ADDITIVE ACTIONS ON TORIC VARIETIES

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ABSTRACT. By an additive action on an algebraic variety X of dimension n we mean a regular action $\mathbb{G}_a^n \times X \to X$ with an open orbit of the commutative unipotent group $\mathbb{G}_a^n$. We prove that if a complete toric variety X admits an additive action, then it admits an additive action normalized by the acting torus. Normalized additive actions on a toric variety X are in bijection with complete collections of Demazure roots of the fan Σ_X. Moreover, any two normalized additive actions on X are isomorphic.

INTRODUCTION

Let X be an irreducible algebraic variety of dimension n over an algebraically closed field $\mathbb{K}$ of characteristic zero and let $\mathbb{G}_a = (\mathbb{K}, +)$ be the additive group of the field. Consider the commutative unipotent group $\mathbb{G}_a^n = \mathbb{G}_a \times \ldots \times \mathbb{G}_a$ (n times). By an additive action on X we mean a regular action $\mathbb{G}_a^n \times X \to X$ with an open orbit. Equivalently, one may consider algebraic varieties with an additive action as equivariant embeddings of the vector group $(\mathbb{K}^n, +)$.

A systematic study of additive actions was initiated by Hassett and Tschinkel [19]. They established a remarkable correspondence between additive actions on the projective space $\mathbb{P}^n$ and local $(n + 1)$-dimensional commutative associative algebras with unit. This correspondence has allowed the authors to classify additive actions on $\mathbb{P}^n$ for small n. The same technique was used by Sharoiko [23] to prove that an additive action on a non-degenerate projective quadric is unique. Further modification of Hassett-Tschinkel correspondence led to characterization of additive actions on arbitrary projective hypersurfaces, in particular, on degenerate projective quadrics [5, 6].

The study of additive actions was originally inspired by problems of arithmetic geometry. Motivated by Manin's conjecture, Chambert-Loir and Tschinkel [8] gave asymptotic formulas for the number of rational points of bounded height on smooth projective equivariant compactifications of the vector group. More generally, asymptotic formulas for the number of rational points of bounded height on quasi-projective equivariant embeddings of the vector group are obtained in [9]. Note that for smooth projective toric varieties Manin's conjecture was proved by Batyrev and Tschinkel [7] using another technique.

In [1] all generalized flag varieties G/P admitting an additive action are found. Here G is a semisimple linear algebraic group and P is a parabolic subgroup. It turns out that if G/P admits an additive action, then the parabolic subgroup P is maximal.

Feigin [15] proposed a construction based on the PBW-filtration to degenerate an arbitrary generalized flag variety G/P to a variety with an additive action. Recently Fu-Hwang [17] and Devyatov [14] have proved that if G/P is not isomorphic to the projective space, then up to isomorphism there is at most one additive action on G/P. Classification of additive actions on singular del Pezzo surfaces is obtained by Derenthal and Loughran [13].

The problem of classification of additive actions on toric varieties is raised in [6, Section 6]. Some instructive examples of such actions are given in [19, Proposition 5.5]. It is natural to divide the problem into two parts. The first one deals with additive actions on a toric variety X of dimension n normalized by the acting torus T. In this case an additive action splits into n pairwise commuting $\mathbb{G}_a$-actions on X normalized by T. It is proved in [12] that $\mathbb{G}_a$-actions on a toric variety X normalized by T are in bijection with some vectors defined in terms of the fan Σ_X associated with X. We call such vectors Demazure roots of a fan. Cox [10] observed that normalized $\mathbb{G}_a$-actions on a toric variety can be interpreted as certain $\mathbb{G}_a$-subgroups of automorphisms of the Cox ring $R(X)$ of the variety X. In turn, such subgroups correspond to homogeneous locally nilpotent derivations of the Cox ring. In these terms the Demazure root is nothing but the degree of the derivation.

After presenting some background on $\mathbb{G}_a$-actions and Demazure roots (Section 1) and on toric varieties and Cox rings (Section 2), we prove in Theorem 3.4 that additive actions on a toric variety X normalized by the acting torus T are in bijection with complete collections of Demazure roots of the fan Σ_X; see Definition 3.3 for a precise definition of a complete collection of Demazure roots. In Theorem 3.6 we show that any two normalized additive actions on X are isomorphic.

The second part of the problem concerns non-normalized additive actions. Theorem 4.1 states that if a complete toric variety admits an additive action, then it admits an additive action normalized by the acting torus.

It is well known that a toric variety is projective if and only if its fan is a normal fan of a convex polytope. In Section 5 we characterize polytopes corresponding to projective toric varieties with an additive action.

In the last section we give explicit examples of additive actions on toric varieties in terms of their Cox rings and formulate several open problems.

1. $\mathbb{G}_a$-ACTIONS AND DEMAZURE ROOTS

Consider an irreducible affine variety X with an effective action of an algebraic torus T. Let M be the character lattice of T and N be the lattice of one-parameter subgroups of T. Let $A = \mathbb{K}[X]$ be the algebra of regular functions on X. It is well known that there is a bijection between faithful T-actions on X and effective M-gradings on A. In fact, the algebra A is graded by a semigroup of lattice points in a convex polyhedral cone $\omega \subseteq M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$. We have

$$A = \bigoplus_{m \in \omega_M} A_m \chi^m,$$

where $\omega_M = \omega \cap M$ and χ^m is the character of the torus T corresponding to a point m.

