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Additive Bases of Vector Spaces over Prime Fields

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It is shown that for any $t > c_p \log n$ linear bases $B_1, ..., B_t$ of Z_p^n their union (with repetitions) $\bigcup_{i=1}^t B_i$ forms an additive basis of Z_p^n; i.e., for any $x \in Z_p^n$ there exist $A_1 \subset B_1, ..., A_t \subset B_t$ such that $x = \sum_{i=1}^t \sum_{y \in A_i} y$.

1. INTRODUCTION

Let Z_p^n be the n-dimensional linear space over the prime field Z_p. An additive basis of Z_p^n is a multiset $\{x_1, ..., x_m\} \subset Z_p^n$, such that any $x \in Z_p^n$ is representable as a 0–1 combination of the x_i's. Let $f(p, n)$ denote the minimal integer t, such that for any t (linear) bases $B_1, ..., B_t$ of Z_p^n, the union (with repetitions) $\bigcup_{i=1}^t B_i$ forms an additive basis of Z_p^n.

The problem of determining or estimating $f(p, n)$, besides being interesting in its own right, is naturally motivated by the study of universal

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flows in graphs (see [JLPT]). The authors of [JLPT] conjectured that $f(p, n)$ is bounded above by a function of p alone.

Clearly $f(p, n) \geq p-1$, as the union of $p-2$ identical copies of the same basis does not form an additive basis. For $p \geq 3$ and $n \geq 2$, this trivial lower bound may be improved to $f(p, n) \geq p$. It clearly suffices to show this for $n=2$. Let $\{a_1, a_2\}$ be any basis of Z_p^2, and consider $p-2$ copies of $\{a_1, a_2\}$ and one copy of $\{a_1+a_2, a_1-a_2\}$. As $-a_2$ is not in the additive span of these $p-1$ bases we obtain $f(p, 2) \geq p$.

In this paper we give two proofs of the following.

THEOREM 1.1. $f(p, n) \leq c(p) \log n$.

In Section 2 we use exponential sums to show that $f(p, n) \leq 1+(p^2/2) \log 2pn$. The algebraic method in Section 3 gives the somewhat better bound $f(p, n) \leq (p-1) \log n + p - 2$. The final Section 4 contains some concluding remarks and open problems.

2. ADDITIVE SPANNING AND EXPONENTIAL SUMS

Let $B_1, \ldots, B_t$ be any $t > (p^2/2) \log 2pn$ bases of Z_p^n. Denote by $\{x_1, \ldots, x_m\}$, $m = tn$, their union with repetitions, and for any $x \in Z_p^n$, let $N(x) = |\{(\varepsilon_1, \ldots, \varepsilon_m): \sum_{j=1}^m \varepsilon_j x_j = x, \varepsilon_j \in \{0, 1\}\}|$.

We shall show that $N(x) > 0$ for all $x \in Z_p^n$. For $x, y \in Z_p^n$, $x \cdot y$ is their standard inner product, and for $a \in Z_p$ let $e(a) = e^{2\pi i a/p}$.

Following Baker and Schmidt [BS, p. 471] we represent $N(x)$ as an exponential sum,

$$\begin{aligned} N(x) &= \sum_{\varepsilon \in \{0,1\}^*} \frac{1}{p^n} \sum_{y \in Z_p^n} e\left(y \cdot \left(\sum_{j=1}^m \varepsilon_j x_j - x\right)\right) \\ &= \frac{1}{p^n} \sum_{y \in Z_p^n} \overline{e(y \cdot x)} \sum_{\varepsilon \in \{0,1\}^m} e\left(y \cdot \sum_{j=1}^m \varepsilon_j x_j\right) \\ &= \frac{1}{p^n} \sum_{y \in Z_p^n} \overline{e(y \cdot x)} \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_m=0}^{1} \prod_{j=1}^m e(\varepsilon_j y \cdot x_j) \\ &= \frac{2^m}{p^n} \sum_{y \in Z_p^n} \overline{e(y \cdot x)} \prod_{j=1}^m \frac{1+e(y \cdot x_j)}{2}. \end{aligned}$$

