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Pao-Liu Chow

Wayne State University, plchow@math.wayne.edu

José-Luis Menaldi

Wayne State University, menaldi@wayne.edu

Maurice Robin

Institut National de Recherche en Informatique et en Automatique

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ADDITIVE CONTROL OF STOCHASTIC LINEAR SYSTEMS WITH FINITE HORIZON*

PAO-LIU CHOW†, JOSÉ-LUIS MENALDI‡ AND MAURICE ROBIN‡

Abstract. We consider a dynamic system whose state is governed by a linear stochastic differential equation with time-dependent coefficients. The control acts additively on the state of the system. Our objective is to minimize an integral cost which depends upon the evolution of the state and the total variation of the control process. It is proved that the optimal cost is the unique solution of an appropriate free boundary problem in a space-time domain. By using some decomposition arguments, the problems of a two-sided control, i.e. optimal corrections, and the case with constraints on the resources, i.e. finite fuel, can be reduced to a simpler case of only one-sided control, i.e. a monotone follower. These results are applied to solving some examples by the so-called method of similarity solutions.

Key words. dynamic programming, stochastic processes, free boundary problems, degenerate second order parabolic equations

Introduction. In this paper, we wish to control a linear stochastic differential equation in the sense of Itô by using additive strategies, i.e. the evolution of the state is subjected to

$$(1) \quad \begin{aligned} y(s) = x + \nu(s-t) + \int_t^s (a(\lambda)y(\lambda) + b(\lambda)) d\lambda \\ + \int_t^s \sigma(\lambda) dw(\lambda-t) \quad \text{for every } s \geq t, \end{aligned}$$

where $a(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ are given deterministic functions, $(w(s), s \geq 0)$ is a standard Wiener process, x is the initial state at the time t and $(\nu(s), s \geq 0)$ stands for the control which is a progressively measurable process with locally bounded variation.

The expected cost takes the form

$$(2) \quad J_{xt}(\nu) = E \left\{ \int_t^T f(y(s), s) ds + c(t)\nu(0) + \int_t^T c(s) d|\nu|(s-t) \right\},$$

with $f(\cdot, \cdot)$, $c(\cdot)$ given, $|\nu|$ denoting the variation of the process ν and T being the finite horizon. Hence, the optimal cost function is

$$(3) \quad u(x, t) = \inf \{ J_{xt}(\nu) : \nu \} \quad \text{for every } x, t.$$

The entire paper is devoted to the one-dimensional case, i.e. x belonging to \mathbb{R} ; however, most of the results can be extended to multidimensional situations.

A formal application of the dynamic programming principle yields the complementary problem

$$(4) \quad \begin{aligned} \max \{ Au - f, |Du| - c \} = 0 \quad \text{in } \mathbb{R} \times [0, T], \\ u(\cdot, T) = 0 \quad \text{in } \mathbb{R}, \end{aligned}$$

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† Department of Mathematics, Wayne State University, Detroit, Michigan 48202.

‡ INRIA, Domaine de Voluceau, B.P. 105, Rocquencourt, 78153 Le Chesnay, Cedex, France.

for the optimal cost (3), where the operators

$$\begin{aligned}
 (5) \quad Au &= -\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2(t) \frac{\partial^2 u}{\partial x^2} - (a(t)x + b(t)) \frac{\partial u}{\partial x}, \\
 Du &= -\frac{\partial u}{\partial x},
 \end{aligned}$$

and $|\cdot|$ denotes the absolute value of a real number.

It is clear that (4) can be regarded either as a variational inequality or as a free boundary problem. In contrast with the classical aspect of the problem (4) we mention that among our assumptions it is allowed to have degeneracy, i.e., $\sigma(t) = 0$, and that we are interested in the characteristics of an optimal policy of the control as well as a possible computation of that optimal strategy. Moreover, we seek a suitable decomposition of (4) into problems, typically of the form

$$\begin{aligned}
 (6) \quad \max \{Au - f, Du - c\} &= 0 \quad \text{in } \mathbb{R} \times [0, T[, \\
 u(\cdot, T) &= 0 \quad \text{in } \mathbb{R}.
 \end{aligned}$$

Also, we wish to be able to treat the case with constraints on the resources, i.e. in the minimization (3) we add a condition:

$$(7) \quad \text{the total variation of } \nu \text{ on } [0, T] \text{ is bounded by a constant } K,$$

where K stands for the total resources available.

On the other hand, we will see that the problem (6), commonly referred to as the “monotone follower,” can be obtained as a limit-case of a quasi-variational inequality.

As the main result of this paper, we should mention the characterization of the optimal cost function as the unique solution of the problem (4) or (6) in a certain sense; the proof of the existence of an optimal control; the construction of an optimal control of Markovian type; the reduction to problems of the form (6); and lastly, some properties of regularity for the optimal cost, e.g. locally Lipschitzian derivative of u , even without assuming uniform ellipticity of the operator A in (5).

This problem is motivated by our interest in studying the optimal control of a dissipative dynamical system under uncertainty. In the simplest model, one considers the automatic cruise control of an aircraft under an uncertain wind condition. The equation (1) is the equation of motion, where $y(s)$ is the speed; $a(s) < 0$ the coefficient of air resistance; $b(s)$ the thrust force; the white-noise term the dynamic force due to the shifting wind condition, and the formal derivative $\dot{\nu}$ represents the control in the form of a corrective thrust force. We wish to find an optimal control policy ν over the flight time T so that, given a finite amount of fuel for correction, the flight speed will deviate as little as possible to a desirable cruising speed at a minimum fuel cost. This fact is expressed by the equations (2) and (3). The system (1)–(3) has another interesting interpretation in the context of optimal harvesting of randomly fluctuating resource Ludwig [36]. In this case, the equation (1) stands for a controlled linear growth model for the size y of a population, say, in a fishery, where $a > 0$ is the birth rate; the terms b and $(\sigma\dot{w})$ are, respectively, the mean and fluctuating rates of migration, and $\dot{\nu}$ denotes the harvesting rate. For instance, in a finite horizon, we would like to determine the harvesting rate in order to maintain the population size as close as possible to an equilibrium size at a minimum cost.

Let us remark that, when the rate function $a \equiv 0$, similar kinds of problems have been considered by several authors, in particular Bather and Chernoff [4], [5], Benes, Shepp and Witsenhausen [6], Borodowski et al. [12], Bratus [13], Chernousko [17],

[18], Gorbunov [22], Harrison and Taksar [24], Harrison and Taylor [25], Jacka [26], Shreve et al. [55], Karatzas [27], [28], and [41], [42]. The connection with optimal stopping is deeply investigated in Karatzas and Shreve [29], [30].

The methods to be used throughout this article are suggested by the techniques presented in the books of Bensoussan and Lions [8], [9], Fleming and Rishel [20], Friedman [21], Kinderlehrer and Stampacchia [31], and Krylov [32].

We organize the contents of the paper as follows:

1. Statement of the problems and assumptions
2. The dynamic programming approach
 - 2.1. Some estimates
 - 2.2. Characterization of the optimal cost
3. The free boundary
 - 3.1. Variational inequality
 - 3.2. Optimal decision
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5. Optimal corrections
 - 5.1. Reduction
 - 5.2. General comments
6. Examples
 - 6.1. Unlimited resources
 - 6.2. Finite resources

1. Statement of the problem and assumptions. Let (Ω, \mathcal{F}, P) be a probability space, $(w(t), t \geq 0)$ be a standard Wiener process in \mathbb{R} and $(\mathcal{F}^t, t \geq 0)$ be a filtration satisfying the usual conditions with respect to $(w(t), t \geq 0)$, i.e., $(\mathcal{F}^t, t \geq 0)$ is an increasing right continuous family of completed σ -subalgebras of \mathcal{F} and $(w(t), t \geq 0)$ is a martingale with respect to $(\mathcal{F}^t, t \geq 0)$.

Denote by \mathcal{V} the set of controls $\nu(\cdot)$ which are progressively measurable random processes from $[0, +\infty)$ into $\bar{\mathbb{R}}$ (extended real numbers), right continuous having left limits (cad-lag), nonnegative and increasing, i.e.,

$$(1.1) \quad \nu(0) \geq 0, \quad \nu(s) - \nu(t) \geq 0 \quad \text{for every } s \geq t \geq 0.$$

The state of the dynamic system is described by the following stochastic equation

$$(1.2) \quad \begin{aligned} dy(s) &= d\nu(s-t) + (a(s)y(s) + b(s)) ds + \sigma(s) dw(s-t), & s \geq t, \\ y(t) &= x + \nu(0), \end{aligned}$$

where $a(s)$, $b(s)$ and $\sigma^2(s)$ stand for the drift and the covariance terms, and x is the initial state at the time t . Note that $y(s) = y_{x,t}(s)$ is a cad-lag random process adapted to $(\mathcal{F}^{s-t}, s \geq t)$.

To each control ν in \mathcal{V} , we associate a cost given by the payoff functional

$$(1.3) \quad \begin{aligned} J_{x,t}(\nu) = E \left\{ \int_t^T f(y(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds + c(t)\nu(0) \right. \\ \left. + \int_t^T c(s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\nu(s-t) \right\}, \end{aligned}$$

where f , α , c and T are respectively, the running cost, the discount factor, the instantaneous cost per unit of fuel and the finite horizon.

Our purpose is to characterize the optimal cost

$$(1.4) \quad \hat{u}(x, t) = \inf \{ J_{x,t}(\nu) : \nu \text{ in } \mathcal{V} \}$$

and to construct an optimal control \hat{v} , i.e.

$$(1.5) \quad \hat{v} \text{ in } \mathcal{V} \text{ such that } \hat{u}(x, t) = J_{xt}(\hat{v})$$

for each initial state (x, t) . This problem corresponds to several simple models, e.g., control of a spaceship with unlimited fuel (cf. Bather and Chernoff [4]), optimal control with no turning back (cf. Barron and Jensen [1]), monotone follower problem (cf. in Benes et al. [6, problem # 2], Karatzas [27], [28]), optimal correction problem (cf. Chernousko [17], [18], Borodovskii et al. [12], Bratus [13], Gorbunov [22], optimal control of a dam (cf. Bather [3], Faddy [19]), control of Brownian motion (cf. Rath [50], [51], Chernoff and Petkau [16]) and inventory theory (cf. Bather [2], Menaldi and Rofman [45]).

A similar study will be made for the optimal cost

$$(1.6) \quad \hat{v}(x, z, t) = \inf \{J_{xt}(\nu) : \nu \text{ in } \mathcal{V}, \nu(T) \leq z\},$$

where the positive constant z stands for the total amount of fuel available. This is associated with the previous cases under constraint of resources, e.g., the control of a spaceship with finite fuel available (cf. Bather and Chernoff [5], problem # 3 in Benes et al. [6]).

Let us summarize the technical assumptions as follows:

$$(1.7) \quad T \text{ is a positive constant,}$$

$$(1.8) \quad a(t), b(t), \sigma(t), \alpha(t), c(t) \text{ are Lipschitz functions from } [0, T] \text{ into } \mathbb{R} \text{ and either } c(t) \geq c_0 > 0 \text{ for every } t \text{ or } c(t) = 0 \text{ for every } t,$$

$f(x, t)$ is a nonnegative continuous function from $\mathbb{R} \times [0, T]$ into \mathbb{R} such that there exist constants $m \geq 1, 0 \leq c \leq C$ satisfying

$$(1.9) \quad \begin{aligned} c|x^+|^m - C &\leq f(x, t) \leq C(1 + |x|^m), \\ |f(x, t) - f(x', t)| &\leq C(1 + |x|^{m-1} + |x'|^{m-1})|x - x'|, \\ |f(x, t) - f(x, t')| &\leq C(1 + |x|^m)|t - t'|, \\ 0 \leq \frac{\partial^2 f}{\partial x^2}(x, t) &\leq C(1 + |x|^q), \quad q = (m - 2)^+, \end{aligned}$$

for every x, x', t, t' ,

where $(\cdot)^+$ denotes the positive part of a real number, i.e., $x^+ = x$ if $x \geq 0$ and $x^+ = 0$ if $x \leq 0$. Note that $\sigma(t)$ could vanish, even everywhere, and then the problem could be degenerate and even deterministic. On the other hand, since the horizon T is finite, without loss of generality, the function $\alpha(t)$ may be assumed to satisfy for every t

$$(1.10) \quad \alpha(t) \geq \alpha_0, \quad \alpha_0 \text{ is positive large enough.}$$

Let us introduce two penalized problems associated with (1.4) as follows: $\varepsilon > 0$,

$$(1.11) \quad \mathcal{V}_\varepsilon \text{ is the set of all controls } \nu(\cdot) \text{ in } \mathcal{V} \text{ such that } \nu(t) \text{ is Lipschitz continuous and}$$

$$0 \leq \frac{d\nu}{dt}(t) \leq \frac{1}{\varepsilon} \text{ for almost every } t \geq 0,$$

$$(1.12) \quad \hat{u}^\varepsilon(x, t) = \inf \{J_{xt}(\nu) : \nu \text{ in } \mathcal{V}_\varepsilon\}$$

and

\mathcal{V}_* is the set of impulse controls, i.e., $\nu(\cdot)$ in \mathcal{V} such that there exist stopping times $(\theta_j, j = 1, 2, \dots)$ $0 \leq \theta_j \leq \theta_{j+1}$, for every $j = 1, 2, \dots$, and adapted random variables $(\xi_j, j = 1, 2, \dots)$ satisfying

$$\nu(t) = \sum_{j=1}^{\infty} \xi_j I(\theta_j \leq t) \quad \text{for every } t \geq 0,$$

where $I(\theta_j \leq t)$ is the characteristic function of the set $(\theta_j \leq t)$,

$$J_{xt}^\varepsilon(\nu) = J_{xt}(\nu) + \varepsilon E \left\{ \sum_{j=1}^{\infty} \exp \left(- \int_t^{t+\theta_j} \alpha(s) ds \right) \right\},$$

$$\hat{u}_\varepsilon(x, t) = \inf \{ J_{xt}^\varepsilon(\nu) : \nu \text{ in } \mathcal{V}_* \}.$$

Notice that (1.12) and (1.15) correspond respectively to a classical stochastic control problem (cf. Fleming and Rishel [20]) and an impulse control problem (cf. Bensoussan and Lions [9]). The term “penalized” is used to indicate that the formal Dynamic Programming equations associated with the problems (1.12) and (1.15) are indeed two possible penalizations of the equation (2.4) below.

On the other hand, for $z \geq 0$ and ν in \mathcal{V} , define a cost

$$F(x, z, t, \nu) = E \left\{ \int_t^\tau f(y(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds + c(t)\nu(0)I(t < \tau) + \int_t^\tau c(s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\nu(s-t) + u^0(y(\tau), \tau) \exp \left(- \int_t^\tau \alpha(s) ds \right) \right\},$$

where $\tau = \tau_{zt}$ is the first exit time from $[\nu \leq z]$ of the process $\nu(s)$, i.e.

$$\tau = \inf \{ s \in [t, T] : \nu(s-t) > z \},$$

$y^0(s)$ is given by (1.2) with $\nu = 0$, and

$$u^0(x, t) = E \left\{ \int_t^T f(y^0(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right\}$$

represents the cost of free evolution. It is clear that the optimal cost (1.6) corresponding to finite fuel conditions, satisfies

$$\hat{v}(x, z, t) = \inf \{ F(x, z, t; \nu) : \nu \text{ in } \mathcal{V} \}.$$

The relation will be used to reduce the problem with constrained resources to the case without constraint.

To conclude this section, let us observe that it is possible to obtain the same optimal cost (1.4) by minimizing the functional cost (1.3), denoted now by $J_{xt}(\mathcal{A})$, over all system controls \mathcal{A} , where \mathcal{A} is a set including the probability space (Ω, \mathcal{F}, P) , the filtration, the Wiener process and the control $(\mathcal{F}^t, w(t), \nu(t), t \geq 0)$. The same idea corresponds to identifying the state process $y_{xt}(s)$ with its probability law P_{xt} on the sample space D of the cad-lag functions. The probability law P_{xt} is characterized by

the conditions

$$\begin{aligned}
 P_{xt}(X_t = x) &= 1, \\
 (1.20) \quad \varphi(X_s, s) - \int_t^s &\left[\frac{\partial \varphi}{\partial s} + \sigma^2(\lambda) \frac{\partial^2 \varphi}{\partial x^2} + (a(\lambda)X_\lambda + b(\lambda)) \frac{\partial \varphi}{\partial x} \right] (X_\lambda, \lambda) d\lambda \\
 &- \int_t^s \varphi(X_\lambda, \lambda) d\nu(\lambda) = M_s
 \end{aligned}$$

is a martingale in $t \leq s \leq T$, for every smooth function φ in $\mathbb{R} \times [0, T]$.