A derivation ∂ of an algebra A is said to be *locally nilpotent* (LND) if for every $f \in A$ there exists $k \in \mathbb{N}$ such that $\partial^k(f) = 0$. For any LND ∂ on A the map $\varphi_\partial : \mathbb{G}_a \times A \to A$, $\varphi_\partial(s, f) = \exp(s\partial)(f)$, defines a structure of a rational $\mathbb{G}_a$-algebra on A. This induces a regular action $\mathbb{G}_a \times X \to X$, where $X = \operatorname{Spec} A$. In fact, any regular $\mathbb{G}_a$-action on X arises this way; see [16, Section 1.5]. A derivation ∂ on A is said to be *homogeneous* if it respects the M-grading. If $f, h \in A \setminus \ker \partial$ are homogeneous, then $\partial(fh) = f\partial(h) + \partial(f)h$ is homogeneous too and $\deg \partial(f) - \deg f = \deg \partial(h) - \deg h$. So any homogeneous derivation ∂ has a well-defined *degree* given as $\deg \partial = \deg \partial(f) - \deg f$ for any homogeneous $f \in A \setminus \ker \partial$. It is easy to see that an LND on A is homogeneous if and only if the corresponding $\mathbb{G}_a$-action is normalized by the torus T in the automorphism group $\operatorname{Aut}(X)$; cf. [16, Section 3.7].

Let X be an affine toric variety, i.e., a normal affine variety with an action of a torus T with an open orbit. In this case

$$A = \bigoplus_{m \in \omega_M} \mathbb{K}\chi^m = \mathbb{K}[\omega_M]$$

is the semigroup algebra. Recall that for a given cone $\omega \subset M_\mathbb{Q}$, its *dual cone* is defined by

$$\sigma = \{p \in N_\mathbb{Q} \,|\, \langle p, v \rangle \geqslant 0 \ \forall v \in \omega\},$$

where $\langle, \rangle$ is the pairing between dual spaces $N_\mathbb{Q}$ and $M_\mathbb{Q}$. Let $\sigma(1)$ be the set of rays of a cone σ and p_ρ be the primitive lattice vector on a ray ρ. For $\rho \in \sigma(1)$ we set

$$\mathfrak{R}_\rho := \{e \in M \,|\, \langle p_\rho, e \rangle = -1 \text{ and } \langle p_{\rho'}, e \rangle \geqslant 0 \ \forall \rho' \in \sigma(1), \, \rho' \neq \rho\}.$$

One easily checks that the set $\mathfrak{R}_\rho$ is infinite for each $\rho \in \sigma(1)$. The elements of the set $\mathfrak{R} := \bigsqcup_\rho \mathfrak{R}_\rho$ are called the *Demazure roots* of the cone σ. Let $e \in \mathfrak{R}_\rho$. Then ρ is called the *distinguished ray* of the root e. One can define the homogeneous LND on the algebra A by the rule

$$\partial_e(\chi^m) = \langle p_\rho, m \rangle \chi^{m+e}.$$

In fact, every homogeneous LND on A has a form $\alpha \partial_e$ for some $\alpha \in \mathbb{K}$, $e \in \mathfrak{R}$; see [21, Theorem 2.7]. In other words, $\mathbb{G}_a$-actions on X normalized by the acting torus are in bijection with Demazure roots of the cone σ.

Example 1.1. Consider $X = \mathbb{K}^n$ with the standard action of the torus $(\mathbb{K}^\times)^n$. It is a toric variety with the cone $\sigma = \mathbb{Q}_{\geqslant 0}^n$ having rays $\rho_i = \langle p_i \rangle_{\mathbb{Q}_{\geqslant 0}}$ with $p_1 = (1, 0, \ldots, 0), \ldots, p_n = (0, \ldots, 0, 1)$. The dual cone ω is $\mathbb{Q}_{\geqslant 0}^n$ as well. In this case we have

$$\mathfrak{R}_{\rho_i} = \{(c_1, \ldots, c_{i-1}, -1, c_{i+1}, \ldots, c_n) \,|\, c_j \in \mathbb{Z}_{\geqslant 0}\},$$
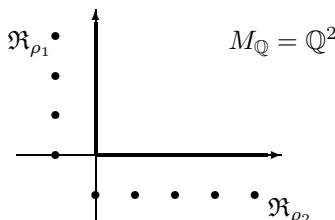
where $c_j = \langle p_j, e \rangle$. Denote $x_1 = \chi^{(1,0,\dots,0)}, \dots, x_n = \chi^{(0,\dots,0,1)}$. Then $\mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]$. It is easy to see that the homogeneous LND corresponding to the root $e = (c_1, \dots, c_n) \in \mathfrak{R}_{\rho_i}$ is

$$(*) \qquad \partial_e = x_1^{c_1} \dots x_{i-1}^{c_{i-1}} x_{i+1}^{c_{i+1}} \dots x_n^{c_n} \frac{\partial}{\partial x_i}.$$

This LND gives rise to the $\mathbb{G}_a$-action

$$x_i \mapsto x_i + s x_1^{c_1} \dots x_{i-1}^{c_{i-1}} x_{i+1}^{c_{i+1}} \dots x_n^{c_n}, \quad x_j \mapsto x_j, \quad j \neq i, \quad s \in \mathbb{G}_a.$$

2. Toric varieties and Cox rings

We keep the notation of Section 1. Let X be a toric variety of dimension n with an acting torus T and Σ be the corresponding fan of convex polyhedral cones in $N_{\mathbb{Q}}$; see [18] or [11] for details.

Let $\Sigma(1)$ be the set of rays of the fan Σ and p_ρ be the primitive lattice vector on a ray ρ. For $\rho \in \Sigma(1)$ we consider the set $\mathfrak{R}_\rho$ of all vectors $e \in M$ such that

(1) $\langle p_\rho, e \rangle = -1$ and $\langle p_{\rho'}, e \rangle \geqslant 0 \ \ \forall \rho' \in \Sigma(1), \rho' \neq \rho$;

(2) if σ is a cone of Σ and $\langle v, e \rangle = 0$ for all $v \in \sigma$, then the cone generated by σ and ρ is in Σ as well.

Note that (1) implies (2) if Σ is a fan with convex support. This is the case if X is affine or complete.