Therefore

$$\left|N(x) - \frac{2^m}{p^n}\right| \leq \frac{2^m}{p^n} \sum_{0 \neq y \in Z_p^n} \prod_{j=1}^m \left|\frac{1+e(y \cdot x_j)}{2}\right|. \tag{2.1}$$

(The same estimate is also used in [BS].) Next we estimate the right hand side of (2.1). For any fixed basis B of Z_p^n, and $y \in Z_p^n$ let $P_B(y) = \prod_{b \in B} |(1+e(y \cdot b))/2|$.

Since $P_B(y)$ depends only on the list of inner products $(y \cdot b : b \in B)$, it follows that the multiset $\{P_B(y): y \in Z_p^m\}$ is independent of the choice of the basis B. Choosing $B = \{b_1, ..., b_n\}$ to be the standard basis of Z_p^n, and noting that for $y = (y_1, ..., y_n)$

$$\left|\frac{1+e(b_j \cdot y)}{2}\right| = \left|\frac{1+e(y_j)}{2}\right| = \left|\cos\frac{\pi y_j}{p}\right|,$$

we obtain

$$\begin{aligned}
\sum_{y \in Z_p^n} \left|\prod_{j=1}^{m} \frac{1+e(y \cdot x_j)}{2}\right| &= \sum_{y \in Z_p^n} \prod_{i=1}^{t} P_{B_i}(y) \\
&\leqslant \sum_{y \in Z_p^n} P_B(y)^t = \sum_{y \in Z_p^n} \prod_{j=1}^{n} \left|\cos\frac{\pi y_j}{p}\right|^t \\
&= \left(\sum_{k=0}^{p-1} \left|\cos\frac{\pi k}{p}\right|^t\right)^n \leqslant \left(1+(p-1)\cos^t\frac{\pi}{p}\right)^n \\
&\leqslant \left(1+p\left(1-\frac{\pi^2}{4p^2}\right)^{(p^2/2)\log 2pn}\right)^n \\
&< \left(1+\frac{1}{2n}\right)^n < e^{1/2}. \qquad (2.2)
\end{aligned}$$

Combining (2.1) and (2.2) we obtain

$$\left|N(x) - \frac{2^m}{p^n}\right| \leqslant \frac{2^m}{p^n}(e^{1/2}-1) < \frac{2^m}{p^n}.$$

Hence $N(x) > 0$ for all $x \in Z_p^n$. ∎

3. Permanents and Vector Sums

In this section we present a second proof of Theorem 1.1, with a somewhat better estimate for $c(p)$. Specifically, we prove the following proposition.

Proposition 3.1. *Let* $A_1 = \{\underline{a}^{11}, ..., \underline{a}^{1n}\}$, $A_2 = \{\underline{a}^{21}, ..., \underline{a}^{2n}\}$, ..., $A_l = \{\underline{a}^{l1}, ..., \underline{a}^{ln}\}$ *be* l *bases of the vector space* Z_p^n. *If*

$$\left(1-\frac{1}{p-1}\right)^{l-p+2} n < 1 \qquad (3.1)$$

then for any vector $\underline{b} \in Z_p^n$ *there are* $\varepsilon_{ij} \in \{0, 1\}$ $(1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant n)$, *such that* $\sum_{i,j} \varepsilon_{ij} \underline{a}^{ij} = \underline{b}$. *In particular, the conclusion holds provided* $l \geqslant (p-1)\log n + p - 2$.

The proof presented here differs considerably from the one given in Section 2 and is based on some simple properties of permanents over finite fields. The basic method resembles the one used in [AT], but several additional ideas are incorporated.

It is convenient to split the proof into several lemmas. We start with the following simple lemma (which appears in a similar context in [AFK]).