More details about this formulation for stochastic control problems can be found in Nisio [49], Bensoussan and Lions [8], Lions and Menaldi [35].

2. The dynamic programming approach. Consider the differential operators

$$(2.1) \quad Au = -\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2(t) \frac{\partial^2 u}{\partial x^2} - (a(t)x + b(t)) \frac{\partial u}{\partial x} + \alpha(t)u,$$

and

$$(2.2) \quad Bu = -\frac{\partial u}{\partial x} - c(t).$$

A heuristic application of the dynamic programming to the penalized problem (1.11), (1.12) yields the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned}
 (2.3) \quad Au + \frac{1}{\varepsilon} (Bu)^+ &= f \quad \text{in } \mathbb{R} \times [0, T[, \\
 u(\cdot, T) &= 0 \quad \text{in } \mathbb{R}
 \end{aligned}$$

to be satisfied by the optimal cost \hat{u}^ε defined in (1.12). Then, as ε tends to zero, (2.3) becomes

$$\begin{aligned}
 (2.4) \quad (Au - f) \vee Bu &= 0 \quad \text{in } \mathbb{R} \times [0, T[, \\
 u(\cdot, T) &= 0 \quad \text{in } \mathbb{R},
 \end{aligned}$$

where $x \vee y$ denotes the maximum of the two real numbers x and y . Equation (2.4) will be used to characterize the optimal cost \hat{u} given by (1.4).

On the other hand, the quasi-variational inequality associated with the impulse control problem (1.14), (1.15) is

$$\begin{aligned}
 (2.5) \quad (Au - f) \vee (u - Mu - \varepsilon) &= 0 \quad \text{in } \mathbb{R} \times [0, T[, \\
 u(\cdot, T) &= 0 \quad \text{in } \mathbb{R},
 \end{aligned}$$

where

$$(2.6) \quad Mu(x, t) = \inf \{ \xi c(t) + u(x + \xi, t) : \xi \geq 0 \},$$

which is satisfied by the optimal cost \hat{u}_ε defined in (1.15). Moreover, \hat{u}_ε is indeed the maximum solution of (2.5). Thus, as ε tends to zero, (2.5) becomes

$$\begin{aligned}
 (2.7) \quad (Au - f) \vee (u - Mu) &= 0 \quad \text{in } \mathbb{R} \times [0, T[, \\
 u(\cdot, T) &= 0 \quad \text{in } \mathbb{R}.
 \end{aligned}$$

Hence, the optimal cost \hat{u} given by (1.4) will be the maximum solution of the equation

(2.7). For details on the two penalized problems one is referred to the books of Fleming and Rishel [20], Bensoussan and Lions [9] and to the works [38], [41] and [52].

2.1. Some estimates. First of all, we will deduce some a priori estimates for the optimal costs (1.4), (1.12) and (1.15).

THEOREM 2.1. *Under the assumptions (1.7), . . . , (1.10) the optimal cost \hat{u} defined by (1.4) is a nonnegative continuous function such that for some constants $0 < c \leq C$, the same $m \geq 1$ of hypothesis (1.9), and every $(x, t), (x', t')$ in $\mathbb{R} \times [0, T]$ we have*

$$\begin{aligned}
 (2.8) \quad & c|x^+|^m - C \leq \hat{u}(x, t) \leq C(1 + |x|^m), \\
 & |\hat{u}(x, t) - \hat{u}(x', t)| \leq C(1 + |x|^{m-1} + |x'|^{m-1})|x - x'|, \\
 & 0 \leq \frac{\partial^2 \hat{u}}{\partial x^2}(x, t) \leq C(1 + |x|^q), \quad q = (m - 2)^+,
 \end{aligned}$$

so, \hat{u} is convex in the first variable. Moreover, if \hat{u} satisfies

$$\begin{aligned}
 (2.9) \quad \hat{u}(x, t) \leq E \left\{ \int_t^{t'} f(y^0(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right. \\
 \left. + \hat{u}(y^0(t'), t') \exp \left(- \int_t^{t'} \alpha(s) ds \right) \right\},
 \end{aligned}$$

for every $t' \geq t \geq 0$, x in \mathbb{R} and $y^0(s)$ given by (1.2) with $\nu = 0$, i.e. the dynamic programming in the weak sense, then we have

$$(2.10) \quad |\hat{u}(x, t) - \hat{u}(x, t')| \leq C(1 + |x|^m)|t - t'|,$$

for every x in \mathbb{R} , t, t' in $[0, T]$ and some constant C .

Proof. Since f has m -polynomial growth as x tends to positive infinity, $\nu(s) \geq 0$ and for the vanishing control $\nu = 0$

$$J_{xt}(0) \leq C(1 + |x|^m),$$

one can restrain the set of admissible controls to those satisfying

$$(2.11) \quad \int_t^T |y_{xt}(s)|^m ds \leq C(1 + |x|^m)$$

for the same $m \geq 1$ of (1.9) and a suitable constant C independent of x, t and ν . Similarly, every admissible control ν may satisfy without loss of generality, the inequality

$$(2.12) \quad \int_t^T |(y_{xt}(s))^+|^m ds \geq c|x^+|^m - C$$

for some constants $0 < c \leq C$ independent of x, t and ν . It is clear that from (2.11) and (2.12) one deduces the first condition of (2.8).

Now, using the fact that for some constant C and for every t, x, x' and ν the following estimate holds

$$(2.13) \quad \int_t^T |y_{xt}(s) - y_{x't}(s)|^m ds \leq C|x - x'|^m,$$

and starting with

$$|\hat{u}(x, t) - \hat{u}(x', t)| \leq \sup \{ |J_{x_t}(\nu) - J_{x'_t}(\nu)| : \nu \text{ in } \mathcal{V} \text{ satisfying (2.9)} \},$$

$$|J_{x_t}(\nu) - J_{x'_t}(\nu)| \leq CE \left\{ \int_t^T (1 + |y_{x_t}(s)|^{m-1} + |y_{x'_t}(s)|^{m-1}) |y_{x_t}(s) - y_{x'_t}(s)| ds \right\},$$

where C is a constant independent of x, t and ν , we obtain the second estimate of (2.8) after applying Hölder's inequality.

In order to get the estimate (2.10), we observe that

$$(2.14) \quad J_{x_t}(\nu) = E \left\{ \int_0^{T-t} f(y(t+s), t+s) \exp \left(- \int_0^s \alpha(t+\lambda) d\lambda \right) ds + c(t)\nu(0) \right. \\ \left. + \int_0^{T-t} c(t+s) \exp \left(- \int_0^s \alpha(t+\lambda) d\lambda \right) d\nu(s) \right\}$$

and if $c(s)$ is strictly positive, the set of admissible controls can be restricted to those continuous at $T-t$ and satisfying for every x, t

$$(2.15) \quad E\{\nu(T-t)\} \leq C(1 + |x|^m),$$

for a suitable constant C independent of x, t and ν . If $y(s)$ and $y'(s)$ denote the evolutions associated respectively to x, t, ν and x, t', ν , we have

$$(2.16) \quad E\{|y(t+s) - y'(t'+s)|^m\} \leq C|t-t'|^m, \text{ for every } s \text{ in } [0, T-t]$$

and some constant C independent of x, t, t' and ν . Hence, starting with

$$\hat{u}(x, t) - \hat{u}(x, t') \leq \sup \{ J_{x_t}(\nu) - J_{x'_t}(\nu) : \nu \text{ in } \mathcal{V} \text{ satisfying (2.11) and (2.15)} \}, \quad t' \leq t$$

and in view of (2.14), (1.9), (1.8), for some constant C ,

$$J_{x_t}(\nu) - J_{x'_t}(\nu) \leq CE \left\{ \left[\int_t^T (1 + |y(s)|^m) ds + \nu(T-t) \right] |t-t'| \right. \\ \left. + \int_0^{T-t} (1 + |y(t+s)|^{m-1} + |y'(t'+s)|^{m-1}) |y(t+s) - y'(t'+s)| ds \right\},$$

we deduce, for a constant C independent of x, t and t' ,

$$(2.17) \quad \hat{u}(x, t') - \hat{u}(x, t) \leq C(1 + |x|^m)|t-t'|, \quad t' \leq t,$$

after using Hölder's inequality and (2.16). To obtain a similar inequality for $t' > t \geq 0$ we shall use (2.9) as follows. From Itô's formula applied to a sequence of smooth functions convergent to

$$x \rightarrow \hat{u}(x, t')$$

we get, for some constant $C > 0$,

$$E\{\hat{u}(y^0(t'), t')\} \leq CE \left\{ \int_t^{t'} (1 + |y^0(s)|^m) ds \right\} + \hat{u}(x, t'),$$

in view of (2.8). Since there is a constant C_0 such that

$$E\{|y^0(s)|^m\} \leq C_0(1 + |x|^m), \quad s \text{ in } [t, t'],$$

the dynamic programming property (2.9) yields

$$\hat{u}(x, t) - \hat{u}(x, t') \leq C(1 + |x|^m)|t-t'|, \quad t' > t.$$

It is clear that this last estimate and (2.18) imply (2.10).

To estimate the second derivative of $\hat{u}(x, t)$ in x , let us proceed as in Krylov [27]. From

$$\begin{aligned} & \hat{u}(x + \Delta x, t) - 2\hat{u}(x, t) + \hat{u}(x - \Delta x, t) \\ & \leq \sup \{ (J_{x+\Delta x}(\nu) - 2J_x(\nu) + J_{x-\Delta x}(\nu)) : \nu \text{ in } \mathcal{V} \text{ satisfying (2.9)} \}, \end{aligned}$$

where the subscript t has been omitted in the functional $J_{xt}(\nu)$, and the equalities

$$\begin{aligned} f(z + r\Delta x, s) - 2f(z, s) + f(z - r\Delta x, s) &= |\Delta x|^2 \int_0^1 d\lambda \int_{-\lambda}^\lambda \frac{\partial^2 f}{\partial x^2}(z + \mu r\Delta x, s) |r|^2 d\mu, \\ y_{x+\Delta x}(s) &= y_x(s) + \Delta x \exp \left(\int_t^s a(\lambda) d\lambda \right), \end{aligned}$$

for every $x, z, \Delta x, r$ and s , we deduce

$$\hat{u}(x + \Delta x, t) - 2\hat{u}(x, t) + \hat{u}(x - \Delta x, t) \leq C(1 + |x|^q) |\Delta x|^2,$$

where $q = (m - 2)^+$, C is a suitable constant independent of (x, t) in $\mathbb{R} \times [0, T]$ and Δx in $[-1, 1]$. Hence an upper bound for the second derivative in x of $\hat{u}(x, t)$ is obtained.

To complete this proof, we need to show that the optimal cost $\hat{u}(x, t)$ is a convex function in the first variable x . Since the functional $J_{xt}(\nu)$ is simultaneously convex in (x, ν) and the set of controls \mathcal{V} is a convex set, we have

$$(2.18) \quad \hat{u}(\theta x + (1 - \theta)x', t) \leq \theta J_{xt}(\nu) + (1 - \theta) J_{x't}(\nu')$$

for every t, x, x', ν, ν' and $0 \leq \theta \leq 1$. Thus, the inequality (2.18) implies the convexity of function \hat{u} . \square

COROLLARY 2.1. *Under the same assumptions of Theorem 2.1 the optimal cost $\hat{u}^\varepsilon(x, t)$ corresponding to the penalized problem (1.11), (1.12), is a nonnegative continuous function satisfying conditions (2.8) and (2.10) uniformly in $\varepsilon > 0$. Furthermore, the optimal cost $\hat{u}_\varepsilon(x, t)$, corresponding to the penalized problem (1.14), (1.15), is a nonnegative continuous function satisfying conditions (2.8), and (2.10) except the bound of the second derivative, uniformly in $\varepsilon > 0$.¹ \square*

Remark 2.1. The optimal cost $\hat{u}_\varepsilon(x, t)$ given by (1.15) is not convex in general. However a discretization in the time variable t allows us to adopt a technique of Scarf [49] in order to show that $\hat{u}_\varepsilon(x, t)$ is ε -convex in x , i.e. for every (x, t) in $\mathbb{R} \times [0, T]$

$$(2.19) \quad \varepsilon + \hat{u}_\varepsilon(x + z, t) - \hat{u}_\varepsilon(x, t) \geq z \frac{\partial \hat{u}_\varepsilon}{\partial x}(x, t) \quad \text{for every } z \geq 0$$

and any $\varepsilon > 0$. On the other hand, a lower bound for the second derivative in x of $\hat{u}_\varepsilon(x, t)$ may be deduced by using the dynamic programming equation (2.3) and the nondegeneracy of $\sigma(t)$.

Define the subset of admissible controls

$$(2.20) \quad \mathcal{V}_0 \text{ is the set of all controls } \nu(\cdot) \text{ in } \mathcal{V} \text{ such that } \nu(t) \text{ is uniformly Lipschitz continuous on } [0, +\infty], \text{ i.e. } 0 \leq d\nu(t)/dt \leq C, \text{ for almost every } t \text{ and some constant } C.$$

THEOREM 2.2. *Let the assumptions (1.7), \dots , (1.10) hold. The infimum of the functional $J_{xt}(\nu)$, given by (1.3), over the sets (a) all controls ν in \mathcal{V} , (b) all Lipschitz controls ν in \mathcal{V}_0 , (c) all impulse controls ν in \mathcal{V}_* , is always the same. Moreover, the*

¹ Note that \hat{u}^ε and \hat{u}_ε satisfy the condition (2.9). See Theorems 2.3 and 2.4.

functions \hat{u}^ε and \hat{u}_ε , given by (1.12) and (1.15), converge to the optimal cost \hat{u} pointwise in $\mathbb{R} \times [0, T]$.²

Proof. Denote by $y(s), y'(s)$ the output corresponding to controls ν, ν' in \mathcal{V} given by (1.2). Using Gronwall's inequality, we obtain

$$(2.21) \quad \int_t^T |y(s) - y'(s)|^m ds \leq |\nu(0) - \nu'(0)|^m + C \int_0^{T-t} |\nu(s) - \nu'(s)|^m ds$$

for a constant C independent of ν, ν', x , and t .

Suppose an arbitrary control ν in \mathcal{V} is given. We define

$$(2.22) \quad \nu_n(t) = \begin{cases} (1 - nt)\nu(0) + n^2t \int_0^{1/n} \nu(s) ds & \text{if } 0 \leq t \leq 1/n, \\ n \int_{t-1/n}^t \nu(s) ds & \text{otherwise} \end{cases}$$

and

$$(2.23) \quad \nu_-(t) = \begin{cases} \nu(0) & \text{if } t = 0, \\ \lim_{s \uparrow t} \nu(s) & \text{otherwise.} \end{cases}$$

Since $\nu(\cdot)$ is a cad-lag process, $\nu_n(s)$ converges, for any fixed ω , to $\nu_-(s)$ for every s , as n approaches infinity. Moreover, except for a countable set in s , we have $\nu_-(s) = \nu(s)$. This fact and the estimate (2.21) imply

$$(2.24) \quad J_{xt}(\nu_n) \rightarrow J_{xt}(\nu) \quad \text{as } n \rightarrow \infty.$$

Hence

$$(2.25) \quad \hat{u}(x, t) = \inf \{J_{xt}(\nu) : \nu \text{ in } \mathcal{V}_0\},$$

because ν_n given by (2.22) belongs to \mathcal{V}_0 .

Now, suppose ν is an arbitrary Lipschitz control in \mathcal{V}_0 and define

$$(2.26) \quad \nu_n(s) = \nu\left(\frac{i}{n}\right) \quad \text{if } \frac{i}{n} \leq s < \frac{i+1}{n}, \quad i = 0, 1, \dots,$$

which is an impulse control in \mathcal{V}_* . Thus from (2.21) and (2.24) we deduce

$$(2.27) \quad \hat{u}(x, t) = \inf \{J_{xt}(\nu) : \nu \text{ in } \mathcal{V}_*\}.$$

To complete the proof, in view of Theorem 2.1, we only need to show that the optimal costs \hat{u}^ε and \hat{u}_ε , given respectively by (1.12) and (1.15) satisfy for every (x, t) in $\mathbb{R} \times [0, T]$

$$(2.28) \quad \hat{u}^\varepsilon(x, t) \rightarrow \hat{u}(x, t) \quad \text{as } \varepsilon \downarrow 0,$$

$$(2.29) \quad \hat{u}_\varepsilon(x, t) \rightarrow \hat{u}(x, t) \quad \text{as } \varepsilon \downarrow 0,$$

where \hat{u} is the optimal cost (1.4). The first convergence (2.28) is deduced from equalities (2.25) and

$$(2.30) \quad \mathcal{V}_0 = \bigcup \{\mathcal{V}_\varepsilon : \varepsilon > 0\}.$$

To prove the convergence (2.29), we use (2.27) and the fact that for every ν in \mathcal{V}_*

² The convergence is also uniform over every compact subset of $\mathbb{R} \times [0, T]$.

such that $J_{xt}^\varepsilon(\nu)$ is finite,

$$(2.31) \quad J_{xt}^\varepsilon(\nu) \rightarrow J_{xt}(\nu) \quad \text{as } \varepsilon \downarrow 0,$$

where the limit is decreasing. \square

Remark 2.2. The estimates of Theorem 2.1 allow us to obtain a locally uniform convergence of the first derivative in x of the optimal cost $\hat{u}^\varepsilon(x, t)$ defined by (1.12). Moreover, some weak convergence of the first and second derivatives of \hat{u}^ε and \hat{u}_ε holds.