The elements of the set $\mathfrak{R} := \bigsqcup_\rho \mathfrak{R}_\rho$ are called the *Demazure roots* of the fan Σ; cf. [12, Définition 4] and [22, Section 3.4]. Again elements $e \in \mathfrak{R}$ are in bijection with $\mathbb{G}_a$-actions on X normalized by the acting torus; see [12, Théoreme 3] and [22, Proposition 3.14]. If X is affine, the $\mathbb{G}_a$-action given by a Demazure root e coincides with the action corresponding to the locally nilpotent derivation ∂_e of the algebra $\mathbb{K}[X]$ as defined in Section 1. Let us denote by H_e the image in $\mathrm{Aut}(X)$ of the group $\mathbb{G}_a$ under this action. Thus H_e is a one-parameter unipotent subgroup normalized by T in $\mathrm{Aut}(X)$.

We recall basic facts from toric geometry. There is a bijection between cones $\sigma \in \Sigma$ and T-orbits $\mathcal{O}_\sigma$ on X such that $\sigma_1 \subseteq \sigma_2$ if and only if $\mathcal{O}_{\sigma_2} \subseteq \overline{\mathcal{O}_{\sigma_1}}$. Here $\dim \mathcal{O}_\sigma = n - \dim\langle\sigma\rangle$. Moreover, each cone $\sigma \in \Sigma$ defines an open affine T-invariant subset U_σ on X such that $\mathcal{O}_\sigma$ is the unique closed T-orbit on U_σ and $\sigma_1 \subseteq \sigma_2$ if and only if $U_{\sigma_1} \subseteq U_{\sigma_2}$.

Let ρ_e be the distinguished ray corresponding to a root e, p_e be the primitive lattice vector on ρ_e, and R_e be the one-parameter subgroup of T corresponding to p_e. Denote by X^{H_e} the set of H_e-fixed points on X.

Our aim is to describe the action of H_e on X.

Proposition 1. *For every point $x \in X \setminus X^{H_e}$ the orbit $H_e x$ meets exactly two T-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ on X, where $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. The intersection $\mathcal{O}_2 \cap H_e x$ consists of a single point, while*

$$\mathcal{O}_1 \cap H_e x = R_e y \quad \text{for any} \quad y \in \mathcal{O}_1 \cap H_e x.$$

Proof. It follows from the proof of [22, Proposition 3.14] that the affine charts U_σ, where $\sigma \in \Sigma$ is a cone containing ρ_e, are H_e-invariant, and the complement of their union is contained in X^{H_e}. This reduces the proof to the case X is affine. Then the assertion is proved in [3, Proposition 2.1]. $\square$

A pair of T-orbits $(\mathcal{O}_1, \mathcal{O}_2)$ on X is said to be H_e-*connected* if $H_e x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ for some $x \in X \setminus X^{H_e}$. By Proposition 1, $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ for such a pair (up to permutation) and $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. Since the torus normalizes the subgroup H_e, any point of $\mathcal{O}_1 \cup \mathcal{O}_2$ can actually serve as the point x.

Lemma 2.1. *A pair of T-orbits $(\mathcal{O}_{\sigma_1}, \mathcal{O}_{\sigma_2})$ is H_e-connected if and only if $e|_{\sigma_2} \leq 0$ and σ_1 is a facet of σ_2 given by the equation $\langle v, e \rangle = 0$.*

Proof. The proof again reduces to the affine case, where the assertion is [3, Lemma 2.2]. $\square$

Now we recall basic ingredients of the Cox construction; see [2, Chapter 1] for more details. Let X be a normal variety with free finitely generated divisor class group $\mathrm{Cl}(X)$ and only constant invertible regular functions. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ which maps onto $\mathrm{Cl}(X)$ isomorphically. The *Cox ring* of the variety X is defined as

$$R(X) = \bigoplus_{D \in K} H^0(X, D),$$

where $H^0(X, D) = \{ f \in \mathbb{K}(X)^\times \mid \mathrm{div}(f) + D \geqslant 0 \} \cup \{0\}$ and multiplication on homogeneous components coincides with multiplication in the field of rational functions $\mathbb{K}(X)$ and extends to $R(X)$ by linearity. It is easy to see that up to isomorphism the graded ring $R(X)$ does not depend on the choice of the subgroup K.

Example 2.2. It is proved in [10] that if X is toric, then $R(X)$ is a polynomial algebra $\mathbb{K}[x_1, \ldots, x_m]$, where the variables x_i correspond to T-invariant prime divisors D_i on X or, equivalently, to the rays ρ_i of the fan Σ_X. The $\mathrm{Cl}(X)$-grading on $R(X)$ is given by $\deg(x_i) = [D_i]$.

Suppose that the Cox ring $R(X)$ is finitely generated. Then $\overline{X} := \mathrm{Spec}\, R(X)$ is a normal affine variety with an action of the torus $H_X := \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$. There is an open H_X-invariant subset $\widehat{X} \subseteq \overline{X}$ such that the complement $\overline{X} \setminus \widehat{X}$ is of codimension at least two in $\overline{X}$, there exists a good quotient $p_X : \widehat{X} \to \widehat{X} /\!/ H_X$, and the quotient space $\widehat{X} /\!/ H_X$ is isomorphic to X; see [2, Construction 1.6.3.1]. So we have the following diagram:

$$\widehat{X} \xrightarrow{\ i\ } \overline{X} = \mathrm{Spec}\, R(X)$$
$$\downarrow {}^{/\!/ H_X}$$
$$X$$

If X is toric, then $\overline{X}$ is isomorphic to $\mathbb{K}^m$, and $\overline{X} \setminus \widehat{X}$ is a union of some coordinate planes in $\mathbb{K}^m$ of codimension at least two [10].

By [2, Theorem 4.2.3.2], any $\mathbb{G}_a$-action on a variety X can be lifted to a $\mathbb{G}_a$-action on the variety $\overline{X}$ commuting with the action of the torus H_X.

