for M. However, we assert that M is not holomorphically separable and does not admit a smooth envelope of holomorphy. To verify this, note that any holomorphic function on M can be holomorphically extended to X and hence it must be constant, because X is a compact connected complex manifold of dimension $\geqq 2$. Thus, M is never holomorphically separable. Moreover, assume that there exists a smooth envelope of holomorphy of M. Then, since every Stein manifold can be realized as a closed complex submanifold of some $\mathbf{C}^N$, we have a holomorphic imbedding $F : M \to \mathbf{C}^N$. But, since any holomorphic function on M is now constant as mentioned above, F must be also constant. Clearly, this is a contradiction. Therefore the condition (A) is not satisfied for this manifold M.

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