Remark 2.3. A similar result to Theorem 2.2 can be found in Menaldi, Quadrat and Rofman [40], Menaldi and Rofman [46].

2.2. Characterization of the optimal cost. Denote by V_m the function space,

v belongs to V_m if $v: \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is such that

$$(2.32) \quad \begin{aligned} |v(x, t)| + \left| \frac{\partial v}{\partial t}(x, t) \right| &\leq C(1 + |x|^m), \\ \left| \frac{\partial v}{\partial x}(x, t) \right| &\leq C(1 + |x|^{m-1}), \end{aligned}$$

for almost every (x, t) and some constant C ,

and by L_{loc}^∞ the space of measurable real functions which are locally essentially bounded in $\mathbb{R} \times]0, T[$.

Consider the following partial differential equation:

$$(2.33) \quad \text{Find } \hat{u}^\varepsilon \text{ in } V_m \text{ such that } \partial^2 \hat{u}^\varepsilon / \partial x^2 \text{ belongs to } L_{loc}^\infty, \hat{u}^\varepsilon(x, T) = 0 \text{ for every } x \text{ in } \mathbb{R} \text{ and } A\hat{u}^\varepsilon + (1/\varepsilon)(B\hat{u}^\varepsilon)^+ = f, \text{ a.e. in } \mathbb{R} \times]0, T[,$$

where operators A and B are defined by (2.1) and (2.2).

THEOREM 2.3. *Assume the hypotheses (1.7), \dots , (1.10) hold. Then (2.33) has one and only one solution, which is given explicitly as the optimal cost (1.12). Moreover, the inequality (2.9) is valid and if*

$$(2.34) \quad x_\varepsilon^*(t) = \inf \left\{ x: \frac{\partial \hat{u}^\varepsilon}{\partial x}(x, t) + c(t) > 0 \right\},$$

we have for every (x, t) in $\mathbb{R} \times [0, T]$

$$(2.35) \quad \begin{aligned} A\hat{u}^\varepsilon = f \quad \text{and} \quad B\hat{u}^\varepsilon \leq 0 &\quad \text{if } x \geq x_\varepsilon^*(t), \\ A\hat{u}^\varepsilon + \frac{1}{\varepsilon} B\hat{u}^\varepsilon = f \quad \text{and} \quad B\hat{u}^\varepsilon \geq 0 &\quad \text{if } x \leq x_\varepsilon^*(t). \end{aligned}$$

Proof. First we suppose that $\sigma(t)$ is nondegenerate, i.e.

$$(2.36) \quad \sigma^2(t) \geq \mu > 0 \quad \text{for every } t \text{ in } [0, T].$$

Then standard techniques in partial differential equations prove that problem (2.33) has a smooth solution. Moreover, classical arguments of stochastic control (e.g. Bensoussan and Lions [8], Fleming and Rishel [20], Krylov [32]) permit us to identify the unique solution of (2.33) with the optimal cost (1.12). Also, the dynamic programming principle holds. In particular (2.9) is true.

To study the degenerate case, i.e., dropping (2.36), we regularize the differential operator (2.1),

$$(2.37) \quad A_\eta = A - \frac{1}{2} \eta \frac{\partial^2}{\partial x^2}, \quad \eta > 0.$$

Because the estimates (2.8) and (2.10) of Theorem 2.1 hold uniformly in η as η tends to zero, we can pass to the limit in η and obtain a solution of the problem (2.33). The uniqueness follows for instance, from the weak maximum principle for degenerate elliptic equations (e.g., Bony [11]). To show that the solution of (2.33) is the optimal cost (1.12), we observe that for every (x, t) in $\mathbb{R} \times [0, T]$, every control ν in \mathcal{V} and some constant $C > 0$,

$$(2.38) \quad E\{|y^\eta(s) - y(s)|^m\} \leq C\eta^{m/2}, \quad \eta \text{ positive,}$$

where $y^\eta(s)$ and $y(s)$ are the evolutions associated with $(\sigma^2 + \eta)^{1/2}$ and σ respectively in the state equation (1.2).

As a consequence of convexity, we have

$$B\hat{u}^\varepsilon(x, t) \leq B\hat{u}^\varepsilon(x', t) \quad \text{if } x \leq x'$$

for every fixed t in $[0, T]$. This implies the last conditions (2.35). \square

Remark 2.4. Using a convolution kernel it is possible to establish that for every function $u(x, t)$,

$$(2.39) \quad u \text{ in } V_m, Au = h \text{ in } \mathcal{D}'(\mathbb{R} \times [0, T]) \text{ with } h(x, t) \text{ continuous in } x \text{ and measurable in } t,$$

where $\mathcal{D}'(\mathbb{R} \times]0, T[)$ denotes the space of distributions on $\mathbb{R} \times]0, T[$, we can apply Itô's formula for every Lipschitz continuous control ν in \mathcal{V}_0 , i.e.

$$(2.40) \quad \begin{aligned} u(x, t) - E \left\{ u(y(T), T) \exp \left(- \int_t^T \alpha(s) ds \right) \right\} \\ = E \left\{ \int_t^T \left[h - \dot{\nu}(s) \frac{\partial u}{\partial x} \right] (y(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right\}, \end{aligned}$$

where $\dot{\nu}$ is the derivative of the Lipschitz control ν . \square

Now, consider the problem:

Find u_ε in V_m such that

$$(2.41) \quad \begin{aligned} u_\varepsilon(x, T) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}, \\ Au_\varepsilon &\leq f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[), \\ u_\varepsilon &\leq \varepsilon + Mu_\varepsilon \quad \text{in } \mathbb{R} \times [0, T], \end{aligned}$$

where M denotes the operator (2.6).

THEOREM 2.4. *Suppose the assumptions (1.7), \dots , (1.10) hold. Then the quasi-variational inequality (2.41) has a maximum solution \hat{u}_ε , which is given explicitly as the optimal cost (1.15). Moreover, the inequality (2.9) is valid.*

Proof. First, for fixed ψ in V_m and $\eta > 0$, consider the problem

Find u in V_m such that

$$(2.42) \quad \begin{aligned} u(x, T) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}, \\ Au + \frac{1}{\eta} (u - \psi)^+ &= f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[). \end{aligned}$$

It is clear that as in Theorem 2.3, we can show that the equation (2.42) has a unique

solution $u = u(x, t; \psi, \eta)$ which satisfies

$$(2.43) \quad u(\psi, \eta) = \inf \left\{ G(\delta) : \delta \text{ is adapted, } 0 \leq \delta \leq \frac{1}{\eta} \right\},$$

where

$$G(\delta) = E \left\{ \int_t^T [f(y^0(s), s) + \delta(s)\psi(y^0(s), s)] \exp \left[- \int_t^s (\alpha(\lambda) + \delta(\lambda)) d\lambda \right] ds \right\},$$

and $y^0(s)$ is given by (1.2) with $\nu = 0$.

Similar to Theorem 2.1 we can prove that $u(\psi, \eta)$ belongs to V_m uniformly as η tends to zero. Therefore,

$$(2.44) \quad u(\psi, \eta) \rightarrow u(\psi) \quad \text{as } \eta \rightarrow 0,$$

in a decreasing fashion and with a local uniformity in $\mathbb{R} \times [0, T]$. Moreover, the limit function $u = u(\psi)$ is the unique solution of the variational inequality:

Find u in V_m such that

$$(2.45) \quad \begin{aligned} u(x, T) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}, \\ Au &\leq f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[), \\ u &\leq \psi \quad \text{in } \mathbb{R} \times [0, T[, \\ Au &= f \quad \text{in } \mathcal{D}'([u < \psi]), \end{aligned}$$

where $[u < \psi]$ denotes the set of points satisfying $u(x, t) < \psi(x, t)$, and also

$$(2.46) \quad u(\psi) = \inf \{ F(\theta) : \theta \text{ is stopping time, } t \leq \theta \leq T \},$$

with

$$F(\theta) = E \left\{ \int_t^\theta f(y^0(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds + \psi(y^0(\theta), \theta) \exp \left(- \int_t^\theta \alpha(s) ds \right) I(\theta < T) \right\},$$

and $I(\theta < T)$ is the characteristic function of the set $[\theta < T]$. We remark that (2.42) is referred to as the penalized problem associated to the variational inequality (2.45). Also the control problem (2.46) is called an optimal stopping time problem (e.g. Bensoussan and Lions [8], Friedman [21], Kinderlehrer and Stampacchia [31]). Notice that the running cost f is unbounded and the operator A could be degenerate (cf. [37], [41], [42] and [52]).

Now, observe that

$$(2.47) \quad \psi \leq \varphi \quad \text{implies} \quad u(\varphi) \leq u(\psi).$$

We may define the decreasing sequence of function

$$(2.48) \quad u^n = u(\psi), \quad \psi = \varepsilon + Mu^{n-1}, \quad n = 1, 2, \dots,$$

where u^0 is the unique solution in V_m of the equation

$$(2.49) \quad \begin{aligned} Au^0 &= f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[), \\ u^0(x, T) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}. \end{aligned}$$

Standard techniques (e.g. Bensoussan [7], Bensoussan and Lions [9], and [38], or [52]) show that $u^n = \hat{u}_\varepsilon^n$,

$$(2.50) \quad \hat{u}_\varepsilon^n = \inf \{J_{xt}^\varepsilon(\nu) : \nu \text{ in } \mathcal{V}_*^n\},$$

where \mathcal{V}_*^n denotes the subset of impulse control \mathcal{V}_* given by (1.13) such that $\theta_j = +\infty$ for every $j \geq n$.

Thus, as in Theorem 2.1, we can prove that functions (2.50) remain in V_m uniformly as n approaches infinity. Hence the limit function

$$(2.51) \quad u_\varepsilon^* = \lim_n \hat{u}_\varepsilon^n$$

solves the quasi-variational inequality (2.41). It is clear that

$$(2.52) \quad u_\varepsilon^* \geq \hat{u}_\varepsilon,$$

with \hat{u}_ε denoting the optimal cost (1.15).

A crucial point is to deduce that

$$(2.53) \quad \hat{u}_\varepsilon \geq u_\varepsilon \quad \text{for every solution } u_\varepsilon \text{ of (2.41).}$$

Indeed, let ν be any impulse control, i.e.

$$\nu(s) = \sum_{j=1}^{\infty} \xi_j I(\theta_j \leq s),$$

which may satisfy

$$(2.54) \quad \int_0^{T-t} |\nu(s)|^m ds \leq C(1 + |x|^m),$$

for a suitable constant independent of (x, t) , and

$$(2.55) \quad \theta_j = T \quad \text{for every } j \geq N(\omega) \quad \text{some random index,}$$

without loss of generality. Since u_ε solves (2.41), we obtain

$$(2.56) \quad u_\varepsilon(x, t) \leq J_{xt}^\varepsilon(\nu_n) + E \left\{ u_\varepsilon(y(\theta_n), \theta_n) \exp \left(- \int_t^{t+\theta_n} \alpha(s) ds \right) \right\},$$

for

$$\nu_n(s) = \sum_{j=1}^n \xi_j I(\theta_j \leq s).$$

As n tends to infinity in (2.56) and by virtue of (2.54), (2.55), we get

$$u_\varepsilon(x, t) \leq J_{xt}^\varepsilon(\nu)$$

which implies (2.53). From this, the equality must hold in (2.52) and the optimal cost (1.15) is the maximum solution of (2.41). \square

Remark 2.5. Under the same assumptions of Theorem 2.4, we can prove that the optimal cost (1.15) is the unique solution of problem (2.41) together with the condition

$$(2.57) \quad A\hat{u}_\varepsilon = f \quad \text{in } \mathcal{D}'([\hat{u}_\varepsilon < \varepsilon + M\hat{u}_\varepsilon]),$$

where $[\hat{u}_\varepsilon < \varepsilon + M\hat{u}_\varepsilon]$ is the set of all points satisfying $\hat{u}_\varepsilon(x, t) < \varepsilon + M\hat{u}_\varepsilon(x, t)$. For a complete treatment of impulse control problems of nondegenerate diffusion processes with bounded running cost, we refer to the book of Bensoussan and Lions [9]. Similar problems are studied in [37], [38] and [52], [53], and some discrete approximations

are described in Bensoussan and Robin [10], Capuzzo–Dolcetta and Matzeu [14] and in general in Kushner [33].

Going back to the initial problem (1.4), consider the set of conditions:

Find u in V_m such that

$$(2.58) \quad \begin{aligned} u(x, T) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}, \\ Au &\leq f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[), \\ u &\leq Mu \quad \text{in } \mathbb{R} \times]0, T[. \end{aligned}$$

Notice that for every u in V_m ,

$$(2.59) \quad u \leq Mu \quad \text{in } \mathbb{R} \times]0, T[$$

is equivalent to

$$(2.60) \quad Bu \leq 0 \quad \text{a.e. in } \mathbb{R} \times]0, T[,$$

where the operators A , B and M are defined by (2.1), (2.2) and (2.6).

THEOREM 2.5. *Let the assumptions (1.7), \dots , (1.10) hold. Then problem (2.58) admits a maximum solution \hat{u} , which is given explicitly as the optimal cost (1.4) and satisfies (2.8) and (2.10). Moreover, defining*

$$(2.61) \quad x^*(t) = \inf \left\{ x : \frac{\partial \hat{u}}{\partial x}(x, t) + c(t) > 0 \right\},$$

we have for almost every (x, t) in $\mathbb{R} \times]0, T[$

$$(2.62) \quad \begin{aligned} A\hat{u} &= f \quad \text{and} \quad B\hat{u} \leq 0 \quad \text{if } x \geq x^*(t), \\ A\hat{u} &\leq f \quad \text{and} \quad B\hat{u} = 0 \quad \text{if } x \leq x^*(t). \end{aligned}$$

Proof. The first part is obtained from Theorem 2.4 by letting ε tend to zero. It is clear that we also apply Theorem 2.2 and Corollary 2.1. Note that \hat{u}^ε satisfies the dynamic programming principle (2.9).

In order to prove (2.62), we approximate the optimal cost (1.4) by the equation (2.33). Since the estimates (2.8) and (2.10) hold uniformly in $\varepsilon > 0$, for the solution \hat{u}^ε of (2.33), we deduce

$$(2.63) \quad \lim_{\varepsilon \downarrow 0} B\hat{u}^\varepsilon \leq 0,$$

which implies

$$(2.64) \quad B\hat{u} \leq 0 \quad \text{in } \mathbb{R} \times]0, T[.$$

Let (x, t) be a point in $\mathbb{R} \times]0, T[$ at which $A\hat{u}, A\hat{u}^\varepsilon, 0 < \varepsilon \leq 1$ exist and are such that $x > x^*(t)$. Since \hat{u} is convex, we have

$$B\hat{u} < 0 \quad \text{at } (x, t);$$

hence, for ε sufficiently small

$$B\hat{u}_\varepsilon < 0 \quad \text{at } (x, t),$$

and from the equation (2.33), we deduce

$$A\hat{u} = f \quad \text{at } (x, t).$$

This verifies (2.62) and the proof is completed. Note that the idea of this theorem can be traced back to [41]. \square

3. The free boundary. Define the differential operator

$$(3.1) \quad A'u = -\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2(t) \frac{\partial^2 u}{\partial x^2} - (a(t)x + b(t)) \frac{\partial u}{\partial x} + (\alpha(t) - a(t))u$$

and the substitutions

$$(3.2) \quad w = -\frac{\partial u}{\partial x} - c(t),$$

$$(3.3) \quad g = \frac{dc}{dt} - (\alpha(t) - a(t))c(t) - \frac{\partial f}{\partial x},$$

for the given functions u and f .