LEMMA 3.2. *Let* $P = P(x_1, ..., x_m)$ *be a multilinear polynomial with* m *variables* $x_1, ..., x_m$ *over a commutative ring with identity* R; *i.e.,* $P = \sum_{U \subseteq \{1,...,m\}} a_U \cdot \prod_{i \in U} x_i$, *where* $a_U \in R$. *If* $P(x_1, ..., x_m) = 0$ *for each* $(x_1, ..., x_m) \in \{0, 1\}^m$ *then* $P \equiv 0$, *i.e.,* $a_U = 0$ *for all* $U \subseteq \{1, ..., m\}$.

Proof. We apply induction on m. The result is trivial for $m = 1$. Assuming it holds for $m-1$ we prove it for m. Clearly $P(x_1, ..., x_m) = P_1(x_1, ..., x_{m-1})\, x_m + P_2(x_1, ..., x_{m-1})$, where P_1 and P_2 are multilinear polynomials in $x_1, ..., x_{m-1}$. Moreover, it is easy to see that P_1 and P_2 satisfy the hypotheses of the lemma for $m-1$. By the induction hypothesis $P_1 \equiv P_2 \equiv 0$, completing the proof. ∎

The next lemma shows a connection between a permanent of a matrix and the possible sums of subsets of its set of columns. This connection is crucial for our proof.

LEMMA 3.3. *Let* $A = (a_{ij})$ *be an* m *by* m *matrix over the finite prime field* Z_p. *Suppose that* $\mathrm{Per}(A) \neq 0$ *(over* Z_p*). Then for any vector* $\underline{c} = (c_1, ..., c_m) \in Z_p^n$ *there are* $\varepsilon_1, ..., \varepsilon_m \in \{0, 1\}$ *such that* $\sum_{j=1}^m \varepsilon_j a_{ij} \neq c_i$ *for all* $1 \leqslant i \leqslant m$. *In other words, for any vector* $\underline{c}$ *there is a subset of the columns of* A *whose sum differs from* $\underline{c}$ *in each coordinate.*

Proof. Suppose the lemma is false and let $A = (a_{ij})$ and $\underline{c}$ be a counter-example. Consider the polynomial $P = P(x_1, ..., x_m) = \prod_{i=1}^m (\sum_{j=1}^m a_{ij} x_j - c_i)$. By assumption, $P(x_1, ..., x_m) = 0$ for each $(x_1, ..., x_m) \in \{0, 1\}^m$. Let $\bar{P} = \bar{P}(x_1, ..., x_m)$ be the multilinear polynomial obtained from P by writing P as a sum of monomials and replacing each monomial $a_{ij} \prod_{i \in U} x_i^{\delta}$, where $U \subseteq \{1, ..., m\}$ and $\delta_i > 0$, by $a_U \prod_{i \in U} x_i$. Clearly $\bar{P}(x_1, ..., x_m) = P(x_1, ..., x_m) = 0$ for each $(x_1, ..., x_m) \in \{0, 1\}^m$. By Lemma 3.2 we conclude that $\bar{P} \equiv 0$. However, this is impossible, since the coefficient of $\prod_{i=1}^m x_i$ in $\bar{P}$ (which equals the coefficient of that product in P) is $\mathrm{Per}\, A \neq 0$. This completes the proof. ∎

For a (column) vector $\underline{v} = (v_1, ..., v_n) \in Z_p^n$ let us denote by $\underline{v}^* = \underline{v}^*(p)$ the (column) vector in $Z_p^{(p-1)n}$ defined by $\underline{v}^*_{(i-1)n+j} = v_j$ for all $1 \leqslant i \leqslant p-1$,

$1 \leqslant j \leqslant n$. Thus $\underline{v}^*$ is simply the tensor product of $\underline{v}$ with a vector of $(p-1)$ 1's. Clearly $\underline{v}^* = \underline{v}^*(p)$ depends on $\underline{v}$ as well as on p, but since p remains fixed during this section we usually omit it and simply write $\underline{v}^*$.