If X is toric and a $\mathbb{G}_a$-action is normalized by the acting torus T, then the lifted $\mathbb{G}_a$-action on $\mathbb{K}^m$ is normalized by the diagonal torus $(\mathbb{K}^\times)^m$. Conversely, if X is either affine or complete, any $\mathbb{G}_a$-action on $\mathbb{K}^m$ normalized by the torus $(\mathbb{K}^\times)^m$ and commuting with the subtorus H_X induces a $\mathbb{G}_a$-action on X. This shows that $\mathbb{G}_a$-actions on X normalized by T are in bijection with locally nilpotent derivations of the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ that are homogeneous with respect to the standard grading by the lattice $\mathbb{Z}^m$ and have degree zero with respect to the $\mathrm{Cl}(X)$-grading.

3. Normalized additive actions

Let X be an irreducible algebraic variety of dimension n. Consider a commutative unipotent algebraic group $\mathbb{G}_a^n = \mathbb{G}_a \times \ldots \times \mathbb{G}_a$ (n times).

Definition 3.1. An *additive action* on a variety X is a regular action $\mathbb{G}_a^n \times X \to X$ with an open orbit.

Let X be a normal variety admitting an additive action. Then X contains an open $\mathbb{G}_a^n$-orbit W isomorphic to the affine space $\mathbb{K}^n$. By [4, Lemma 1], any invertible function on X is constant and the divisor class group $\mathrm{Cl}(X)$ is freely generated by classes $[D_1], \ldots, [D_l]$ of the prime divisors such that $X \setminus W = D_1 \cup \ldots \cup D_l$. In particular, the Cox ring $R(X)$ introduced in Section 2 is well defined for such a variety X.

Now we assume that X is toric and an additive action $\mathbb{G}_a^n \times X \to X$ is normalized by the acting torus T. Then the group $\mathbb{G}_a^n$ is a direct product of n subgroups $\mathbb{G}_a$ each normalized by T. They correspond to pairwise commuting homogeneous locally nilpotent derivations on the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ having degree zero with respect to the $\mathrm{Cl}(X)$-grading. In turn, such derivations up to scalar are in bijection with Demazure roots of the fan Σ_X.

Consider a set of homogeneous derivations ∂_e of the polynomial algebra $\mathbb{K}[x_1, \ldots, x_m]$ corresponding to some Demazure roots e of the fan Σ_X.

Lemma 3.2. *Derivations ∂_e and $\partial_{e'}$ commute if and only if either $\rho_e = \rho_{e'}$ or $\langle p_e, e' \rangle = \langle p_{e'}, e \rangle = 0$.*

Proof. We have $\partial_e \partial_{e'} = \partial_{e'} \partial_e$ if and only if $\partial_e \partial_{e'}(x_j) = \partial_{e'} \partial_e(x_j)$ for all $j = 1, \ldots, n$. Now the lemma follows from a direct computation with formula $(*)$. $\qquad \square$

Definition 3.3. A set $e_1, \ldots, e_n$ of Demazure roots of a fan Σ of dimension n is called a *complete collection* if $\langle p_i, e_j \rangle = -\delta_{ij}$ for all $1 \le i, j \le n$.

In this case, the vectors $p_1, \ldots, p_n$ form a basis of the lattice N, and $-e_1, \ldots, -e_n$ is the dual basis of the dual lattice M.

The following result may be considered as a combinatorial description of normalized additive actions on toric varieties.

Theorem 3.4. *Let X be a toric variety. Then additive actions on X normalized by the acting torus T are in bijection with complete collections of Demazure roots of the fan Σ_X.*

Proof. As we have seen, a normalized additive action on X determines n pairwise commuting homogeneous locally nilpotent derivations of the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$. They have the form ∂_e for some Demazure roots e.

Lemma 3.5. *Homogeneous locally nilpotent derivations $\partial_1, \ldots, \partial_n$ of the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ corresponding to Demazure roots $e_1, \ldots, e_n$ define a normalized additive action on X if and only if $e_1, \ldots, e_n$ form a complete collection.*

Proof. Assume first that the derivations $\partial_1, \ldots, \partial_n$ give rise to the additive actions $\mathbb{G}_a^n \times X \to X$. If some e_i and e_j with $i \ne j$ correspond to the same ray of the fan Σ_X, then the $\mathbb{G}_a^n$-action changes at most $n-1$ coordinates of the ring $\mathbb{K}[x_1, \ldots, x_m]$, and any $\mathbb{G}_a^n$-orbit on X cannot be n-dimensional. Then Lemma 3.2 implies that $\langle p_i, e_j \rangle = 0$ for $i \ne j$. By definition, we have $\langle p_i, e_i \rangle = -1$, and thus $e_1, \ldots, e_n$ form a complete collection.

Conversely, if $e_1, \ldots, e_n$ is a complete collection, then the corresponding homogeneous locally nilpotent derivations commute. They define a $\mathbb{G}_a^n$-action on $\mathbb{K}[x_1, \ldots, x_m]$, and hence on $\mathbb{K}^m$. This action descends to X. We have to show that the $\mathbb{G}_a^n$-action on X has an open orbit. For this purpose it suffices to check that the group $\mathbb{G}_a^n \times H_X$ acts on $\mathbb{K}^m$ with an open orbit.

By construction, the group $\mathbb{G}_a^n$ changes exactly n of the coordinates $x_1, \ldots, x_m$, while the weights of the remaining $m - n$ coordinates with respect to the Cl(X)-grading form a basis of the lattice of characters of the torus H_X. This shows that the stabilizer of the point $(1, \ldots, 1) \in \mathbb{K}^m$ in the group $\mathbb{G}_a^n \times H_X$ is trivial. Since $\dim(\mathbb{G}_a^n \times H_X) = n + m - n = m$, we conclude that the $(\mathbb{G}_a^n \times H_X)$-orbit of the point $(1, \ldots, 1)$ is open in $\mathbb{K}^m$. $\qquad\square$

This completes the proof of Theorem 3.4. $\qquad\square$

Corollary 1. A toric variety X admits a normalized additive action if and only if there is a complete collection of Demazure roots of the fan Σ_X.