If u solves (2.4), then by taking formal derivative with respect to the variable x , we can deduce the equation

$$(3.4) \quad \begin{aligned} (A'w - g) \vee w &= 0 \quad \text{in } \mathbb{R} \times [0, T[, \\ w(\cdot, T) &= 0 \quad \text{in } \mathbb{R}, \end{aligned}$$

to be satisfied by the optimal cost \hat{u} , defined in (1.4), through the transformation (3.2). It is clear that (3.4) represents a classical variational inequality in the unknown w (e.g. Bensoussan and Lions [8], Friedman [21], Kinderlehrer and Stampacchia [31]). In this connection with optimal stopping, we refer to Bather and Chernoff [4], Karatzas [28] and more recently to Karatzas and Shreve [29], [30]. Moreover, the solution w of (3.4) has a stochastic representation as the optimal cost of a stopping time problem, i.e.

$$(3.5) \quad w(x, t) = \inf \{S_{x_t}(\theta) : t \leq \theta \leq T, \text{ stopping time}\},$$

where

$$(3.6) \quad S_{x_t}(\theta) = E \left\{ \int_t^\theta g(y^0(s), s) \exp \left(- \int_t^s (\alpha(\lambda) - a(\lambda)) d\lambda \right) ds \right\},$$

and the process $y^0(s) = y_{x_t}^0(s)$ is given by (1.2) with the control $\nu = 0$.

Then, with the function $w(x, t)$ we can define the moving boundary $x(t)$, $0 \leq t < T$, by

$$(3.7) \quad x(t) = \inf \{x : w(x, t) < 0\}.$$

As in Bather and Chernoff [4], Benes et al. [6], Karatzas [27], Menaldi and Robin [41], the reflected diffusion process on the half-space $[x \geq x(t)]$ will prove an optimal control for the original problem (1.4), (1.5). It is clear that the use of the variational inequality (3.4) will help us to obtain enough regularity of the free boundary (3.7) in order to be able to construct the reflected diffusion process.

First we consider the problem (3.4) and next the optimal control related to the free boundary (3.7).

3.1. Variational inequality. Consider the penalizing function

$$(3.8) \quad \beta(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq 0, \\ \lambda^2 & \text{if } 0 \leq \lambda \leq 1, \\ 2\lambda - 1 & \text{if } \lambda \geq 1, \end{cases}$$

and the set of controls $\tilde{\mathcal{V}}_\varepsilon$, $\varepsilon > 0$, defined by

(3.9) (η, ξ) belongs to $\tilde{\mathcal{V}}_\varepsilon$ if $\eta(t), \xi(t)$ are progressively measurable random processes from $[0, +\infty[$ into \mathbb{R} such that for every $t \geq 0$ and λ in \mathbb{R} ,

$$\lambda \eta(t) - \frac{1}{\varepsilon} \beta(\lambda) \leq \xi(t) \leq \frac{1}{\varepsilon}.$$

Note that if (η, ξ) belongs to $\tilde{\mathcal{V}}_\varepsilon$, then by looking at the graph of $\beta(\lambda)$, we deduce for every $t \geq 0$,

(3.10)
$$0 \leq \eta(t) \leq \frac{2}{\varepsilon}, \quad 0 \leq \xi(t) \leq \frac{1}{\varepsilon}.$$

In order to be able to derive the variational inequality (3.4), we introduce another penalized problem,

(3.11)
$$\hat{u}^\varepsilon(x, t) = \inf \{ \tilde{J}_{xt}(\eta, \xi) : (\eta, \xi) \text{ in } \tilde{\mathcal{V}}_\varepsilon \},$$

(3.12)
$$\tilde{J}_{xt}(\eta, \xi) = E \left\{ \int_t^T (f(y(s), s) + c(s)\eta(s) + \xi(s)) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right\},$$

with $s \geq t$,

(3.13)
$$y(s) = x + \int_t^s (a(\lambda)y(\lambda) + b(\lambda) + \eta(\lambda - t)) d\lambda + \int_0^{s-t} \sigma(\lambda + t) dw(\lambda),$$

i.e., the equation (1.2) for

(3.14)
$$\nu(s) = \int_0^s \eta(\lambda) d\lambda, \quad s \geq 0.$$

The Hamilton-Jacobi-Bellman equation associated with the above penalized problem is precisely the following:

Find \hat{u}^ε in V_m such that $\partial^2 \hat{u}^\varepsilon / \partial x^2$ belongs to L_{loc}^∞ ,

(3.15)
$$\hat{u}^\varepsilon(x, T) = 0 \quad \text{for every } x \text{ in } \mathbb{R},$$

$$A\hat{u}^\varepsilon + \frac{1}{\varepsilon} \beta(B\hat{u}^\varepsilon) = f, \quad \text{a.e. in } \mathbb{R} \times]0, T[,$$

where the operators A, B are given by (2.1), (2.2) and the spaces V_m, L_{loc}^∞ are defined in (2.32).

THEOREM 3.1. *Under the hypotheses (1.7), . . . , (1.10) the optimal cost \hat{u}^ε defined by (3.11) is a nonnegative continuous function such that for some constants $0 < c \leq C$, the same $m \geq 1$ for the assumption (1.9), and every $0 < \varepsilon \leq 1, (x, t), (x', t')$ in $\mathbb{R} \times [0, T]$ we have*

(3.16)
$$c|x^+|^m - C \leq \hat{u}^\varepsilon(x, t) \leq C(1 + |x|^m),$$

$$|\hat{u}^\varepsilon(x, t) - \hat{u}^\varepsilon(x', t)| \leq C(1 + |x|^{m-1} + |x'|^{m-1})|x - x'|,$$

$$|\hat{u}^\varepsilon(x, t) - \hat{u}^\varepsilon(x, t')| \leq C(1 + |x|^m)|t - t'|,$$

$$0 \leq \frac{\partial^2 \hat{u}^\varepsilon}{\partial x^2}(x, t) \leq C(1 + |x|^q), \quad q = (m - 2)^+,$$

so, \hat{u}^ε is convex in the first variable. Moreover, the partial differential equation (3.15)

has one and only one solution, which is precisely the function \hat{u}^ε . Furthermore, \hat{u}^ε converges to the optimal cost \hat{u} , given by (1.4), as ε approaches zero.

Proof. Use the same technique of Theorems 2.1, 2.2 and 2.3. \square

Now, we differentiate (3.15) with respect to the variable x and let ε tend to zero, to obtain the variational inequality (3.4). Due to the lack of an a priori estimate of the mixed derivative of \hat{u}^ε in x, t , we prefer to use a weak formulation of (3.4) in the sense of Mignot and Puel [48]. However, that estimate will be obtained later on by means of the interpretation (3.5).

Consider the weighted norm, $p > 2m + 1$,

$$(3.17) \quad \|v\|_p = \left(\int_{\mathbb{R}} |v(x)|^2 (1 + |x|^2)^{-p} dx \right)^{1/2}$$

and the Hilbert spaces

$$(3.18) \quad H \text{ is the set of all real measurable functions } v \text{ on } \mathbb{R} \text{ such that } \|v\|_p \text{ is finite,}$$

$$(3.19) \quad V \text{ is the set of all real measurable functions } v \text{ on } \mathbb{R} \text{ with a derivative } v' \text{ such that } \|v\|_p \text{ and } \|v'\|_{p-1} \text{ are finite.}$$

Identifying H and its dual, we denoted by $\langle \cdot, \cdot \rangle$ the pairing between V' , the dual, and V . The natural inner product in H is

$$(3.20) \quad (u, v) = \int_{\mathbb{R}} u(x)v(x)(1 + |x|^2)^{-p} dx,$$

with the corresponding norm $|\cdot| = \|\cdot\|_p$, for a fixed p . Define the bilinear form, for t in $[0, T]$,

$$(3.21) \quad a(t, u, v) = \int_{\mathbb{R}} \left[\sigma^2(t) \left(\frac{\partial u}{\partial x}(x) \right) \left(\frac{\partial v}{\partial x}(x) - 2px(1 + |x|^2)^{-1}v(x) \right) - (a(t)x + b(t)) \left(\frac{\partial u}{\partial x}(x) \right) v(x) + (\alpha(t) - a(t))u(x)v(x) \right] (1 + |x|^2)^{-p} dx,$$

which is continuous and strictly positive on V . Notice that for any smooth function $u(t) = u(t, x)$, we have

$$(3.22) \quad \left(-\frac{\partial u}{\partial t}(t), v \right) + a(t, u(t), v) = \langle A'u(t), v \rangle$$

for every v in V , with A' the differential operator (3.1).

Let $L^2(0, T; X)$ be the classical space of all square integrable functions on $]0, T[$ with values in a Hilbert space X . Introduce the problem:

Find w in $L^2(0, T; V)$, $w \leq 0$ such that

$$(3.23) \quad \int_0^T \left[\left\langle -\frac{\partial v}{\partial t}(t), v(t) - w(t) \right\rangle + a(t, w(t), v(t) - w(t)) \right] dt + \frac{1}{2} |v(T)|^2 \cong \int_0^T (g(t), v(t) - w(t)) dt,$$

for every v in $L^2(0, T; V)$, with $\partial v / \partial t$ in $L^2(0, T; V')$ and $v \leq 0$,

where the function $g(t) = g(t, x)$ is given by (3.3).

If \hat{u} is the optimal cost (1.4), define

$$(3.24) \quad \hat{w} = -\frac{\partial \hat{u}}{\partial x} - c(t).$$

THEOREM 3.2. *Let the assumptions (1.7), \dots , (1.10) hold. Suppose also that $\sigma(t)$ is nondegenerate, i.e. (2.36). Then the function \hat{w} given by (3.24) is the maximum solution of the weak variational inequality (3.23).*

Proof. Note that from Mignot and Puel [48], we know that the problem (3.23) admits a maximum solution \hat{w} . This weak solution is actually a strong solution, i.e., it is smooth in t and satisfies (3.4) in a pointwise (a.e.) sense. However, the point is to identify that solution with (3.24).

Denote by $\beta'(\lambda)$ the derivative of the function (3.8),

$$(3.25) \quad \hat{w}^\varepsilon = -\frac{\partial \hat{u}^\varepsilon}{\partial x} - c(t),$$

with \hat{u}^ε being the optimal cost (3.11). Since \hat{u}^ε solves (3.15) and $\sigma(t)$ is nondegenerate, we are able to differentiate the equation (3.15) to obtain

$$(3.26) \quad \begin{aligned} A'\hat{w}^\varepsilon - \frac{1}{\varepsilon} \beta'(\hat{w}^\varepsilon) \frac{\partial \hat{w}^\varepsilon}{\partial x} &= g \quad \text{in } \mathbb{R} \times [0, T], \\ \hat{w}^\varepsilon(T, x) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}. \end{aligned}$$

The facts that $\hat{u}^\varepsilon(t, x)$ is convex in x and $\beta(\lambda)$ increasing, $\beta'(0) = 0$, imply that

$$\beta'(\hat{w}^\varepsilon) \geq 0, \quad \hat{w}^\varepsilon \beta'(\hat{w}^\varepsilon) \geq 0, \quad \frac{\partial \hat{w}^\varepsilon}{\partial x} \geq 0 \quad \text{in } \mathbb{R} \times [0, T].$$

Thus, an integration by parts in (3.26) gives

$$(3.27) \quad \begin{aligned} \int_0^T \left[\left\langle -\frac{\partial v}{\partial t}(t), v(t) - \hat{w}^\varepsilon(t) \right\rangle + a(t, \hat{w}^\varepsilon(t), v(t) - \hat{w}^\varepsilon(t)) \right] dt + \frac{1}{2} |v(T)|^2 \\ \cong \int_0^T (g(t), v(t) - \hat{w}^\varepsilon(t)) dt \end{aligned}$$

for every v in $L^2(0, T; V)$, with $\partial v / \partial t$ in $L^2(0, T; V')$, and $v \leq 0$. Since the estimates (3.16) ensure that

$$\hat{w}^\varepsilon \rightarrow \hat{w} \text{ weakly in } L^2(0, T; V),$$

we have

$$\liminf_{\varepsilon \downarrow 0} \int_0^T a(t, \hat{w}^\varepsilon(t), \hat{w}^\varepsilon(t)) dt \cong \int_0^T a(t, \hat{w}(t), \hat{w}(t)) dt.$$

Therefore, by means of the following bound, for some constant $C > 0$,

$$\beta(\hat{w}^\varepsilon) \leq \varepsilon(f - A\hat{u}^\varepsilon) \leq \varepsilon C(1 + |x|^m), \quad \varepsilon > 0,$$

derived from Theorem 3.1, we take the limit in (3.27) as ε tends to zero in order to deduce that function \hat{w} , given by (3.24), is a solution of the weak variational inequality formulation (3.23).

Now, we prove that for any solution w of the problem (3.23)

$$(3.28) \quad w \leq \hat{w}^\varepsilon \quad \text{for every } \varepsilon > 0.$$

Indeed, if $z = w - \hat{w}^\epsilon$ from (3.23) and (3.26) we obtain

$$(3.29) \quad \int_0^T \left[\left\langle -\frac{\partial v}{\partial t}(t), v(t) - z(t) \right\rangle + a(t, z(t), v(t) - z(t)) \right] dt + \frac{1}{2} |v(T)|^2 \cong \int_0^T (q(t), v(t) - z(t)) dt,$$

for every v in $L^2(0, T; V)$, with $\partial v/\partial t$ in $L^2(0, T; V')$ and $v \leq \hat{w}^\epsilon$, where

$$(3.30) \quad q(x, t) = \frac{1}{\epsilon} \beta'(\hat{w}^\epsilon(x, t)) \frac{\partial \hat{w}^\epsilon}{\partial x}(x, t).$$

Thus, by taking $v = \hat{w}^\epsilon - \lambda \theta$, λ any arbitrary positive number, in (3.29), we may deduce

$$(3.31) \quad \int_0^T \left[\left\langle \frac{\partial \theta}{\partial t}(t), z(t) \right\rangle + a(t, z(t), \theta(t)) \right] dt \leq \int_0^T (q(t), \theta(t)) dt$$

for every $\theta = \theta(t)$ such that

$$(3.32) \quad \theta \text{ belongs to } L^2(0, T; V), \partial \theta/\partial t \text{ belongs to } L^2(0, T; V'), \theta(0) = 0 \text{ and } \theta \geq 0.$$

Therefore, introducing θ_η as the solution of

$$(3.33) \quad \eta \frac{\partial \theta_\eta}{\partial t} + \theta_\eta = z^+, \quad \theta_\eta(0) = 0,$$

we see that θ_η satisfies (3.32). Hence, from (3.31) with $\theta = \theta_\eta$, $\eta > 0$, we obtain

$$\int_0^T [a(t, z(t), \theta_\eta(t)) - (q(t), \theta_\eta(t))] dt \leq 0.$$

Since $\theta_\eta \rightarrow z^+$ in $L^2(0, T; V)$, we have

$$\int_0^T [a(t, z(t), z^+(t)) - (q(t), z^+(t))] dt \leq 0.$$

But, $z(t) \geq 0$ implies $\hat{w}^\epsilon \leq w$, and note $w \leq 0$. We have $q(t) = 0$. Thus

$$\int_0^T a(t, z(t), z^+(t)) dt \leq 0.$$

This means $z^+(t) = 0$, i.e. (3.28). This completes the proof. \square

Recall the function space V_{m-1} as in (2.32), i.e.

v belongs to V_{m-1} if $v: \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is locally Lipschitz continuous such that

$$(3.34) \quad \begin{aligned} |v(x, t)| + \left| \frac{\partial v}{\partial t}(x, t) \right| &\leq C(1 + |x|^{m-1}), \\ \left| \frac{\partial v}{\partial x}(x, v) \right| &\leq C(1 + |x|^q), \quad q = (m-2)^+, \end{aligned}$$

for every x, t and some constant C

and the space of distributions $\mathcal{D}'(\mathbb{R} \times]0, T[)$. Consider the variational inequality:

Find w in V_{m-1} such that

$$(3.35) \quad \begin{aligned} w(x, T) &= 0 \quad \text{for every } x \text{ in } \mathbb{R}, \\ A'w &\leq g \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[), \\ w &\leq 0 \quad \text{in } \mathbb{R} \times]0, T[. \end{aligned}$$

Suppose that

$$(3.36) \quad \left| \frac{\partial f}{\partial x}(x, t) - \frac{\partial f}{\partial x}(x, t') \right| \leq C(1 + |x|^{m-1})|t - t'|,$$

for every x, t and some constant C .

From this together with (1.9) we get $\partial f / \partial x$ belonging to V_{m-1} .

THEOREM 3.3. *Assume the hypotheses (1.7), \dots , (1.10) and (3.36) hold. Then the variational inequality (3.35) admits a maximum solution \hat{w} , which is given explicitly as the optimal stopping cost (3.5) and (3.24) is true. Moreover, we have*

$$(3.37) \quad A'\hat{w} = g \quad \text{in } \mathcal{D}'([\hat{w} < 0])$$

and \hat{w} is the unique solution of (3.35) and (3.37) simultaneously.