A simple corollary of Lemma 3.3 is the following.

COROLLARY 3.4. *Let $\underline{a}^1, ..., \underline{a}^{(p-1)n}$ be $(p-1)n$ vectors in Z_p^n. Let A be the $(p-1)n$ by $(p-1)n$ matrix whose columns are the vectors $\underline{a}^{1*}, ..., \underline{a}^{(p-1)n*}$. If* Per $A \neq 0$ *then any vector $\underline{b} \in Z_p^n$ is a sum of a certain subset of the vectors $\underline{a}^1, ..., \underline{a}^{(p-1)n}$.*

Proof. Let $\underline{c} = (c_1, ..., c_{(p-1)n}) \in Z_p^{(p-1)n}$ be a vector satisfying $\{c_{(i-1)n+j}: 1 \leqslant i \leqslant p-1\} = Z_p \setminus \{b_j\}$ for each j, $1 \leqslant j \leqslant n$. By Lemma 3.3 there are $\varepsilon_1, ..., \varepsilon_{(p-1)n} \in \{0, 1\}$ such that for any $1 \leqslant i \leqslant p-1$ and any $1 \leqslant j \leqslant n$

$$\sum_{l=1}^{(p-1)n} \varepsilon_l \underline{a}^l_{(i-1)n+j} \neq c_{(i-1)n+j}.$$

However, since the left hand side in the last equality is simply $\sum_{l=1}^{(p-1)n} \varepsilon_l \underline{a}^l_j$ this shows that $\sum_{l=1}^{(p-1)n} \varepsilon_l \underline{a}^l_j \notin Z_p \setminus \{b_j\}$ for each $1 \leqslant j \leqslant n$. Consequently, $\sum_{l=1}^{(p-1)n} \varepsilon_l \underline{a}^l = \underline{b}$, completing the proof. ∎

The last corollary implies that in order to prove Proposition 3.1 it suffices to show that from any sequence of $l \cdot n$ vectors consisting l bases of Z_p^n one can choose $(p-1)n$ distinct members $\underline{d}^1, ..., \underline{d}^{(p-1)n}$ of the sequence such that the permanent of the matrix whose columns are $\underline{d}^{1*}, ..., \underline{d}^{(p-1)n*}$ is nonzero (over Z_p). In what follows we show that this is always possible provided (3.1) holds.

LEMMA 3.5. *Let $D = \{\underline{d}^1, ..., \underline{d}^n\}$ be a basis of Z_p^n, and let A_D be a $(p-1)n$ by $(p-1)n$ matrix whose columns are the vectors $\underline{d}^{1*}, ..., \underline{d}^{n*}$, each appearing $p-1$ times. Then* Per $A_D \neq 0$.

Proof. Let $E = \{\underline{e}^1, ..., \underline{e}^n\}$ be the standard basis of Z_p^n, and let A_E be the $(p-1)n$ by $(p-1)n$ matrix whose columns are $\underline{e}^{1*}, ..., \underline{e}^{n*}$, each appearing $(p-1)$ times. One can easily check that Per A_E is simply the number of perfect matchings in the union of n pairwise disjoint complete bipartite graphs $K_{p-1,p-1}$, which is $((p-1)!)^n \neq 0$ (in Z_p). Since D is a basis, each column of A_E is a linear combination of the columns of A_D. By the multilinearity of the permanent function it follows that Per A_E is a linear combination (over Z_p) of permanents of matrices whose columns are columns of A_D. Since Per $A_E \neq 0$, we conclude that there is a $(p-1)n$ by $(p-1)n$ matrix M, each column of which is d^{i*} for some $1 \leqslant i \leqslant n$, satisfying Per $M \neq 0$. However, if the same column appears in M p times or more,

than Per M is divisible by $p!$, and is thus 0. It follows that no column appears in M more than $(p-1)$ times, and hence M equals A_D up to a permutation of the columns. Thus $\operatorname{Per} A_D = \operatorname{Per} M \neq 0$, completing the proof. ∎

