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ADDITIVE DERIVATIONS OF SOME OPERATOR ALGEBRAS

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1. Introduction

All algebras and vector spaces in this note will be over $\mathbf{F}$ where $\mathbf{F}$ is either the real field or the complex field. Let $\mathscr{A}$ be an algebra and $\mathscr{A}_1$ any subalgebra of $\mathscr{A}$. An additive (linear) mapping D: $\mathscr{A}_1 \to \mathscr{A}$ is called an additive (linear) derivation if

$$(1) \qquad D(ab) = aD(b) + D(a)b$$

holds for all pairs $a, b \in \mathscr{A}_1$. Let X be a normed linear space. By $\mathscr{B}(X)$ we mean algebra of bounded linear operators on X. We denote by $\mathscr{F}(X)$ the subalgebra of bounded finite rank operators. We shall call a subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$ standard provided $\mathscr{A}$ contains $\mathscr{F}(X)$.

This research is motivated by the well-known results in [2], [3].

THEOREM 1.1. *Let X be a normed space and let $\mathscr{A}$ be a standard operator algebra on X. Then every linear derivation D: $\mathscr{A} \to \mathscr{B}(X)$ is of the form*

$$D(A) = AT - TA$$

for some $T \in \mathscr{B}(X)$.

THEOREM 1.2. *Let $\mathscr{A}$ be a semi-simple Banach algebra. Let D: $\mathscr{A} \to \mathscr{A}$ be an additive derivation. Then $\mathscr{A}$ contains a central idempotent e such that $e\mathscr{A}$ and $(1 - e)\mathscr{A}$ are closed under D, $D|_{(1-e)\mathscr{A}}$ is continuous and $e\mathscr{A}$ is finite dimensional.*

Using these two results one can easily see that every additive derivation D: $\mathscr{B}(X) \to \mathscr{B}(X)$, where X is an infinite dimensional Banach space, is inner. In this note we shall give a complete description of all additive derivations on

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$\mathscr{B}(X)$ in the case that X is finite dimensional. In particular we shall see that in this case there exists an additive derivation D: $\mathscr{B}(X) \to \mathscr{B}(X)$ which is not inner. Assuming that X is an infinite dimensional Hilbert space we will succeed to prove an analogue of Theorem 1.1 for additive derivations.

We shall need some facts about additive derivations f: $\mathbf{F} \to \mathbf{F}$ where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$. Every such derivation vanishes at every algebraic number. On the other hand, if $t \in \mathbf{F}$ is transcendental then there is an additive derivation f: $\mathbf{F} \to \mathbf{F}$ which does not vanish at t [4]. It follows that a non-trivial additive derivation f: $\mathbf{F} \to \mathbf{F}$ is not continuous. It is well known that a noncontinuous additive function f: $\mathbf{F} \to \mathbf{F}$ is unbounded on an arbitrary neighborhood of zero [1].

2. Additive Derivations of Standard Operator Algebras

We shall begin this section by proving a lemma which will be needed in the sequel.

LEMMA 2.1. *Let X be a normed space and let D: $\mathscr{B}(X) \to \mathscr{B}(X)$ be an additive derivation. Then there exists an additive derivation f: $\mathbf{F} \to \mathbf{F}$ such that*

$$D(tI) = f(t)I \tag{2}$$

holds for all $t \in \mathbf{F}$.

Proof. For an arbitrary operator $A \in \mathscr{B}(X)$ and for an arbitrary number t we have

$$D(tA) = D((tI)A) = tD(A) + D(tI)A.$$

On the other hand,

$$D(tA) = D(A(tI)) = AD(tI) + tD(A).$$

Comparing the two expressions, so obtained, for $D(tA)$ we arrive at

$$D(tI)A = AD(tI).$$

Thus, the operator $D(tI)$ commutes with an arbitrary operator $A \in \mathscr{B}(X)$. It follows that $D(tI) \in \mathbf{F}I$. It is easy to see that the mapping f: $\mathbf{F} \to \mathbf{F}$ defined by (2) is an additive derivation.

The proof of this lemma implies that an additive derivation D: $\mathscr{B}(X) \to \mathscr{B}(X)$ is linear derivation if and only if f is a trivial derivation.

Suppose now that a Banach space X is finite dimensional. We are going to obtain the general form of additive derivations on $\mathscr{B}(X)$, that is, on the algebra of all $n \times n$ matrices.

Let D be an additive derivation on the algebra of all $n \times n$ matrices. Lemma 2.1 implies the existence of an additive derivation f on $\mathbf{F}$ such that $D(tI) = f(t)I$ holds for all $t \in \mathbf{F}$. A simple calculation shows that a mapping E on the algebra of all $n \times n$ matrices defined by

$$E\big((a_{ij})\big) = D\big((a_{ij})\big) - \big(f(a_{ij})\big)$$

is a linear derivation. Thus, E is an inner derivation. We have obtained the following result.