Proof. First, suppose that $\sigma(t)$ is nondegenerate, i.e. (2.36). Then, as was described in the proof of Theorem 2.4, by applying the classical results we deduce that the function \hat{w} , defined by (3.5), solves the variational inequality (3.35), (3.37).

On the other hand, by means of the assumption (3.36) we can show that

$$(3.38) \quad |\hat{w}(x, t) - \hat{w}(x, t')| \leq C(1 + |x|^{m-1})|t - t'|,$$

for every x, t and some constant C .

Therefore, if w is a solution of the weak variational inequality (3.23), we claim that $w = \hat{w}$, the unique solution of problem (3.35), (3.37), i.e. the optimal cost (3.5). Indeed, an integration by parts in (3.35), (3.37) yields

$$(3.39) \quad \begin{aligned} \int_0^T \left[\left(-\frac{\partial \hat{w}}{\partial t}(t), v(t) - \hat{w}(t) \right) + a(t, \hat{w}(t), v(t) - \hat{w}(t)) \right] dt \\ \cong \int_0^T (g(t), v(t) - \hat{w}(t)) dt, \end{aligned}$$

for every v in $L^2(0, T; V)$ such that $v \leq 0$,

after using the property (3.22). Hence, adding (3.39) with $v = w$, to (3.23) with $v = \hat{w}$, we get

$$- \int_0^T a(t, w(t) - \hat{w}(t), w(t) - \hat{w}(t)) \cong 0$$

which implies $w = \hat{w}$.

To study the degenerate case, i.e. to drop assumption (2.36), we regularize the problem by changing $\sigma(t)$ into

$$(3.40) \quad \sigma_\eta(t) = (\sigma^2(t) + \eta)^{1/2}, \quad \eta > 0.$$

Since the estimate (3.38) is uniform in η and the expression (3.5) is stable as η tends to zero, we can complete the proof in a similar way as in [37], [38]. \square

Remark 3.1. If $\hat{u}(x, t)$ is the optimal cost (1.4), then

$$(3.41) \quad \frac{\partial \hat{u}}{\partial x}(x, t) = \sup \{G_{xt}(\theta) : 0 \leq \theta \leq T, \text{ stopping time}\},$$

where

$$(3.42) \quad G_{xt}(\theta) = \left\{ \int_t^\theta \frac{\partial f}{\partial x}(y^0(s), s) \exp\left(-\int_t^s (\alpha(\lambda) - a(\lambda)) d\lambda\right) ds + c(\theta) \exp\left(-\int_t^\theta (\alpha(\lambda) - a(\lambda)) d\lambda\right) \right\},$$

and $y^0(s) = y_{xt}^0$ is the process (1.2) with $\nu = 0$.

Remark 3.2. The Theorem 3.2 holds without assuming the nondegeneracy condition (2.36). On the other hand, if we do not include regularity in t for the definition of the space V_{m-1} , i.e., $v(x, t)$ continuous in (x, t) but locally Lipschitz only in the variable x , then, the conclusion of Theorem 3.3 is true without the hypothesis (3.36) on f . Clearly, in that case, the optimal cost $\hat{w}(x, t)$ is continuous in (x, t) but locally Lipschitz only in the variable x .

3.2. Optimal decision. First we give an abstract result about the existence of an optimal policy.

THEOREM 3.4. *Under the assumptions (1.7), \dots , (1.10), and $m > 1$, there exists an optimal control \hat{v} in \mathcal{V} of the initial problem (1.4), (1.5).*

Proof. Let t be fixed in $[0, T[$ and consider the norm

$$(3.43) \quad \|\nu\|_m = \left(E \left\{ \int_0^{T-t} |\nu(s)|^m ds \right\} \right)^{1/m}.$$

Noting that for $0 \leq s < T - t$

$$(3.44) \quad (E\{|\nu(s)|^m\})^{1/m} \leq (T - t - s)^{-1/m} \|\nu\|_m, \quad 0 \leq s < T - t,$$

and the linear character of the state equation (1.2) we have:

If $(\nu_n, n = 0, 1, \dots)$ is a sequence in \mathcal{V} such that

$$(3.45) \quad \|\nu_n - \nu_0\|_m \rightarrow 0, \quad E\{|\nu_n(0) - \nu_0(0)|^m\} \rightarrow 0, \quad |E\{\nu_n(T-t) - \nu_0(T-t)\}| \rightarrow 0,$$

then $J_{xt}(\nu_n) \rightarrow J_{xt}(\nu_0)$ as $n \rightarrow \infty$,

and

$$(3.46) \quad \text{the mapping } \nu \rightarrow J_{xt}(\nu) \text{ is convex from } \mathcal{V} \text{ into } \mathbb{R},$$

where $J_{xt}(\nu)$ is the functional (1.3).

By means of the hypotheses (1.8), relative to $c(t)$, and (1.9) we deduce that

$$(3.47) \quad J_{xt}(\nu) \rightarrow +\infty \text{ as } \|\nu\|_m \rightarrow \infty \text{ and, unless } c(t) = 0 \text{ for every } t, \text{ also as } E\{|\nu(T-t)|\} \rightarrow \infty.$$

Thus, there is a sequence $(\nu'_n, n = 1, 2, \dots)$ in \mathcal{V} and a ν'_0 in $L^m(]0, T-t[\times \Omega)$, the space of m -integrable functions, such that as n goes to infinity

$$(3.48) \quad J_{xt}(\nu'_n) \rightarrow \hat{u}(x, t), \quad \nu'_n \rightarrow \nu'_0 \text{ weakly in } L^m, \text{ and } \|\nu'_n\|_m + |E\{\nu'_n(T-t)\}| \leq C, \text{ for some constant } C.$$

Hence, we can define $(\nu_n, n = 1, 2, \dots)$ in \mathcal{V} as a convex combination of $(\nu'_n, n = 1, 2, \dots)$,

$$(3.49) \quad \nu_n = \sum_{i=n}^{n+k} \alpha_i^n \nu'_i, \quad \alpha_i^n \text{ in } [0, 1] \text{ with } \sum_{i=n}^{n+k} \alpha_i^n = 1$$

and a nonnegative increasing function $q(s), 0 \leq s \leq T - t$ satisfying

$$(3.50) \quad \nu_n \rightarrow \nu'_0 \text{ strongly in } L^m, \text{ and } E\{\nu_n(s)\} \rightarrow q(s) \text{ for every } s \text{ in } [0, T - t].$$

Moreover, if N is a countable subset of $[0, T - t[$, a similar argument to the previous one and the inequality (3.44) allow us to show that $\nu_n(s)$ is strongly convergent in $L^m(\Omega)$ for every s in N ; in particular we may assume that

$$\nu_n(s) \rightarrow \nu'_0(s) \text{ strongly in } L^m(\Omega) \text{ and almost surely in } \Omega, \text{ for every rational in } [0, T - t[.$$

Clearly, $\nu'_0(\cdot)$ is nonnegative, increasing and progressively measurable. Define

$$(3.51) \quad \nu_0(s) = \inf \{ \nu'_0(s') : s' > t, s' \text{ rational} \}$$

which is right continuous having left-hand limits, adapted and $\nu_0 = \nu'_0$ in $L^m(]0, T[\times \Omega)$, $\nu_0(0) = \nu'_0(0)$. Hence, for an eventual subsequence if necessary, from (3.50) we have

$$(3.52) \quad \nu_n \rightarrow \nu_0 \text{ strongly in } L^m, \nu_n(0) \rightarrow \nu_0(0) \text{ strongly in } L^m(\Omega), \text{ and } E\{\nu_n(T - t)\} \rightarrow q(T - t), \text{ as } n \rightarrow \infty,$$

and if

$$(3.53) \quad \nu_0(T - t) = \sup \{ \nu_0(s) : 0 \leq s < T - t \}$$

then

$$E\{\nu_0(s)\} = q(s)$$

provided both functions are continuous at s in $[0, T - t]$. Since

$$E \left\{ \int_t^T c(s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\hat{\nu}(s - t) \right\} = \int_t^T c(s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) dq(s - t),$$

where

$$(3.54) \quad \hat{\nu}(s) = \begin{cases} \nu_0(s) & \text{if } 0 \leq s \leq T - t, \\ \nu_0(T - t) & \text{if } s \geq T - t, \end{cases}$$

we may deduce from (3.45) that $\hat{\nu}$ belongs to \mathcal{V} and

$$(3.55) \quad J_{xt}(\nu_n) \rightarrow J_{xt}(\hat{\nu}).$$

But, based on the convexity properties (3.46), (3.49), we have

$$J_{xt}(\nu_n) \leq \sum_{i=n}^{n+k} \alpha_i^n J_{xt}(\nu'_i)$$

and from (3.48), for every $\varepsilon > 0$,

$$J_{xt}(\nu'_i) \leq \hat{u}(x, t) + \varepsilon \text{ if } i \geq n(\varepsilon)$$

which implies

$$J_{xt}(\nu_i) \leq \hat{u}(x, t) + \varepsilon \text{ for every } i \geq n(\varepsilon).$$

Therefore, we obtain with (3.55)

$$J_{xt}(\hat{v}) = \hat{u}(x, t)$$

and the proof is completed. \square

Now we give a constructive approach of the optimal control through the free boundary (3.7).

Let $\hat{u}(x, t)$ be the optimal cost (1.4), for $0 \leq t < T$ define

$$(3.56) \quad x^*(t) = \inf \left\{ x: \frac{\partial \hat{u}}{\partial x}(x, t) + c(t) > 0 \right\}$$

and suppose

$$(3.57) \quad x^*(t) \text{ is finite and can be extended to a continuous function on } [0, T].$$

Some sufficient conditions to ensure (3.57) will be given later on. Note that in order to determine the free boundary (3.56) we need only to know the function $\hat{w}(x, t)$, which is the unique solution of the variational inequalities (3.35), (3.37).

THEOREM 3.5. *Let the hypotheses (1.7), \dots , (1.10) and (3.57) hold. Then there exists a control \hat{v} in \mathcal{V} whose associated state $y(s) = y_{xt}(s, \hat{v})$, defined by the stochastic equation (1.2), satisfies*

$$(3.58) \quad y(s) \geq x^*(s), \text{ for every } t \leq s \leq T, \int_t^T I(y(s) > x^*(s)) d\hat{v}(s-t) = 0, \\ I(\cdot) \text{ denotes the characteristic function, and } \hat{v}(0) = (x^*(t) - x)^+.$$

Moreover, the process \hat{v} is continuous, uniquely determined by the conditions (1.2), (3.58) and finally, the control \hat{v} is optimal, i.e., (1.5) is valid.

Proof. It is clear that $y(s)$ is the reflected diffusion on the continuation set $[y \geq x^*(s)]$ with initial value $x \vee x^*(t)$ at the time t . Since we assume $x^*(t)$, $0 \leq t \leq T$ to be only continuous, it is necessary to make precise the classical arguments about the existence of the reflected diffusion. Indeed, let $(x_\epsilon(s), 0 < \epsilon \leq 1)$ be a smooth approximation of $x^*(s)$, i.e. $x_\epsilon(s)$ has a continuous derivative $\dot{x}_\epsilon(s)$, $x_\epsilon(t) = x^*(t)$ and

$$(3.59) \quad x_\epsilon(s) \rightarrow x^*(s), \text{ uniformly in } [t, T] \text{ as } \epsilon \rightarrow 0.$$

We define the processes $(z_\epsilon(s), \eta_\epsilon(s), t \leq s \leq T)$, which are continuous and progressively measurable, as the unique solution of the stochastic equations, $t \leq s \leq T$

$$(3.60) \quad z_\epsilon(s) = (x - x^*(t))^+ + \int_t^s (a(\lambda)z_\epsilon(\lambda) + b(\lambda)) d\lambda + \int_t^s \sigma(\lambda) dw(\lambda - t) \\ + \int_t^s (a(\lambda)x_\epsilon(\lambda) + \dot{x}_\epsilon(\lambda)) d\lambda + \int_t^s I(z_\epsilon(\lambda) = 0) d\eta_\epsilon(\lambda),$$

$$\eta_\epsilon(0) = 0, \quad \eta_\epsilon(s) - \eta_\epsilon(\lambda) \geq 0 \text{ for every } T \geq s \geq \lambda \geq t,$$

$$z_\epsilon(s) \geq 0 \text{ for every } T \geq s \geq t, \text{ and } \int_t^T I(z_\epsilon(s) > 0) d\eta_\epsilon(s) = 0.$$

Thus, if $y_\varepsilon(s) = z_\varepsilon(s) + x_\varepsilon(s)$, $t \leq s \leq T$, we have

$$\begin{aligned}
 y_\varepsilon(s) &= x \vee x^*(t) + \int_t^s (a(\lambda)y_\varepsilon(\lambda) + b(\lambda)) d\lambda + \int_t^s \sigma(\lambda) dw(\lambda - t) \\
 &\quad + \int_t^s I(y_\varepsilon(\lambda) = x_\varepsilon(\lambda)) d\eta_\varepsilon(\lambda), \\
 (3.61) \quad y_\varepsilon(s) &\geq x_\varepsilon(s), \quad \text{for every } T \geq s \geq t, \\
 \int_t^T I(y_\varepsilon(s) > x_\varepsilon(s)) d\eta_\varepsilon(s) &= 0.
 \end{aligned}$$

Since

$$y_\varepsilon(s) - y_{\varepsilon'}(s) = \int_t^s a(\lambda)(y_\varepsilon(\lambda) - y_{\varepsilon'}(\lambda)) d\lambda + \eta_\varepsilon(s) - \eta_{\varepsilon'}(s),$$

an integration by parts yields

$$\begin{aligned}
 |y_\varepsilon(s) - y_{\varepsilon'}(s)|^2 &= 2 \int_t^s a(\lambda)|y_\varepsilon(\lambda) - y_{\varepsilon'}(\lambda)|^2 d\lambda \\
 &\quad + 2 \int_t^s (y_\varepsilon(\lambda) - y_{\varepsilon'}(\lambda)) d\eta_\varepsilon(\lambda) - 2 \int_t^s (y_\varepsilon(\lambda) - y_{\varepsilon'}(\lambda)) d\eta_{\varepsilon'}(\lambda).
 \end{aligned}$$

But the last two terms are equal to

$$\begin{aligned}
 &2 \int_t^s (x_\varepsilon(\lambda) - y_{\varepsilon'}(\lambda)) d\eta_\varepsilon(\lambda) + 2 \int_t^s (x_{\varepsilon'}(\lambda) - y_\varepsilon(\lambda)) d\eta_{\varepsilon'}(\lambda) \\
 &\cong 2 \int_t^s (x_\varepsilon(\lambda) - x_{\varepsilon'}(\lambda)) d\eta_\varepsilon(\lambda) + 2 \int_t^s (x_{\varepsilon'}(\lambda) - x_\varepsilon(\lambda)) d\eta_{\varepsilon'}(\lambda).
 \end{aligned}$$

Hence, by Gronwall's inequality we deduce

$$(3.62) \quad |y_\varepsilon(s) - y_{\varepsilon'}(s)|^2 \leq C(\eta_\varepsilon(T) + \eta_{\varepsilon'}(T)) \sup \{|x_\varepsilon(s) - x_{\varepsilon'}(s)| : t \leq s \leq T\},$$

for every $t \leq s \leq T$ and some deterministic constant C depending on T . Similarly, taking some $q \geq 1 + x_\varepsilon(s)$, for every $t \leq s \leq T$, $0 < \varepsilon \leq 1$, we obtain

$$(3.63) \quad |y_\varepsilon(s) - q|^2 + \eta_\varepsilon(s) \leq \exp\left(2 \int_t^T |a(\lambda)| d\lambda\right),$$

for every s in $[t, T]$. Now, letting ε go to zero and using the estimates (3.62), (3.63), we get two continuous and progressively measurable $(y(s), \eta(s), t < s < T)$ such that

$$\begin{aligned}
 y(s) &= x \vee x^*(t) + \int_t^s (a(\lambda)y(\lambda) + b(\lambda)) d\lambda + \int_t^s \sigma(\lambda) dw(\lambda - t) \\
 &\quad + \int_t^s I(y(\lambda) = x^*(\lambda)) d\eta(\lambda), \\
 (3.64) \quad \eta(0) &= 0, \quad \eta(s) - \eta(\lambda) \geq 0, \quad y(s) \geq x^*(s)
 \end{aligned}$$

for every $T \geq s \geq \lambda \geq t$, and $\int_t^T I(y(s) > x^*(s)) d\eta(s) = 0$.