LEMMA 3.6. *Let* $A_1 = \{\underline{a}^{11}, \underline{a}^{12}, \ldots, \underline{a}^{1n}\}, \ldots, A_l = \{\underline{a}^{l1}, \underline{a}^{l2}, \ldots, \underline{a}^{ln}\}$ *be* l *bases of* Z_p^n *and let* $S = (\underline{s}_1, \ldots, \underline{s}_{ln})$ *be the sequence of length* $l \cdot n$ *of vectors in* $Z_p^{(p-1)n}$ *given by* $\underline{s}_{(i-1)n+j} = \underline{a}^{ij*}$ *for all* $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant n$. *Suppose that for some integer* h

$$\left(1 - \frac{1}{p-1}\right)^{l-h} \cdot (p-1) \cdot n < h + 1. \tag{3.2}$$

Then there are $(p-1)n$ *distinct indices* $1 \leqslant i_1 < i_2 < \cdots < i_{(p-1)n} \leqslant ln$ *such that the matrix whose columns are* $\{\underline{s}_{ij}\colon 1 \leqslant j \leqslant (p-1)n\}$ *has a nonzero permanent.*

Proof. Given a $(p-1)n$ by $(p-1)n$ matrix B whose columns are members of S, we call a column of B a *repeated column* if the same member of S appears in at least one additional column of B. Let $c(B)$ denote the total number of repeated columns of B. Our objective is to construct a matrix with no repeated columns whose permanent is nonzero. To this end, we construct a sequence of matrices $B_1, B_2, \ldots$, with nonzero permanents as follows. Let B_1 be the $(p-1)n$ by $(p-1)n$ matrix whose columns are $\underline{s}_1, \ldots, \underline{s}_n$, each appearing $(p-1)$ times. By Lemma 3.5 $\operatorname{Per} B_1 \neq 0$, and clearly, all the $(p-1)n$ columns of B_1 are repeated columns. Since A_2 is a basis, each column of B_1 is a linear combination of $\underline{s}_{n+1}, \ldots, \underline{s}_{2n}$. Let us replace all but one of the $p-1$ occurrences of each $\underline{s}_i$ in B_1 by the linear combination of $\underline{s}_{n+1}, \ldots, \underline{s}_{2n}$ expressing it. By the multilinearity of the permanent function, this enables us to write $\operatorname{Per} B_1 \neq 0$ as a linear combination of permanents of matrices whose columns are all from the set $\{\underline{s}_1, \ldots, \underline{s}_{2n}\}$. Obviously, at least one of these matrices has a nonzero permanent. Let B_2 be such a matrix. Then, there are at least n nonrepeated columns of B_2, since each of the n vectors $\underline{s}_1, \ldots, \underline{s}_n$ appears precisely once in it. Hence, $c(B_2) \leqslant (1 - 1/(p-1))(p-1)n$. It is also clear that no $\underline{s}_i$ appears more than $p-1$ times as a column of B_2, as $\operatorname{Per}(B_2) \neq 0$. Assume, by induction, that we have already constructed, for each $i \leqslant k$, a $(p-1)n$ by $(p-1)n$ matrix B_{i+1}, each column of which belongs to the set $\underline{s}_1, \ldots, \underline{s}_{(i+1)n}$, satisfying

$$\operatorname{Per}(B_{i+1}) \neq 0 \qquad \text{and} \qquad c(B_{i+1}) \leqslant \left(1 - \frac{1}{p-1}\right)^i (p-1)n. \tag{3.3}$$