The following theorem shows that a normalized additive action on a toric variety is essentially unique. Let us say that two additive actions on a variety X are isomorphic if these actions are conjugate by an automorphism of X.

Theorem 3.6. *Any two normalized additive actions on a toric variety are isomorphic.*

Proof. Define the group Aut(Σ) of automorphisms of a fan Σ as the subgroup of linear transformations γ of the lattice N preserving the fan Σ. Any linear transformation γ determines the dual transformation γ^* of the lattice M. We say that two complete collections of Demazure roots of Σ are *equivalent*, if one can be sent to another by an automorphism of Σ.

Let X be a toric variety. Any automorphism of the fan Σ_X is induced by an automorphism of X; see [11, Theorem 3.3.4]. Thus it suffices to prove that every two complete collections $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ of Demazure roots of Σ_X are equivalent.

For the primitive vectors defined by our collections, we may assume that $p_1 = p_1', \ldots, p_r = p_r'$, while $p_{r+1}, \ldots, p_n, p_{r+1}', \ldots, p_n'$ are pairwise different. Then $p_{r+1}', \ldots, p_n'$ are non-positive linear combinations of $p_1, \ldots, p_n$, and $p_{r+1}, \ldots, p_n$ are non-positive linear combinations of $p_1', \ldots, p_n'$. This implies that, up to renumbering, the vector $p_j', j > r$, has coordinate -1 at p_j and zero coordinates at other $p_s, s > r$.

If there is a ray of the fan Σ_X whose primitive vector p'' is not involved yet, then p'' is a non-positive linear combination of both $p_1, \ldots, p_n$ and $p_1', \ldots, p_n'$. This shows that p'' is a non-positive linear combination of $p_1, \ldots, p_r$.

Consider the automorphism γ of the lattice N which sends the basis $p_1, \ldots, p_n$ to the basis $p_1', \ldots, p_n'$. It is easy to see that γ sends $p_1', \ldots, p_n'$ to $p_1, \ldots, p_n$, and preserves all the vectors p''. Thus γ sends one complete collection of Demazure roots to another.

It remains to check that γ is an automorphism of the fan Σ_X. Consider the vector $p_j, j > r$, and the corresponding Demazure root e_j. Then all vectors $p_i, i \neq j, p_i'$, $i \neq j$, and p'' are contained in the hyperplane $\langle \cdot, e_j \rangle = 0$. Thus we can decompose the transformation γ in the composition of transformations $\gamma_{r+1}, \ldots, \gamma_n$, where γ_j sends p_j to p_j' and fixes the hyperplane $\langle \cdot, e_j \rangle = 0$ pointwise.

It suffices to check that each γ_j is an automorphism of the fan Σ_X. Equivalently, we have to check that for any cone σ in Σ_X containing p_j there is a cone in Σ_X generated by the same rays, but with ρ_j replaced by ρ'_j. If the cone σ contains both p_j and p'_j, the assertion is clear. If σ does not contain p'_j, it is generated by p_j and some cone σ_0 contained in the hyperplane $\langle \cdot, e_j \rangle = 0$. By the symmetric arguments, the cone σ_0 is contained in the hyperplane $\langle \cdot, e'_j \rangle = 0$ as well. Condition (2) from the definition of a Demazure root of a fan Σ implies that the cone generated by p'_j and σ_0 belongs to Σ as well. This completes the proof of Theorem 3.6. $\square$

4. Arbitrary additive actions

Let X be a complete toric variety with an acting torus T. It is well known that the automorphism group $\operatorname{Aut}(X)$ is a linear algebraic group with T as a maximal torus; see [12], [10]. In particular, $\operatorname{Aut}(X)$ contains a maximal unipotent subgroup U, and all such subgroups are conjugate in $\operatorname{Aut}(X)$.

Let G be the product of the semisimple part and the unipotent radical of $\operatorname{Aut}(X)$.

Theorem 4.1. *Let X be a complete toric variety with an acting torus T. The following conditions are equivalent:*

(1) *There exists an additive action on X normalized by the acting torus T.*
(2) *There exists an additive action on X.*
(3) *A maximal unipotent subgroup U of the automorphism group $\operatorname{Aut}(X)$ acts on X with an open orbit.*
(4) *The subgroup G acts on X with an open orbit.*

Proof. Clearly, condition (1) implies (2). Since any unipotent subgroup of $\operatorname{Aut}(X)$ is contained in a maximal unipotent subgroup, implication (2) $\Rightarrow$ (3) holds.

Now we prove implication (3) $\Rightarrow$ (1). Let a maximal unipotent subgroup U act on X with an open orbit. Since all maximal unipotent subgroups are conjugate in $\operatorname{Aut}(X)$ and U coincides with the unipotent radical of a Borel subgroup in $\operatorname{Aut}(X)$ [20, Section 21.3], we may assume that U is normalized by T. Then U is generated by one-parameter subgroups U_e normalized by T and corresponding to some Demazure roots e.

Let $\mathcal{O}$ be an open U-orbit in X. Being an orbit of a unipotent group, $\mathcal{O}$ is isomorphic to the affine space $\mathbb{K}^n$. Since U is normalized by T, the subset $\mathcal{O}$ is a T-invariant open affine chart on X. It implies that there are coordinates $x_1, \ldots, x_n$ on $\mathcal{O}$ such that the T-action on $\mathcal{O}$ is the standard action of T on $\mathbb{K}^n$.

The root subgroups U_e acts on $\mathbb{K}^n$ as

$$x_i \mapsto x_i + s x_1^{c_1} \ldots x_n^{c_n}, \quad s \in \mathbb{K}, \quad x_j \mapsto x_j \quad \forall j \neq i,$$

for some i, where the monomial $x_1^{c_1} \ldots x_n^{c_n}$ does not depend on x_i. It corresponds to a locally nilpotent derivation ∂_e given by

$$\partial_e(x_i) = x_1^{c_1} \ldots \widehat{x_i} \ldots x_n^{c_n}, \quad \partial_e(x_j) = 0 \ \forall j \neq i,$$

where $e = (c_1, \ldots, c_n)$ is a Demazure root with $c_i = -1$.