So the process $\hat{\nu}$ is defined by

$$(3.65) \quad \hat{\nu}(s) = \begin{cases} (x^*(t) - x)^+ + \eta(t+s) & \text{if } 0 \leq s \leq T-t, \\ (x^*(t) - x)^+ + \eta(T) & \text{if } s \geq T-t. \end{cases}$$

It remains to prove that $\hat{\nu}$ is optimal. Indeed, let us assume that there is no degeneracy, i.e. (2.36); then the optimal cost (1.4) is smooth enough to apply Itô's formula for a semimartingale (cf. Meyer [47]) in order to get, for every ν in \mathcal{V}

$$(3.66) \quad \begin{aligned} & E \left\{ \hat{u}(t, x + \nu(0)) - \hat{u}(T, y(T)) \exp \left(- \int_t^T \alpha(s) ds \right) \right\} \\ &= E \left\{ \int_t^T A\hat{u}(s, y(s)) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right. \\ &\quad \left. - \int_t^T \frac{\partial \hat{u}}{\partial x}(s, y(s)) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\nu(s-t) \right. \\ &\quad \left. - \sum_{t < s \leq T} [\hat{u}(s, y(s)) - \hat{u}(s, y(s-))] \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) \right\}, \end{aligned}$$

where $y(s-)$ denotes the limit from the left at s . Since $\hat{u}(T, \cdot) = 0$ in \mathbb{R} , $A\hat{u} \leq f$, $-\partial\hat{u}/\partial x \leq c(\cdot)$ in $\mathbb{R} \times [0, T[$ and

$$-[\hat{u}(s, y(s)) - \hat{u}(s, y(s-))] \leq c(s)[y(s) - y(s-)] = c(s)[\nu(s) - \nu(s-)]$$

we deduce

$$(3.67) \quad \hat{u}(t, x) \leq J_{xt}(\nu), \quad \text{for every } \nu \text{ in } \mathcal{V}.$$

Similarly, choosing $\hat{\nu}$ given by (3.65), we obtain from (3.66)

$$(3.68) \quad \hat{u}(t, x) = J_{xt}(\hat{\nu}),$$

after using the fact that

$$\begin{aligned} A\hat{u}(s, y) &= f(s, y) \quad \text{if } y \geq x^*(s), \quad T \geq s \geq 0, \\ \frac{\partial \hat{u}}{\partial x}(s, y) &= -c(s) \quad \text{if } y \leq x^*(s), \quad T \geq s \geq 0. \end{aligned}$$

Until now, we have established the optimality of control $\hat{\nu}$ under the assumption (2.36). In order to remove the nondegeneracy (2.36), let us consider the function $(\hat{u}_\varepsilon, 0 < \varepsilon \leq 1)$ given as the optimal cost (1.4) with a covariance

$$\sigma_\varepsilon(t) = (\sigma^2(t) + \varepsilon)^{1/2}$$

instead of $\sigma(t)$. We have, as ε tends to zero

$$(3.69) \quad \hat{u}_\varepsilon \rightarrow \hat{u}, \partial\hat{u}_\varepsilon/\partial x \rightarrow \partial\hat{u}/\partial x \text{ locally uniform in } \mathbb{R} \times [0, T], \text{ and } \partial^2\hat{u}_\varepsilon/\partial x^2 \text{ locally bounded in } \mathbb{R} \times [0, T].$$

Since (3.67) holds for $\hat{u}_\varepsilon, 0 < \varepsilon \leq 1$, we obtain the same inequality as the limit when ε goes to zero. Now, the Itô's formula (3.66) for the control $q + \hat{\nu}, q > 0$, yields

$$\begin{aligned} \hat{u}_\varepsilon(t, x + \hat{\nu}(0) + q) &= E \left\{ \int_t^T A_\varepsilon \hat{u}_\varepsilon(s, y(s) + q) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right. \\ &\quad \left. - \int_t^T \frac{\partial \hat{u}_\varepsilon}{\partial x}(s, y(s) + q) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\hat{\nu}(s-t) \right\}. \end{aligned}$$

Because of $y(s) \geq x^*(s)$ and

$$0 \leq f(s, y + q) - A_\varepsilon \hat{u}_\varepsilon(s, y + q) \leq C(1 + |y|^m) I \left(\frac{\partial \hat{u}_\varepsilon}{\partial x}(s, x^*(s) + q) = 0 \right),$$

where C is a constant, $I(\cdot)$ denotes the characteristic function, we deduce, by means of (3.69) as ε tends to zero

$$\begin{aligned} \hat{u}(t, x + \hat{v}(0) + q) = E \left\{ \int_t^T f(s, y(s) + q) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds \right. \\ \left. - \int_t^T \frac{\partial \hat{u}}{\partial x}(s, x^*(s) + q) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\hat{v}(s - t) \right\}. \end{aligned}$$

Thus the equality (3.68) follows when q becomes zero. Therefore, the proof is completed. \square

Remark 3.3. A way to approximate the solution $(y(s), \eta(s), t \leq s \leq T)$ of (3.64) is by solving the Itô's equation

$$\begin{aligned} (3.70) \quad dy^\varepsilon(s) &= (a(s)y^\varepsilon(s) + b(s)) ds + \sigma(s) dw(s - t) \\ &+ \frac{1}{\varepsilon} (x^*(s) - y^\varepsilon(s))^+ ds, \quad T \geq s \geq t, \\ y^\varepsilon(t) &= x \vee x^*(t). \end{aligned}$$

Similar to [39], it can be proved that for every $1 \leq p < \infty$,

$$(3.71) \quad \begin{aligned} E(\sup \{|y^\varepsilon(s) - y(s)|^p : t \leq s \leq T\}) &\rightarrow 0, \\ E \left(\sup \left\{ \left| \eta(s) - \frac{1}{\varepsilon} \int_t^s (x^*(\lambda) - y^\varepsilon(\lambda))^+ d\lambda \right|^p : t \leq s \leq T \right\} \right) &\rightarrow 0 \end{aligned}$$

as ε tends to zero. This provides an approximation of the optimal control \hat{v} . \square

Remark 3.4. Considering the solution $\hat{u}(t, x)$ of (2.62), i.e., the optimal cost (1.4), for $x^*(t) \pm \varepsilon$ and letting ε go to zero, we obtain

$$(3.72) \quad \begin{aligned} A\hat{u}(t, x^*(t) +) &= f(t, x^*(t)) \\ &\geq - \frac{\partial \hat{u}}{\partial t}(t, x^*(t) -) + (a(t)x^*(t) + b(t))c(t) + \alpha(t)\hat{u}(t, x^*(t)) \end{aligned}$$

which implies

$$(3.73) \quad \frac{\partial \hat{u}}{\partial t}(t, x^*(t) +) - \frac{\partial \hat{u}}{\partial t}(t, x^*(t) -) \leq -\sigma^2(t) \frac{\partial^2 \hat{u}}{\partial x^2}(t, x^*(t) +) \leq 0.$$

So, the first derivative of $\hat{u}(t, x)$ with respect to t has a nonnegative jump at $x = x^*(t)$ and if that jump vanishes and $\sigma(t) \neq 0$, then the second derivative of \hat{u} with respect to x is continuous throughout the free boundary $x^*(t)$. The last observation can be deduced also from the classical regularity on the function \hat{w} , a solution of the variational inequality (3.35), (3.37). \square

Remark 3.5. Even under degeneracy, it can be proved (cf. [37]) that $A\hat{w}$ is locally bounded, more precisely as in Lewy and Stampacchia [34] we have

$$(3.74) \quad -g^- \leq A\hat{w} \leq g, \quad \text{a.e. in } \mathbb{R} \times]0, T[$$

where \hat{w} is the solution of the variational inequality (3.35), (3.37), i.e., \hat{w} is given by

either (3.5) or (3.24) and g by (3.3). This implies, using the standard regularity results for parabolic partial differential equations,

$$(3.75) \quad \partial \hat{w} / \partial t, \partial^2 \hat{w} / \partial x^2 \text{ are essentially locally bounded in } (x, t) \text{ belonging to } \mathbb{R} \times [0, T] \text{ such that } \sigma(t) \neq 0$$

and also

$$(3.76) \quad \partial \hat{w} / \partial t \text{ is essentially locally bounded in } x \text{ belonging to } \mathbb{R} \text{ for almost every } t \text{ such that } \sigma(t) \neq 0.$$

Clearly, from (3.24), (3.75) and (3.76) we deduce that

$$(3.77) \quad \text{for almost every } t \text{ in } [0, T] \text{ the function } \partial \hat{u} / \partial t \text{ is continuous in the variable } x \text{ belonging to } \mathbb{R}.$$

Note that (3.77) holds under the assumptions (1.7), \dots , (1.10), and that (3.73) is actually an equality. \square

Remark 3.6. Going through the proof of Theorem 3.5 we notice that the continuity of the free boundary $x^*(t)$, given by (3.56), at the end point $t = T$ is not really used. It suffices to suppose

$$(3.78) \quad x^*(t) \text{ is continuous and bounded from above on } [0, T[$$

in lieu of (3.57). \square

Remark 3.7. Define the function $(q(t), 0 \leq t \leq T)$ by

$$(3.79) \quad q(t) = \sup \left\{ x: \frac{\partial f}{\partial x}(x, t) \leq \frac{dc}{dt}(t) - (\alpha(t) - a(t))c(t) \right\}$$

which is bounded in view of the hypotheses (1.8) and (1.9) if $m > 1$. The function (3.79) will provide an upper bound for the free boundary (3.56), more precisely

$$(3.80) \quad \text{if } x^*(t) \text{ is continuous on } [0, T[\text{ then } x^*(t) \leq q(t) \text{ for every } t \text{ in } [0, T[.$$

Indeed, fix (x, t) in $\mathbb{R} \times]0, T[$ such that $x < x^*(t)$. By continuity, there is $\delta > 0$ such that $x' < x^*(t')$ for every $|t' - t| < \delta, |x' - x| < \delta$. Since $\hat{w}(x', t') = 0$, by definition of the free boundary, we get $A\hat{w} = 0$ at (x', t') . This fact and (3.35) yield $g(x', t') \geq 0$. Hence, as x' approaches $x^*(t')$ we deduce $g(x^*(t'), t') \leq 0$ for every $|t' - t| < \delta$. Clearly, this implies (3.80). \square

4. Finite resources. In this section we study the case of a monotone follower problem with a constraint on the resources (1.6).

Let A be the differential operator (2.1) and define

$$(4.1) \quad B'v = \frac{\partial v}{\partial z} - \frac{\partial v}{\partial x} - c(t)$$

for a function $v(x, z, t)$, (x, z) in $\mathbb{R} \times [0, \infty[$, $0 \leq t \leq T$. A heuristic application of the dynamic programming to the problem (1.16), \dots , (1.19) yields the following Hamilton-Jacobi-Bellman equation

$$(4.2) \quad \begin{aligned} (Av - f) \vee B'v &= 0 \quad \text{in } \mathbb{R} \times]0, \infty[\times [0, T[, \\ v(\cdot, \cdot, T) &= 0 \quad \text{in } \mathbb{R} \times [0, \infty[, \\ Av &= f \quad \text{in } \mathbb{R} \times \{0\} \times [0, T[\end{aligned}$$

to be satisfied by the optimal cost \hat{v} given by (1.6).

First of all, we need some a priori estimates.

THEOREM 4.1. *Assume (1.7), \dots , (1.10) hold. Then the optimal cost \hat{v} defined by (1.6) is a nonnegative continuous function such that for some constants $0 < c \leq C$, the same $m \geq 1$ of the hypothesis (1.9), and every $(x, z, t), (x', z', t')$ in $\mathbb{R} \times [0, \infty[\times [0, T]$ we have*

$$\begin{aligned}
 & c|x^+|^m - C \leq \hat{v}(x, z, t) \leq C(1 + |x|^m), \\
 (4.3) \quad & |\hat{v}(x, z, t) - \hat{v}(x', z, t)| \leq C(1 + |x|^{m-1} + |x'|^{m-1})|x - x'|, \\
 & |\hat{v}(x, z, t) - \hat{v}(x, z, t')| \leq C(1 + |x|^m)|t - t'|, \\
 & 0 \leq \frac{\partial^2 \hat{v}}{\partial x^2}(x, z, t) \leq C(1 + |x|^q), \quad q = (m - 2)^+
 \end{aligned}$$

and

$$(4.4) \quad \hat{v}(x, z, t) - \hat{v}(x, z', t) \leq C(1 + |x|^{m-1})(z' - z)^+,$$

so, \hat{v} is convex in the first variable and decreasing in the second variable.³

Proof. The estimate (4.3) is obtained by an analogy to Theorem 2.1. Let us prove (4.4). Indeed, notice that

$$(4.5) \quad \hat{v}(x, z, t) - \hat{v}(x, z', t) \leq \sup \{ (J_{xt}(\nu) - J_{xt}(\nu')) : \nu' \text{ in } \mathcal{V} \text{ satisfying (2.9) and } \nu'(T - t) \leq z' \},$$

where ν is chosen as any measurable function of ν' , with $\nu(T - t) \leq z$. In particular, we take

$$\nu(s) = \begin{cases} \nu'(s) & \text{if } \nu'(s) \leq z, \\ z & \text{if } \nu'(s) \geq z. \end{cases}$$

Hence, using the fact that for $y(s), y'(s)$ denoting the processes associated with ν, ν' , respectively,

$$E\{|y(s) - y'(s)|^m\} \leq C|(z' - z)^+|^m \quad \text{for every } s \text{ in } [t, T],$$

for some constant C independent of x, t, z, z', ν and ν' , we deduce, by virtue of (1.8), (1.9) and Hölder's inequality,

$$(4.6) \quad J_{xt}(\nu) - J_{xt}(\nu') \leq CE \left\{ \int_t^T (1 + |y(s)|^m + |y'(s)|^m) ds \right\} |(z' - z)^+|^m,$$

for another constant C . Finally, since (2.11) is equivalent to

$$(4.7) \quad E \left\{ \int_0^{T-t} |\nu(s)|^m ds \right\} \leq C(1 + |x|^m),$$

for an appropriate constant C , the expressions (4.5) and (4.6) imply (4.4). \square

Denote by V_m the function space,

v belongs to V_m if $v : \mathbb{R} \times [0, \infty[\times [0, T] \rightarrow \mathbb{R}$ is locally Lipschitz continuous such that

$$\begin{aligned}
 (4.8) \quad & |v(x, z, t)| + \left| \frac{\partial v}{\partial t}(x, z, t) \right| \leq C(1 + |x|^m), \\
 & \left| \frac{\partial v}{\partial x}(x, z, t) \right| + \left| \frac{\partial v}{\partial z}(x, z, t) \right| \leq C(1 + |x|^{m-1})
 \end{aligned}$$

for almost every (x, z, t) and some constant C .

³ Note that \hat{v} satisfies the dynamic programming equation.

Note the change of notation with respect to the definition (2.34) in § 2.

Observe that function u^0 , given by (1.18), is also the unique solution of the equation

$$(4.9) \quad \begin{aligned} Au^0 &= f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, T[), \\ u^0(\cdot, T) &= 0 \quad \text{in } \mathbb{R}, \end{aligned}$$

under the regularity (2.34).

Consider the problem:

Find v in V_m such that

$$(4.10) \quad \begin{aligned} v(\cdot, \cdot, T) &= 0 \quad \text{in } \mathbb{R} \times]0, \infty[, \\ v(\cdot, 0, \cdot) &= u^0 \quad \text{in } \mathbb{R} \times]0, T[, \\ Av &\leq f \quad \text{in } \mathcal{D}'(\mathbb{R} \times]0, \infty[\times]0, T[), \\ B'v &\leq 0 \text{ a.e. in } \mathbb{R} \times]0, \infty[\times]0, T[. \end{aligned}$$

Notice that for every v in V_m ,

$$(4.11) \quad B'v \leq 0 \text{ a.e. in } \mathbb{R} \times]0, \infty[\times]0, T[$$

is equivalent to

$$(4.12) \quad v \leq M'v \quad \text{in } \mathbb{R} \times]0, \infty[\times]0, T[,$$

where the operator

$$(4.13) \quad Mv = \inf \{ \xi c(t) + v(x + \xi, z - \xi, t) : 0 \leq \xi \leq z \}.$$

THEOREM 4.2. *Under the assumptions (1.7), \dots , (1.10) the problem (4.10) possesses a maximum solution \hat{v} , which is given explicitly as the optimal cost (1.6). Moreover, we have the following decomposition:*

$$(4.14) \quad \hat{v}(x, z, t) = \hat{u}(x, t) + h(x + z, t) \quad \text{for every } (x, z, t) \text{ in } \mathbb{R} \times]0, \infty[\times]0, T[,$$

where \hat{u} is the unlimited optimal cost (1.4) and

$$(4.15) \quad h = u^0 - \hat{u} \quad \text{in } \mathbb{R} \times]0, T[,$$

with u^0 being defined by (4.9).