Let us show that if $k+2 \leqslant l$ we can construct a matrix B_{k+2} with the same properties. If $c(B_{k+1}) = 0$ simply take $B_{k+2} = B_{k+1}$. Otherwise, replace

each occurrence of each repeated column of B_{k+1} but one, by a linear combination of $\underline{s}_{(k+1)n+1}, \ldots, \underline{s}_{(k+2)n}$ and apply, as before, multilinearity to obtain a matrix B_{k+2} with a nonzero permanent. Since no repeated column can appear in B_{k+1} more than $p-1$ times, we conclude that

$$c(B_{k+2}) \leqslant \left(1-\frac{1}{p-1}\right) c(B_{k+1}) \leqslant \left(1-\frac{1}{p}\right)^{k+1} (p-1)n.$$

In particular, taking $i=l-h$, it follows from (3.2) and (3.3) that there is a matrix B_{l-h+1}, each column of which belongs to the set $\underline{s}_1, \ldots, \underline{s}_{(l-h+1)n}$ such that $\operatorname{per}(B_{l-h+1}) \neq 0$ and $c(B_{l-h+1}) \leqslant (1-1/(p-1))^{l-h}(p-1)n < h+1$.

Thus B_{l-h+1} has at most h repeated columns. Denote these columns by $\underline{b}^l, \underline{b}^{l-1}, \ldots, \underline{b}^{l-h+1}$. For each i, $0 \leqslant i \leqslant h-2$, let us express $\underline{b}^{l-i}$ as a linear combination of $\underline{s}_{(l-i-1)n+1}, \ldots, \underline{s}_{(l-i)n}$. Applying multilinearity once more we obtain a matrix with nonzero permanent and no repeated columns. This completes the proof. ∎

We are now ready to prove Proposition 3.1. Given the l bases $A_1, \ldots, A_l$, where l satisfies (3.1), we apply Lemma 3.6 with $h=p-2$ to conclude that there is a set I of $(p-1)n$ distinct double indices ij such that the matrix whose columns are $\{\underline{a}^{ij*}: ij \in I\}$ has a nonzero permanent. By Corollary 3.4, this implies that for any vector $\underline{b} \in Z_p^n$ there are $\varepsilon_{ij} \in \{0, 1\}$, $(ij \in I)$, such that $\sum_{ij \in I} \varepsilon_{ij} \underline{a}^{ij} = \underline{b}$. This completes the proof of Proposition 3.1. Observe that we actually proved a somewhat stronger result; if l satisfies (3.1) then it is possible to choose a fixed set of $(p-1)n$ of our vectors such that any $\underline{b} \in Z_p^n$ is a sum of a subset of this fixed set. ∎

4. Concluding Remarks and Open Problems

The main open problem is, of course, whether the union of any $c(p)$ linear bases of Z_p^n is an additive basis, where $c(p)$ depends on p alone. The following two results, which follow from our previous proofs of Theorem 1.1, suggest that this, indeed, may be the case.

Proposition 4.1. *For any l bases $B_1, \ldots, B_l$ of Z_p^n, when $l \geqslant p \log (pn)$ there are subsets $A_i \subset B_i$ $(1 \leqslant i \leqslant l)$, such that $\sum_{i=1}^{l} |A_i| \leqslant (p-1)n$ and $\bigcup_{i=1}^{l} A_i$ (with repetitions) is an additive basis of Z_p^n.*

Proposition 4.2. *Let $S=(s_1, s_2, \ldots, s_l)$ be a sequence of vectors in Z_p^n and suppose that each subsequence of $l-(p-1)n$ members of S linearly spans Z_p^n. Then S is an additive basis of Z_p^n.*

The following conjecture about permanents would imply, if true, that $f(p, n) \leqslant p$.

CONJECTURE 4.3. For any p nonsingular n by n matrices $A_1, A_2, \ldots, A_p$ over Z_p, there is an n by $p \cdot n$ matrix C such that

$$\operatorname{Per}\begin{bmatrix} A_1 A_2 \cdots A_p \\ A_1 A_2 \cdots A_p \\ \vdots \qquad \vdots \\ A_1 A_2 \cdots A_p \\ C \end{bmatrix} \neq 0.$$

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