Let U_i be a subgroup of the automorphism group of $\mathbb{K}^n$ sending x_i to $x_i + c$, $c \in \mathbb{K}$, and fixing all x_j, $j \neq i$. We claim that the group U contains all the subgroups U_i, $i = 1, \ldots, n$.

Let us assume the converse. By renumbering we may suppose that for some $r < n$ precisely the subgroups U_j with $j \leq r$ are contained in U. By $r = 0$ we mean that there are no subgroups U_i in U at all.

Since the action of U on $\mathbb{K}^n$ is transitive, for every $i > r$ there is at least one subgroup U_e of the form $x_i \mapsto x_i + sx_1^{c_1} \ldots x_n^{c_n}$ in U. If for all such subgroups the monomial $x_1^{c_1} \ldots x_n^{c_n}$ depends at least on one variable $x_{r+1}, \ldots, x_n$, the subspace $x_{r+1} = \ldots = x_n = 0$ in $\mathbb{K}^n$ is invariant under U, a contradiction. Thus we have a subgroup U_e of the form $x_i \mapsto x_i + sx_1^{c_1} \ldots x_r^{c_r}$ for some $i > r$.

Consider the subspace spanned by the derivations ∂_e over all subgroups U_e contained in U in the Lie algebra of all derivations of the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$. This subspace may be identified with the tangent algebra of the group U. In particular, it is closed under the Lie bracket of derivations.

Let ∂_j with $\partial_j(x_i) = \delta_{ij}$ be the derivation corresponding to U_j and ∂_e correspond to the subgroup U_e of the form $x_i \mapsto x_i + sx_1^{c_1} \ldots x_r^{c_r}$ for some $i > r$.

It is easy to check that the commutant $[\partial_1, \partial_e]$ is a locally nilpotent derivation sending x_i to $c_1 x_1^{c_1-1} \ldots x_r^{c_r}$ and annihilating all other variables. Taking more commutants of ∂_e with $\partial_1, \ldots, \partial_r$, we obtain a locally nilpotent derivation, which sends x_i to a non-zero constant and annihilates all other variables. It shows that the derivation ∂_i is contained in the Lie algebra of U. Thus the subgroup U_i is contained in U, a contradiction with a choice of r.

We conclude that the subgroups U_i, $i = 1, \ldots, n$, are contained in $\mathrm{Aut}(X)$ and generate a commutative unipotent subgroup normalized by T, which acts on $\mathcal{O}$ transitively. Thus implication $(3) \Rightarrow (1)$ is proved.

Since any maximal unipotent subgroup is contained in G, we have $(3) \Rightarrow (4)$.

Let us prove implication $(4) \Rightarrow (3)$. Since X is irreducible, the open G-orbit on X intersects the open T-orbit X_0. Take a point x_0 in X_0 whose G-orbit is open in X.

Note that G coincides with the subgroup of $\mathrm{Aut}(X)$ generated by all root subgroups U_e. By Proposition 1, the intersection of the orbit $U_e \cdot x_0$ with X_0 coincides with the orbit $R_e \cdot x_0$, where R_e is a one-parameter subtorus in T represented by the distinguished vector p_e of the Demazure root e. Thus the orbit $G \cdot x_0$ is open in X if and only if the subtori R_e are not contained in a proper subtorus of T or, equivalently, the vectors p_e's are not contained in a proper subspace of $N_\mathbb{Q}$.

Hence we have a set $p_1, \ldots, p_n$ of linearly independent primitive vectors of the fan Σ_X corresponding to some Demazure roots $e_1, \ldots, e_n$. Let U be a maximal unipotent subgroup of $\mathrm{Aut}(X)$ normalized by T. If the root subgroup U_{e_i} is contained in the unipotent radical of $\mathrm{Aut}(X)$, then U_{e_i} is contained in U as well. If U_{e_i} is in the semisimple part of $\mathrm{Aut}(X)$ and is not in U, then the opposite root subgroup U_{-e_i} is in U. The hyperplane $\langle \cdot, e_i \rangle = 0$ contains all the rays of the fan Σ_X except for ρ_{e_i} and ρ_{-e_i}. Thus replacing the vector p_i by p_{-e_i} in the set $p_1, \ldots, p_n$, we again obtain a linearly independent set.

Proceeding this way, we obtain a linearly independent set $p_1, \ldots, p_n$, where the vectors p_i correspond to some root subgroups in U. Thus U acts on X with an open orbit, and the proof of implication $(4) \Rightarrow (3)$ is completed. $\qquad \square$

Corollary 2. A toric variety X admits an additive action if and only if there is a complete collection of Demazure roots of the fan Σ_X.

Remark 4.2. As we have seen in the proof, condition (4) of Theorem 4.1 is equivalent to existence of linearly independent primitive vectors $p_1, \ldots, p_n$ in the fan Σ_X corresponding to some Demazure roots $e_1, \ldots, e_n$. At the same time we know that condition (1) of Theorem 4.1 means that there is a complete collection of Demazure

roots of the fan Σ_X. It seems to be an interesting problem to prove explicitly that linearly independent vectors $p_1, \ldots, p_n$ correspond to a complete collection of Demazure roots.

Let $\mathfrak{u}$ be the Lie algebra of a maximal unipotent subgroup U of $\mathrm{Aut}(X)$ normalized by T. Consider the Grassmannian $\mathrm{Grass}(n, \mathfrak{u})$ of n-dimensional subspaces of the space $\mathfrak{u}$. Commutative Lie subalgebras of dimension n form a closed subvariety Z in $\mathrm{Grass}(n, \mathfrak{u})$. The adjoint action of the torus T on $\mathfrak{u}$ induces an action of T on $\mathrm{Grass}(n, \mathfrak{u})$, which leaves invariant the subvariety Z. Points in Z fixed by T are precisely T-normalized commutative subalgebras in $\mathfrak{u}$.