Proof. First of all, we remark the dynamic programming equation applies to both optimal control problems (1.4), (1.6), i.e. if

$$(4.16) \quad \begin{aligned} z(s) &= z - \nu(s - t) \quad \text{for every } t \leq s \leq T, \\ J_{xt}(v, \theta) &= E \left\{ \int_t^\theta f(y(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds + c(t)\nu(0)I(t < \theta) \right. \\ &\quad \left. + \int_t^\theta c(s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d\nu(s - t) \right\} \end{aligned}$$

and

$$(4.18) \quad \tau = \inf \{ s \in [t, T] : z(s) < 0 \}$$

then

$$(4.19) \quad \hat{u}(x, t) = \inf \left\{ J_{xt}(v, \theta) + E \left[\exp \left(- \int_t^\theta \alpha(s) ds \right) \hat{u}(y(\theta -), \theta) \right] : \mathcal{A} \right\},$$

$$\begin{aligned}
 \hat{v}(x, z, t) = \inf & \left\{ J_{xt}(\nu, \theta \wedge \tau) \right. \\
 (4.20) \quad & \left. + E \left[\exp \left(- \int_t^{\theta \wedge \tau} \alpha(s) ds \right) \hat{v}(y(\theta \wedge \tau -), z(\theta \wedge \tau -), \theta \wedge \tau) \right] : \mathcal{A} \right\},
 \end{aligned}$$

where $t \leq \theta \leq T$ is any stopping time associated with the system control \mathcal{A} , which includes the probability space (Ω, \mathcal{F}, P) , the filtration, the Wiener process and the control $(\mathcal{T}', w(t), \nu(t), t \geq 0)$.

Next, by virtue of the estimates (4.3), (4.4) we can prove as in § 2 that the optimal cost (1.6) is the maximum solution of the problem (4.10).

Finally, let us prove (4.14). Indeed, using either (1.19) or (4.20) with $\theta = \tau$, and the fact that

$$\hat{v}(\cdot, 0, \cdot) = \hat{u} + h,$$

we obtain

$$\begin{aligned}
 \hat{v}(x, z, t) = \inf & \left\{ J_{xt}(\nu, \tau) + E \left[\exp \left(- \int_t^\tau \alpha(s) ds \right) \hat{u}(y(\tau -), \tau) \right] \right. \\
 (4.21) \quad & \left. + E \left[\exp \left(- \int_t^\tau \alpha(s) ds \right) h(y(\tau -), \tau) \right] : \mathcal{A} \right\}.
 \end{aligned}$$

Since we may assume that $\nu(\cdot)$ is continuous and because of

$$y(\tau -) = x + z + \int_t^\tau (a(s)y(s) + b(s)) ds + \int_t^\tau \sigma(s) dw(s)$$

and

$$Ah = f - A\hat{u} \geq 0$$

we get, by applying Itô's formula

$$(4.22) \quad E \left\{ \exp \left(- \int_t^\tau \alpha(s) ds \right) h(y(\tau -), \tau) \right\} \leq h(x + z, t).$$

Clearly, combining (4.19), (4.21) and (4.22), we deduce

$$(4.23) \quad \hat{v}(x, t) \leq \hat{u}(x, t) + h(x + z, t).$$

On the other hand, denoting by $v(x, z, t)$ the right-hand side of (4.23), we have

$$(4.24) \quad Av(x, z, t) = A\hat{u}(x, t) + f(x + z, t) - A\hat{u}(x + z, t).$$

Denoting by $x^*(t)$ the free boundary (3.56), the equality (4.24) yields

$$Av(x, z, t) \leq A\hat{u}(x, t) \leq f(x, t) \quad \text{if } x + z \geq x^*(t).$$

Because

$$A\hat{u}(x + z, t) - A\hat{u}(x, t) = \int_0^z A' \frac{\partial \hat{u}}{\partial x}(x + \lambda, t) d\lambda,$$

where A' is the operator (3.1), so from (4.24) we obtain

$$Av(x, z, t) = f(x, t) - \int_0^z g(x + \lambda, t) d\lambda \leq f(x, t) \quad \text{if } x + z < x^*(t),$$

in view of Remark 3.7 and the definition (3.3). Hence

$$(4.25) \quad Av \leq f \text{ in } \mathbb{R} \times]0, \infty[\times]0, T[$$

and also

$$(4.26) \quad B'v = B\hat{u} \leq 0 \text{ in } \mathbb{R} \times]0, \infty[\times]0, T[.$$

This implies that v solves the problem (4.10) and since \hat{v} is the maximum solution, the equality must hold in (4.23). \square

COROLLARY 4.1. *If the conditions (1.7), \dots , (1.10) and (3.78) hold, then the control $\hat{v} \wedge z$ is optimal for the problem with the resource constraints (1.6), where \hat{v} is the process defined in Theorem 3.5.*

Proof. The result is straightforward and follows from the decomposition (4.14), the technique of Theorem 3.5 and Remark 3.6. \square

Remark 4.1. An equivalence to Theorem 3.4 can be stated for the problem with the resource constraints (1.6). Moreover, the fact that $f(t, x)$ approaches infinity as x goes to positive infinity is useless in the proof for existence of an optimal control relative to problem (1.6). \square

Remark 4.2. From the expressions (1.4) and (1.6), it follows that

$$(4.27) \quad \hat{v}(x, z, t) \rightarrow \hat{u}(x, t) \text{ as } z \rightarrow +\infty$$

in a decreasing fashion and pointwise in $\mathbb{R} \times]0, T[$. Hence, the equalities (4.14) and (4.15) imply, for every t in $]0, T[$,

$$(4.28) \quad u^0(x, t) - \hat{u}(x, t) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

in a decreasing fashion. This means that for a large initial state x , the optimal cost (1.4) is very close to the cost of the free-control evolution. Clearly, this agrees with the characteristics of the optimal control of Theorem 3.5. \square

5. Optimal corrections. Now, we consider a model of an optimal correction control problem which will be reduced to a problem of the type presented in § 1.

Denote by \mathcal{V} the set of controls $\nu(\cdot)$ which are progressively measurable random processes from $[0, +\infty[$ into \mathbb{R} , right continuous having left limit (cad-lag) and with locally bounded variation. Hence if \mathcal{V}_+ is the set of processes in \mathcal{V} which are nonnegative and increasing, we have the following decomposition

$$(5.1) \quad \mathcal{V} = \mathcal{V}_+ \ominus \mathcal{V}_+,$$

i.e., for every $\nu(\cdot)$ in \mathcal{V} there exist $\nu_1(\cdot), \nu_2(\cdot)$ in \mathcal{V}_+ such that

$$(5.2) \quad \begin{aligned} \nu(t) &= \nu_1(t) - \nu_2(t), & t \geq 0, \\ \nu_1(0) &= (\nu(0))^+, & \nu_2(0) = (\nu(0))^- . \end{aligned}$$

Note the change of notations used in § 1.

The state of the dynamic system is described by (1.2), i.e.,

$$(5.3) \quad \begin{aligned} y(s) &= x + \nu(s-t) + \int_t^s (a(\lambda)y(\lambda) + b(\lambda)) d\lambda \\ &+ \int_t^s \sigma(\lambda) dw(\lambda-t), & s \geq t, \end{aligned}$$

$y(s) = y_{xt}(s, \nu)$ being a cad-lag random process adapted to $(\mathcal{F}^{s-t}, s \geq t)$. A cost

associated to each control ν in \mathcal{V} is given by the payoff functional (1.3), i.e.

$$(5.4) \quad J_{xt}(\nu) = E \left\{ \int_t^T f(y(s), s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) ds + c(t)|\nu(0)| \right. \\ \left. + \int_t^T c(s) \exp \left(- \int_t^s \alpha(\lambda) d\lambda \right) d|\nu|(s-t) \right\},$$

where $a(t)$, $b(t)$, $\sigma(t)$, $c(t)$, $\alpha(t)$, $f(x, t)$ and T satisfy (1.7), (1.8), (1.9), (1.10), and $|\nu|$ denotes the total variation of ν , i.e., $|\nu| = \nu_1 + \nu_2$ given by (5.2). Notice that a better notation could be $J_{xt}(\nu_1, \nu_2)$ in lieu of $J_{xt}(\nu)$, because ν_1, ν_2 are not uniquely determined by ν . However, we prefer to use (5.4).

Our purpose is to characterize the optimal cost

$$(5.5) \quad \hat{u}(x, t) = \inf \{ J_{xt}(\nu) : \nu \text{ in } \mathcal{V} \}$$

and to construct an optimal control $\hat{\nu}$ in \mathcal{V} .

In the first part of this section we treat the problem just stated and then offer some general comments about other extensions of these results.

5.1. Reduction. Let us suppose that $f(x, t)$ is symmetric in the following sense.

$$(5.6) \quad f(x, t) = f(2x_0(t) - x, t), \quad (x, t) \text{ in } \mathbb{R} \times [0, T] \text{ with } x_0(t) \text{ being Lipschitz} \\ \text{continuous in } [0, T] \text{ and satisfying } \dot{x}_0(t) = a(t)x_0(t) + b(t), \quad t \text{ in } [0, T],$$

where $\dot{x}_0(t)$ denotes the derivative of $x_0(t)$. From (5.6) we have

$$(5.7) \quad \frac{\partial f}{\partial x} = 0 \quad \text{at } (x_0(t), t) \text{ for every } t \text{ in } [0, T].$$

Therefore, the function $f(x, t)$ is completely determined by the restriction of $f(x, t)$ on the half-line $x \geq x_0(t)$ for every t in $[0, T]$. The assumptions (1.9) and (5.6) imply

$$(5.8) \quad c|x|^m - C \leq f(x, t) \leq C(1 + |x|^m) \quad \text{in } \mathbb{R} \times [0, T],$$

for some constants $C \geq c > 0, m \geq 1$. Observe that $x_0(t)$ represents the minimal trajectory of the system.

THEOREM 5.1. *Let the assumptions (1.7), \dots , (1.10) and (5.6) hold. Then, if $\hat{u}(x, t)$ denotes the optimal cost (5.5), we have*

$$(5.9) \quad \hat{u}(x, t) = \hat{u}(2x_0(t) - x, t) \quad \text{for every } (x, t) \text{ in } \mathbb{R} \times [0, T],$$

where $x_0(t)$ is given in (5.6).

Proof. Let ν be an arbitrary control in \mathcal{V} and (x, t) be any point in $\mathbb{R} \times [0, T]$. From (5.3) we have for $t \leq s \leq T$

$$y_{xt}(s, \nu) = 2x_0(s) + y_{xt}(s, \nu - 2q) \quad \text{with } q(s) = x_0(s) - \int_t^s a(s)x_0(s) ds.$$

Since

$$q(s) = \int_t^s b(s) ds + x_0(t),$$

we have

$$(5.10) \quad y_{xt}(s, \nu) = 2x_0(s) - \check{y}_z(s, -\nu), \quad z = 2x_0(t) - x,$$

where $\check{y}(s)$ solves an equation similar to (5.3) with a new Wiener process $\check{w}(s-t) = -w(s-t)$ in lieu of $w(s-t)$. Hence

$$\check{y}_{zt}(s, -\nu) = y_{zt}(s, -\nu) \quad \text{in law.}$$

Thereby, we obtain by virtue of (5.6)

$$(5.11) \quad J_{xt}(\nu) = J_{zt}(-\nu),$$

where z is given by (5.10).

Thus, the assertion (5.9) is deduced from (5.11) by taking the infimum over ν in \mathcal{V} \square

Remark 5.1. As in Theorem 2.1, we can prove that under the hypotheses (1.7), \dots , (1.10) and (5.6), there exist constants $C \cong c > 0$, such that for the same $m \cong 1$ of the assumption (1.9) and every $(x, t), (x', t')$ in $\mathbb{R} \times [0, T]$ we have

$$(5.12) \quad \begin{aligned} 0 &\leq \hat{u}(x, t) \leq C(1 + |x|^m), \\ |\hat{u}(x, t) - \hat{u}(x', t)| &\leq C(1 + |x|^{m-1} + |x'|^{m-1})|x - x'|, \\ |\hat{u}(x, t) - \hat{u}(x, t')| &\leq C(1 + |x|^m)|t - t'|, \\ 0 &\leq \frac{\partial^2 \hat{u}}{\partial x^2}(x, t) \leq C(1 + |x|^q), \quad q = (m - 2)^+, \end{aligned}$$

so \hat{u} is convex in the first variable. Actually, $m = 1$ in (5.12) even if $m > 1$ in the assumption (5.8).

Remark 5.2. From Theorem 5.1 we deduce that

$$(5.13) \quad \frac{\partial \hat{u}}{\partial x} = 0 \quad \text{at } (x_0(t), t) \text{ for every } t \text{ in } [0, T],$$

which represents a Neumann boundary condition for the corresponding Hamilton-Jacobi-Bellman equation, i.e. the optimal cost \hat{u} is the solution of the equation

$$(5.14) \quad \begin{aligned} (A\hat{u} - f) \vee B\hat{u} &= 0 \quad \text{if } x \leq x_0(t), 0 \leq t \leq T, \\ \hat{u}(\cdot, T) &= 0 \quad \text{in }]-\infty, x_0(T)], \end{aligned}$$

with the boundary condition (5.13). This implies that the restriction of the optimal cost $\hat{u}(x, t)$ to the half-line $x \leq x_0(t), 0 \leq t \leq T$, is actually the solution of a quasi-variational inequality with Neumann boundary condition, associated with an optimal impulse control problem where the state of the system is a reflected diffusion process (cf. Bensoussan and Lions [9], and [37], [52]). On the other hand, notice that $\hat{u} = f(x_0)$ if $c = 0$ and f is time-independent.

The whole § 3 can be adapted to this case. For instance, define the differential operator

$$(5.15) \quad A'u = -\frac{\partial u}{\partial t} - \sigma^2(t) \frac{\partial^2 u}{\partial x^2} - (a(t)x + b(t) + \dot{x}_0(t)) \frac{\partial u}{\partial x} + (\alpha(t) - a(t))u$$

and the substitutions

$$(5.16) \quad \hat{w}(x, t) = -\frac{\partial \hat{u}}{\partial x}(x - x_0(t), t) - c(t),$$

$$(5.17) \quad g(x, t) = \frac{dc}{dt}(t) - (\alpha(t) - a(t))c(t) - \frac{\partial f}{\partial x}(x - x_0(t), t),$$

for the given functions \hat{u} and f .

Then, the following equation is satisfied by the optimal cost (5.5) through (5.16) and (5.17),

$$\begin{aligned}
 (5.18) \quad & (A'\hat{w} - g) \vee \hat{w} = 0 \quad \text{in }]-\infty, 0] \times [0, T[, \\
 & \hat{w}(\cdot, T) = 0 \quad \text{in }]-\infty, 0], \\
 & \hat{w}(0, \cdot) = 0 \quad \text{in } [0, T].
 \end{aligned}$$

Moreover, the solution \hat{w} of (5.18) admits a stochastic representation as the optimal cost of a stopping time problem, i.e.,

$$(5.19) \quad \hat{w}(x, t) = \inf \{S_{xt}(\theta) : t \leq \theta \leq T, \text{ stopping time}\},$$

where

$$\begin{aligned}
 (5.20) \quad S_{xt}(\theta) &= E \left\{ \int_t^{\theta \wedge \tau} g(y^0(s), s) \exp \left(- \int_t^s (\alpha(\lambda) - a(\lambda)) d\lambda \right) ds \right\}, \\
 \tau &= \inf \{s \geq t : y^0(s) \geq 0\},
 \end{aligned}$$

and the process $y^0(s) = y_{xt}^0(s)$ is given by (5.3) with the control $\nu = 0$.

Next, with the function $\hat{w}(x, t)$ we can define the moving boundary $x^*(t), 0 \leq t < T$, by

$$(5.21) \quad x^*(t) = \inf \{x \leq 0 : \hat{w}(x, t) < 0\}$$

which induces an optimal control.

The precise variational inequality is exactly (3.23) with the space

$$(5.22) \quad V \text{ is the set of all real measurable functions } v \text{ on } [0, \infty[\text{ with a derivative } v' \text{ such that } \|v\|_p \text{ and } \|v'\|_{p-1} \text{ are finite, and } v(0) = 0,$$

where $\|\cdot\|_p$ and (\cdot, \cdot) are the norm and the inner product on $[0, \infty[$ instead of \mathbb{R} . The bilinear form $a(t, u, v)$ is defined as in (3.21) but the integration is over $[0, \infty[$ in lieu of \mathbb{R} , where a term is added in order to use the new definition (5.15) of the operator A' . In a similar way, if the space V_{m-1} is given by (3.34) restricted to $[0, \infty[$, we can state a strong formulation of the variational inequality as follows:

Find w in V_{m-1} such that

$$\begin{aligned}
 (5.23) \quad & w(x, T) = w(0, t) = 0 \quad \text{for every } (x, t) \text{ in }]-\infty, 0] \times [0, T], \\
 & A'w \leq g \quad \text{in } \mathcal{D}'(]-\infty, 0[\times]0, T]), \quad w \leq 0 \quad \text{in }]-\infty, 0] \times [0, T].
 \end{aligned}$$

As in Theorems 3.2 and 3.3 we can prove

THEOREM 5.2. *Under the hypotheses (1.7), \dots , (1.10) the function (5.19) is the maximum solution of the weak variational inequality (3.23) with the changes (5.22). Moreover, if we also suppose (3.36) is true and*

$$(5.24) \quad \text{the derivative of } x_0(t) \text{ is Lipschitz continuous in } [0, T],$$

then the strong version (5.23) of the variational inequality admits a maximum solution, which is precisely the optimal cost (5.19) and the equality

$$(5.25) \quad A'\hat{w} = g \quad \text{in } \mathcal{D}([\hat{w} < 0])$$

holds.

Remark 5.3. Similar results to Theorems 3.4 and 3.5 can be proved. For instance, assuming (1.7), \dots , (1.10) and (3.78), there exists an optimal control $\hat{\nu}$ in \mathcal{V} which is

continuous and uniquely determined by (5.3) and the conditions

$$(5.26) \quad \hat{v}(s) = \hat{v}_+(s) - \hat{v}_-(s) \quad \text{with } \hat{v}_+, \hat{v}_- \text{ in } \mathcal{V}_+,$$

if $z_+(s) = x_0(s) + x^*(s)$ and $z_-(s) = x_0(s) - x^*(s)$, then we impose

$$(5.27) \quad \begin{aligned} \hat{v}_+(0) &= (z_+(t) - x)^+, & \hat{v}_-(0) &= (z_-(t) - x)^-, \\ z_+(s) &\leq y(s) \leq z_-(s) & \text{for every } t \leq s \leq T, \end{aligned}$$

$$(5.28) \quad \begin{aligned} \int_t^T I(y(s) > z_+(s)) d\hat{v}_+(s-t) &= 0, \\ \int_t^T I(y(s) < z_-(s)) d\hat{v}_-(s-t) &= 0, \end{aligned}$$

where $I(\cdot)$ denotes the characteristic function, $y(s)$ the associated state and $x_0(t)$, $x^*(t)$ are given by (5.6), (5.21) respectively, i.e. \hat{v} reproduces the reflected diffusion of $y(s)$ on the interval $[z_+, z_-]$.

5.2. General comments. Most of the results presented herein can be extended to more general situations. Let us mention the following examples:

Extension to multidimensional model. This includes all of § 2 about the dynamic programming equation, the second part of § 3, i.e. § 3.2, about the optimal decision process, all of § 4 about the case of finite resources, the first part of this section, i.e., § 5.1, about the optimal correction problem. Let us mention that one of the main difficulties of the multidimensional case is the smoothness of the free boundary, which is for us an open question.

Extension to partially observed system. Since the model-equation is linear and the system may be degenerate, we can treat a multidimensional model with incomplete information on the state of the system. In particular, a separation principle result can be obtained (cf. [44]).

Extension to nonconvex data. In all of §§ 2, 4 and in the first part of this section, i.e., § 5.2, we may allow the coefficients of the stochastic equation (1.2) to be nonlinear in x , i.e., $\sigma = \sigma(x, t)$, $g = g(x, t)$ in lieu of $ax + b$, and also $\alpha = \alpha(x, t)$, $c = c(x, t)$ and $f = f(x, t)$ to not necessarily be convex in x . In that case, the optimal cost $\hat{u}(x, t)$ is no longer convex in x and the technique of [41] applies.

Extension to diffusion with jumps. All results herein may be extended to a model in which a Poisson integral is added to the stochastic equation (1.2). The technique is similar to that used in [42].

Extension to long term average criterion. When the horizon is infinite, we may consider a model with a long term average cost instead of the cost (2). (See, e.g. [43].)

Nonsymmetric case. It is possible to treat cases in which the reduction (5.9) does not hold. This is the case, for instance, if $f(x, t)$ is not symmetric or the cost $J_{x,t}(v)$ involves $c_1(\cdot)\nu_1(\cdot)$ and $c_2(\cdot)\nu_2(\cdot)$ with $\nu = \nu_1 - \nu_2$.

To conclude, let us mention that decomposable models and problems with the long run average criterion may be treated. Also, a combined version of §§ 4 and 5 can be developed.

6. Examples. To illustrate the results obtained in the previous sections, we shall consider some examples. We assume that the coefficients a, b, α, σ in (1.2) and (1.3) are constant, and the running cost $f(x)$ is time-independent and satisfies the condition (1.9). In addition, let $c(t) \equiv 0$, i.e., the cost for control is negligible. As mentioned in the introduction, for $a < 0$ and $b > 0$, the equation (1.2) may be interpreted as an

automatic cruise control problem. Probabilistically it pertains to the control of the motion of a Brownian particle with viscous damping, or an Ornstein-Uhlenbeck process [56]. In the case that $a > 0$ and $b < 0$, it becomes a simple model for the control of the population of a renewable resource. In either case, the unperturbed equilibrium state is $x_0 = (-b/a) > 0$. We wish to construct the optimal control, in particular, to find the free boundary, so that the mean-square deviation from the equilibrium value x_0 is minimum.

6.1. Unlimited resources. Under the above assumptions, the average cost (1.3) yields

$$(6.1) \quad J_{x_t}(v) = E \left\{ \int_t^T f(y(s)) e^{-\alpha s} ds \right\}.$$

By Theorem 2.5, the optimal cost \hat{u} (1.4) must satisfy

$$(6.2) \quad \begin{aligned} A_0 \hat{u} &= f \quad \text{and} \quad \frac{\partial \hat{u}}{\partial x} \geq 0 \quad \text{if } x \geq x^*(t), \\ A_0 \hat{u} &\leq f \quad \text{and} \quad \frac{\partial \hat{u}}{\partial x} = 0 \quad \text{if } x \leq x^*(t), \quad 0 \leq t \leq T, \end{aligned}$$

where

$$(6.3) \quad A_0 u = -\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} - (ax + b) \frac{\partial u}{\partial x} + \alpha u,$$

$$(6.4) \quad x^*(t) = \inf \left\{ x: \frac{\partial \hat{u}}{\partial x}(x, t) > 0 \right\}.$$

To construct the solution \hat{u} for $x \geq x^*(t)$, we let $s = (T - t)$ so that (5.2) gives the following free-boundary problem

$$(6.5) \quad \begin{aligned} v(x, s) &\equiv \hat{u}(x, T - s), \\ Lv &= \frac{\partial v}{\partial s} - \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial x^2} - (ax + b) \frac{\partial v}{\partial x} + \alpha v = f(x), \\ \frac{\partial v}{\partial x} &\geq 0, \quad \text{for } x > x^*(T - s), \quad 0 \leq s \leq T, \\ v(x, 0) &= 0, \\ \frac{\partial v}{\partial x} \Big|_{x=x^*(T-s)} &= 0, \end{aligned}$$

where $v(x, s) = \hat{u}(x, T - s)$.

Introduce the following change of variables:

$$(6.6) \quad \begin{aligned} \tau &= \frac{e^{2\alpha s} - 1}{2\alpha}, \quad 0 \leq s \leq T, \\ \xi &= \frac{1}{\sigma} (x - x_0) e^{\alpha s}, \quad x_0 = -\frac{b}{a}, \\ \omega &= v e^{\alpha s}, \\ \xi^*(\tau) &= \frac{1}{\sigma} (1 + 2\alpha\tau)^{1/2} \left\{ x^* \left[T - \frac{1}{2\alpha} \ln(1 + 2\alpha\tau) \right] - x_0 \right\}. \end{aligned}$$

In terms of the above variables, it is easy to check that (6.5) reduces to a standard free-boundary problem for a heat equation.

$$\begin{aligned}
 Mw &= \frac{\partial w}{\partial \tau} - \frac{1}{2} \frac{\partial^2 w}{\partial \xi^2} = g(\xi, \tau), \\
 \frac{\partial w}{\partial \xi} &\geq 0, \quad \text{for } \xi \geq \xi^*(\tau), \quad 0 \leq \tau \leq \tau_1 = (e^{2aT} - 1)/2a, \\
 w(\xi, 0) &= 0, \\
 \left. \frac{\partial w}{\partial \xi} \right|_{\xi = \xi^*(\tau)} &= 0,
 \end{aligned}
 \tag{6.7}$$

where

$$\begin{aligned}
 g(\xi, \tau) &= (1 + 2a\tau)^\beta \cdot f\{\sigma\xi(1 + 2a\tau)^{-1/2} + x_0\}, \\
 \beta &= \frac{\alpha - 2a}{2a}.
 \end{aligned}
 \tag{6.8}$$

To solve (6.7) we seek a similarity solution of the form

$$\begin{aligned}
 w(\xi, \tau) &= [\theta(\tau)]^n \varphi(\eta) \quad \text{for some } n \in \mathbb{R}^+, \\
 \eta &= \frac{\xi}{\theta(\tau)}, \quad \tau > 0.
 \end{aligned}
 \tag{6.9}$$

By a straightforward computation, we get

$$Mw = \theta^{n-1} \dot{\theta}(n\varphi - \eta\varphi') - \theta^{n-2} \varphi'' = g(\xi^* \eta, \tau)
 \tag{6.10}$$

or

$$\theta \dot{\theta}(n\varphi - \eta\varphi') - \varphi'' = g(\xi^* \eta, \tau) / \theta^{n-2}.
 \tag{6.11}$$

Now, suppose that f is symmetric about x_0 such that

$$f(x + x_0) = h(x) = |r|^m h(rx) \quad \text{for every } r \in \mathbb{R} - \{0\}.
 \tag{6.12}$$

That is, h is positive and homogeneous of degree m . Then the system (6.7) is reducible to a one-dimensional problem, if we choose

$$\theta \dot{\theta} = \frac{1}{2}, \quad \theta(0) = 0,
 \tag{6.13}$$

so that the free boundary is given by

$$\xi^*(\tau) = \delta \theta(\tau) = \delta \left(\frac{\tau}{2} \right)^{1/2} \quad \text{for some } \beta \in \mathbb{R}, \quad 0 \leq \tau < \tau_1.
 \tag{6.14}$$

In view of (6.8), (6.12)–(6.14), the equation becomes an ordinary differential equation

$$-\frac{1}{2} \varphi'' + \frac{1}{2} (n\varphi - \eta\varphi') = \sigma^m h(\eta),
 \tag{6.15}$$

provided that

$$n = m + 2, \quad \beta = \frac{m}{2}.
 \tag{6.16}$$

Let us summarize the above results:

THEOREM 6.1. *In (1.2) and (1.3), we assume the following:*

(6.17) *a, b, α, σ are constant and c(t) ≡ 0, the conditions (6.12) and (6.16) are satisfied.*

Then, under the transformations (6.6) and (6.9), the free boundary problem (6.5) is reducible to

$$\begin{aligned}
 &-\frac{1}{2}\varphi'' + \frac{1}{2}(n\varphi - \eta\varphi') = \sigma^m h(\eta) \quad \text{for } \eta \geq \delta, \\
 &\varphi'(\delta) = 0, \\
 &\varphi''(\delta) = 0, \\
 &\varphi(\eta) = O(\eta^m) \quad \text{as } \eta \rightarrow \infty.
 \end{aligned}
 \tag{6.18}$$

Remark 6.1. The last two conditions in (6.18) follow from Theorem 2.1. The reduced problem (6.18) is a free boundary value problem in one dimension where δ is to be determined in the process of constructing the solution. A special case, to be considered in what follows, has been solved by Benes, Shepp and Witsenhausen [6].

As a special case, let $m = 2$. By (6.16), we get

$$\beta = 1, \quad n = 4.
 \tag{6.19}$$

Then, setting $\sigma = 1$, (6.18) may be written as

$$\begin{aligned}
 &-\frac{1}{2}\varphi'' + \frac{1}{2}(4\varphi - \eta\varphi') = \eta^2, \quad \eta \geq \delta, \\
 &\varphi(\delta) = \frac{1}{2}\delta^2, \\
 &\varphi'(\delta) = 0, \\
 &\varphi(\eta) = O(\eta^2) \quad \text{as } \eta \rightarrow \infty.
 \end{aligned}
 \tag{6.20}$$

Similar to [6, Problem 2] (with η replaced by $-x$), the solution of (6.20) is given by

$$\varphi(\eta) = \varphi_0(\eta) + b(\delta)\varphi_1(\eta) \int_{\eta}^{\infty} [\varphi_1(\lambda)]^{-2} e^{-\lambda^2/2} d\lambda
 \tag{6.21}$$

where

$$\begin{aligned}
 &\varphi_0(\eta) = (\eta^2 + \frac{1}{2}), \\
 &\varphi_1(\eta) = (\eta^4 + 6\eta^2 + 3), \\
 &b(\delta) = \varphi'_0(\delta) / \left\{ [\varphi_1(\delta)]^{-1} e^{-\delta^2/2} - \varphi'_1(\delta) \int_{\delta}^{\infty} [\varphi_1(\lambda)]^{-2} e^{-\lambda^2/2} d\lambda \right\}.
 \end{aligned}
 \tag{6.22}$$

The parameter δ is determined by the equation

$$\delta^2 + 1 = \frac{4\delta\varphi_1(\delta) \int_{\eta}^{\infty} [\varphi_1(\lambda)]^{-2} e^{-(\lambda^2 - \delta^2)/2} d\lambda}{\varphi_1(\delta)\varphi'_1(\delta) \int_{\eta}^{\infty} [\varphi_1(\lambda)]^{-2} e^{-\lambda^2/2} d\lambda - 1}
 \tag{6.23}$$

which may be solved numerically to yield $\delta = -0.6388 \dots$. In view of (6.5), (6.6), (6.9) and (6.22), the problem (6.2) is solved and the associated free boundary is given by

$$x = \frac{\delta}{2} \left[\frac{1 - e^{-2a(T-t)}}{a} \right]^{1/2} - \frac{b}{a}, \quad 0 \leq t \leq T.
 \tag{6.24}$$

6.2. Finite resources. In the previous case 6.1, suppose the resource ν for control is finite so that $0 \leq \nu(T) \leq z$. The optimal cost $\hat{v}(x, z, t)$ defined by (1.19) can be

decomposed, according to Theorem 4.2, into two simple problems. That is, noting (4.13) and (4.14),

$$(6.25) \quad \hat{v}(x, z, t) = u^0(x + z, t) - [\hat{u}(x + z, t) - \hat{u}(x, t)]$$

where $\hat{u}(x, t)$ is the optimal cost without resource constraint, while $u^0(x, t)$ is the cost of free evolution defined by (1.18). Therefore it must satisfy

$$(6.26) \quad \begin{aligned} A_0 u^0 &= f, \quad 0 \leq t < T, \quad x \in \mathbb{R}, \\ u^0(x, T) &= 0, \\ u^0(x, t) &= O(|x|^m) \quad \text{as } |x| \rightarrow \infty, \end{aligned}$$

where A_0 is defined by (6.3). By the transformation (6.6), (6.26) may be solved to give

$$(6.27) \quad \begin{aligned} u^0(x, t) &= e^{-\alpha(T-t)} \int_0^{(T-t)} \int_{\mathbb{R}} \frac{\exp([\xi(x, t) - \rho]^2 / 2[\tau(t) - \lambda] + 2a\beta\lambda)}{2\pi[\tau(t) - \lambda]} \\ &\quad \times (1 + 2a\lambda)^{\beta} f \left[\sigma(1 + 2a\lambda)^{-1/2} \rho - \frac{b}{a} \right] d\lambda d\rho, \\ \xi(x, t) &= \frac{1}{\sigma} \left(x + \frac{b}{a} \right) e^{a(T-t)}, \\ \tau(t) &= (2a)^{-1} [e^{2a(T-t)} - 1]. \end{aligned}$$

Thus, as a consequence of Theorems 4.2 and 6.1, we have

COROLLARY 6.1. *If, in addition to the hypotheses (6.17), we assume $v \leq z$, then, in view of (6.27), the solution of (6.26) is reducible to a one-dimensional problem (6.18).*

Remark 6.2. Note that the free boundary, given by (6.14), remains unchanged. In particular, for $m = 2$, this problem may be solved explicitly.

We wish to point out that, for the optimal correction problems, the case of vanishing cost, $c = 0$, is less interesting. In this case the optimal policy would be to counteract the noise as long as the resources remain available so that $f(y(t), t)$ is kept to the minimum. However, for $c \neq 0$, the method of similarity transformations (6.6) and (6.9) is no longer applicable. This, of course, is true also for the one-sided control problems. Consequently one must deal with the genuine free-boundary problems for which the analytical solutions are difficult to obtain.

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