Take any commutative n-dimensional subgroup H in U which acts on X with an open orbit, and consider the closure of the T-orbit of the point in Z corresponding to the Lie algebra of H. This closure contains finitely many T-fixed points. If H is not normalized by T, then there are at least two fixed points in the closure. In general, not any subgroup corresponding to such points acts on X with an open orbit; see Examples 6.3 and 6.4 below. We expect that at least one such subgroup does act with an open orbit on X.

5. Projective toric varieties and polytopes

It is well known that there is a correspondence between convex lattice polytopes and projective toric varieties. The aim of this section is to characterize polytopes corresponding to toric varieties that admit an additive action.

We begin with preliminary results; see [11, Chapter 2] and [18, Section 1.5] for more details. Let M be a lattice of rank n and P be a full dimensional convex polytope in the space $M_{\mathbb{Q}}$. We say that P is a *lattice polytope* if all its vertices lie in M.

A subsemigroup $S \subseteq M$ is called *saturated* if S coincides with the intersection of the group $\mathbb{Z}S$ and the cone $\mathbb{Q}_{\geq 0}S$ it generates. A lattice polytope P is *very ample* if for every vertex $v \in P$, the semigroup $S_{P,v} := \mathbb{Z}_{\geq 0}(P \cap M - v)$ is saturated. It is known that for every lattice polytope P and every $k \geq n - 1$ the polytope kP is very ample; see [11, Corollary 2.2.19].

Let us consider M as a lattice of characters of a torus T. Let $P \subseteq M_{\mathbb{Q}}$ be a very ample polytope and $P \cap M = \{m_1, \ldots, m_s\}$. We consider a map

$$T \to \mathbb{P}^{s-1}, \quad t \mapsto (\chi^{m_1}(t) : \ldots : \chi^{m_s}(t))$$

and define a variety X_P as the closure of the image of this map in $\mathbb{P}^{s-1}$. It is known that X_P is a projective toric variety with the acting torus T, and any projective toric variety appears this way.

Definition 5.1. We say that a very ample polytope P is *inscribed in a rectangle* if there is a vertex $v_0 \in P$ such that

(1) the primitive vectors on the edges of P containing v_0 form a basis $e_1, \ldots, e_n$ of the lattice M;
(2) for every inequality $\langle p, x \rangle \leq a$ on P that corresponds to a facet of P not passing through v_0 we have $\langle p, e_i \rangle \geq 0$ for all $i = 1, \ldots, n$.

The following result was communicated to us by Evgeny Feigin as a conjecture.

Theorem 5.2. *Let P be a very ample polytope and X_P be the corresponding projective toric variety. Then X_P admits an additive action if and only if the polytope P is inscribed in a rectangle.*
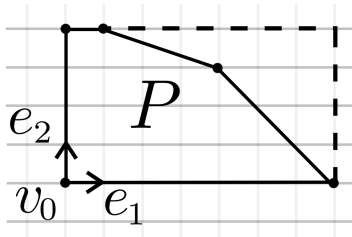
FIGURE 1. A polytope P inscribed in a rectangle

Proof. By Corollary 2, a toric variety X admits an additive action if and only if the fan Σ_X admits a complete collection of Demazure roots. By [11, Proposition 3.1.6], the fan Σ_{X_P} of the toric variety X_P corresponding to the polytope P coincides with the normal fan Σ_P of the polytope P. It is straightforward to check that two conditions of Definition 5.1 are precisely the conditions on $-e_1, \ldots, -e_n$ to be a complete collection of Demazure roots of the fan Σ_P. $\square$

Example 5.3. The segment $P = [0, d]$ in $\mathbb{Q}^1$ with $d \in \mathbb{Z}_{\geq 1}$ is a polytope inscribed in a rectangle, and the variety

$$X_P = \overline{\{(1 : a : \ldots : a^d) \,;\, a \in \mathbb{K}\}} \subseteq \mathbb{P}^d$$

is a rational normal curve of degree d.

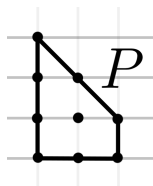**Example 5.4.** The polytope

FIGURE 2. The polytope of a Hirzebruch surface

defines the surface

$$X_P = \overline{\{(1 : a : a^2 : b : ab : a^2b : b^2 : ab^2 : b^3) \,;\, a, b \in \mathbb{K}\}} \subseteq \mathbb{P}^8$$

isomorphic to the Hirzebruch surface.

6. EXAMPLES AND OPEN PROBLEMS

We begin this section with explicit formulas for additive actions on toric varieties in terms of Cox rings.

Example 6.1. The normalized additive action on the projective space $\mathbb{P}^n$ is given by

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0, x_1 + s_1x_0, \ldots, x_n + s_nx_0).$$

The hyperplane $x_0 = 0$ consists of $\mathbb{G}_a^n$-fixed points and thus for $n \geq 2$ the number of $\mathbb{G}_a^n$-orbits on $\mathbb{P}^n$ is infinite.

Example 6.2. The normalized additive action on the product $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ is given by
$$(x_1, x_2, \ldots, x_{2n-2}, x_{2n}) \mapsto (x_1, x_2 + s_1 x_1, \ldots, x_{2n-1}, x_{2n} + s_n x_{2n-1}).$$
This shows that the number of $\mathbb{G}_a^n$-orbits on $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ is 2^n.

Example 6.3. By [19, Proposition 3.2], every additive action on $\mathbb{P}^2$ has the form
$$(x_0, x_1, x_2) \mapsto (x_0, x_1 + bs_1 x_0, x_2 + as_1 x_1 + s_2 x_0)$$
with some fixed $a, b \in \mathbb{K}$. For $a \neq 0$ and $b \neq 0$ we have one isomorphy class of non-normalized additive actions. Every such action has three orbits on $\mathbb{P}^2$. With $a = 0$ it degenerates to a normalized additive action, while with $b = 0$ it degenerates to a $\mathbb{G}_a^2$-action on $\mathbb{P}^2$ with generic one-dimensional orbits, which is not an additive action.

Example 6.4. Let X be the Hirzebruch surface $\mathbb{F}_d$. Its fan is generated by the vectors
$$p_1 = (1, 0), \quad p_2 = (0, 1), \quad p_3 = (-1, d), \quad p_4 = (0, -1)$$
with some $d \in \mathbb{Z}_{\geq 1}$. The Cox ring $\mathbb{K}[x_1, x_2, x_3, x_4]$ carries a $\mathbb{Z}^2$-grading given by
$$\deg(x_1) = (1, 0), \quad \deg(x_2) = (0, 1), \quad \deg(x_3) = (1, 0), \quad \deg(x_4) = (d, 1).$$
Moreover, X is obtained as a geometric quotient of
$$\widehat{X} = \mathbb{K}^4 \setminus (\{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\})$$
by the action of the torus $H_X = (\mathbb{K}^\times)^2$, whose action is given by the $\mathbb{Z}^2$-grading.

In this case the Demazure roots are $(1, 0)$, $(-1, 0)$ and $(k, 1)$ with $0 \leq k \leq d$, and the corresponding homogeneous locally nilpotent derivations are given as
$$x_1 \frac{\partial}{\partial x_3}, \quad x_3 \frac{\partial}{\partial x_1}, \quad x_1^k x_2 x_3^{d-k} \frac{\partial}{\partial x_4}.$$
There are two complete collections of Demazure roots, namely $(1, 0), (d, 1)$ and $(-1, 0), (0, 1)$. They define two normalized additive actions
$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3 + s_1 x_1, x_4 + s_2 x_1^d x_2)$$
and
$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + s_1 x_3, x_2, x_3, x_4 + s_2 x_2 x_3^d),$$
which are rearranged by the automorphism $(x_1, x_2, x_3, x_4) \mapsto (x_3, x_2, x_1, x_4)$.

Applying the results of [12] or [10] we obtain
$$\mathrm{Aut}(X) \cong \mathbb{K}^\times \cdot \mathrm{PSL}(2) \ltimes \mathbb{G}_a^{d+1}.$$

For $d = 1$ a maximal unipotent subgroup of $\mathrm{Aut}(X)$ acts as
$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + ax_3, x_2, x_3, x_4 + bx_1 x_2 + cx_2 x_3).$$
Every point $[a : b] \in \mathbb{P}^2$ defines a $\mathbb{G}_a^2$-action on $\mathbb{F}_1$, which is an additive action if $[a : b] \neq [0 : 1]$. With $[a : b] = [1 : 0]$ we have a normalized additive action, while all other points $[a : b]$ define pairwise isomorphic non-normalized additive actions. Thus there are exactly two isomorphy classes of additive actions on $\mathbb{F}_1$. This result is obtained in [19, Proposition 5.5] by geometric arguments.

In [19, Theorem 6.2], a classification of additive actions on smooth projective threefolds with Picard group of rank 1 is given. The following result may be considered as a generalization of this classification to the case of not necessary smooth varieties of arbitrary dimension in toric setting.

Proposition 2. *Let X be a complete toric variety with* $\operatorname{rk}\operatorname{Cl}(X) = 1$. *Then X admits an additive action if and only if X is a weighted projective space* $\mathbb{P}(1, d_1, \ldots, d_n)$ *for some positive integers d_i.*

Proof. First assume that X admits a (normalized) additive action. We know that the group $\operatorname{Cl}(X)$ is free and thus $\operatorname{Cl}(X) \cong \mathbb{Z}$. This allows us to assume that the primitive vectors on the rays of the fan Σ_X are

$$(1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1), (-a_1, \ldots, -a_n), \quad a_i \geq 0.$$

Since the fan Σ_X is complete we have $a_i > 0$. Putting $d_i = a_i$ we obtain the fan of the weighted projective space $\mathbb{P}(1, d_1, \ldots, d_n)$; see [18, Section 2.2].

Conversely, such a fan possesses a complete collection of Demazure roots, and thus $\mathbb{P}(1, d_1, \ldots, d_n)$ admits an additive action. $\qquad\square$

Remark 6.5. In terms of Cox rings, a normalized additive action on $\mathbb{P}(1, d_1, \ldots, d_n)$ is given by

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0, x_1 + s_1 x_0^{d_1}, \ldots, x_n + s_n x_0^{d_n})$$

and thus the complement of the open $\mathbb{G}_a^n$-orbit on $\mathbb{P}(1, d_1, \ldots, d_n)$ consists of $\mathbb{G}_a^n$-fixed points.

Let us finish this paper with several questions and open problems.

Problem 1. Find all toric varieties which admit a normalized additive action with finitely many orbits.

It is shown in [19, Section 3] that the projective space $\mathbb{P}^n$ with $n \geq 6$ admits infinitely many non-isomorphic additive actions. This rises the following problem.

Problem 2. Describe all additive actions on a given toric variety X. When is the number of isomorphy classes of such actions on X infinite?

As we have shown in this paper, if a toric variety X of dimension n admits an additive action, then the Cox ring $\mathbb{K}[x_1, \ldots, x_m]$ is $\mathbb{Z}^{m-n}$-graded, the degrees of $x_1, \ldots, x_{m-n}$ form a basis of the lattice $\mathbb{Z}^{m-n}$, and $\deg x_{m-n+1}, \ldots, \deg x_m$ are non-positive combinations of these basis vectors. Moreover, if X is either complete or affine, then these conditions are necessary and sufficient. These observations motivate the following problem.

Problem 3. Characterize algebraic varieties admitting an additive action in terms of their Cox rings.

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