THEOREM 2.2. *A mapping D defined on the algebra of all $n \times n$ matrices is an additive derivation if and only if there exists an additive derivation f: $\mathbf{F} \to \mathbf{F}$ and an $n \times n$ matrix (b_{ij}) such that*

$$D\big((a_{ij})\big) = (a_{ij})(b_{ij}) - (b_{ij})(a_{ij}) + \big(f(a_{ij})\big).$$

Putting $(a_{ij}) = tI$ in the above relation one can see that the additive derivation f in the previous theorem is uniquely determined. Thus, if the relations

$$D\big((a_{ij})\big) = (a_{ij})(b_{ij}) - (b_{ij})(a_{ij}) + \big(f(a_{ij})\big)$$
$$D\big((a_{ij})\big) = (a_{ij})(c_{ij}) - (c_{ij})(a_{ij}) + \big(g(a_{ij})\big)$$

hold for all $(a_{ij}) \in \mathscr{B}(X)$, then we have $f = g$ and $(b_{ij}) = (c_{ij}) + tI$ for some $t \in \mathbf{F}$.

Now, we are ready to prove our main theorem.

THEOREM 2.3. *Let X be an infinite dimensional Hilbert space. Then every additive derivation D: $\mathscr{F}(X) \to \mathscr{B}(X)$ is of the form*

$$D(A) = AT - TA$$

for some $T \in \mathscr{B}(X)$.

Proof. Suppose that A is a normal finite rank operator. Then we can find a complete orthonormal set

$$\{x_1, x_2, \dots, x_m\} \cup \{x_\alpha;\ \alpha \in J\}$$

such that Im A is spanned by $\{x_1, x_2, \dots, x_m\}$. Let us choose a pair $\beta, \gamma \in \{1, 2, \dots, m\} \cup J$. We extend the set $\{x_1, x_2, \dots, x_m\}$ to the countable set

$$\{x_n;\, n \in \mathbf{N}\} \subset \{x_1, x_2, \dots, x_m\} \cup \{x_\alpha;\, \alpha \in J\}$$

such that $x_\beta, x_\gamma \in \{x_n;\ n \in \mathbf{N}\}$ is valid. Let us denote the orthogonal complement of the subspace spanned by $\{x_n;\, n \in \mathbf{N}\}$ by Y. For an arbitrary $n \in \mathbf{N}$ we define orthogonal projections P_n, R_n by

$$P_n x_k = x_k \text{ for } k \le n \quad \text{and} \quad R_n x_n = x_n,$$
$$P_n x_k = 0 \text{ for } k > n \quad \text{and} \quad R_n x_k = 0 \text{ for } k \ne n$$
$$P_n|_Y = 0 \quad \text{and} \quad R_n|_Y = 0.$$

Let P: $X \to X$ be an orthogonal projection satisfying $Px_k = x_k$, $k \in \mathbf{N}$, and $P|_Y = 0$. We denote the algebra of all $n \times n$ matrices by M^n. We shall need two more definitions. A mapping φ_n: $M^n \to \mathscr{B}(X)$ is defined as follows:

$$\varphi_n((a_{ij}))\left(\sum_{k \in \mathbf{N}} t_k x_k\right) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} t_k\right) x_i$$

and

$$\varphi_n((a_{ij}))|_Y = 0.$$

We will denote the mapping φ_n^{-1}: Im $\varphi_n \to M^n$ by ψ_n.

It is easy to prove that the mapping E_n: $M^n \to M^n$ given by

$$E_n((a_{ij})) = \psi_n\Big(P_n D\big(\varphi_n((a_{ij}))\big) P_n\Big)$$

is an additive derivation for all $n \in \mathbf{N}$. So we can find matrices $C^n = (c_{ij}^n) \in M^n$ and additive derivations f_n: $\mathbf{F} \to \mathbf{F}$ such that

$$E_n((a_{ij})) = (a_{ij})(c_{ij}^n) - (c_{ij}^n)(a_{ij}) + (f_n(a_{ij}))$$

holds for all $(a_{ij}) \in M^n$. For an arbitrary $(a_{ij}) \in M^n$ we choose $(b_{ij}) \in M^{n+1}$ in the following way:

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \le n \text{ and } j \le n, \\ 0 & \text{if } i = n+1 \text{ or } j = n+1. \end{cases}$$

Comparing

$$E_{n+1}\big((b_{ij})\big) = \psi_{n+1}\Big(P_{n+1}D\big(\varphi_{n+1}((b_{ij}))\big)P_{n+1}\Big)$$

and

$$E_n\big((a_{ij})\big) = \psi_n\Big(P_nD\big(\varphi_n((a_{ij}))\big)P_n\Big)$$

we get $f_{n+1} = f_n = f$ for all $n \in \mathbf{N}$. Moreover, the matrices C^n can be choosen so that

$$c_{ij}^n = c_{ij}^k, \quad \max\{i,j\} \le \min\{n,k\}.$$

Thus, we can denote $c_{ij} = c_{ij}^n$, $n \ge i, j$.

For arbitrary numbers $n, k \in \mathbf{N}$ and $i \ge n, k$ we have

$$(3) \qquad P_iD(R_n)x_k = P_iD(R_n)P_ix_k = \begin{cases} c_{nk}x_n & \text{if } k \ne n, \\ -\sum\limits_{\substack{r=1,\\ r\ne n}}^{i} c_{rn}x_r & \text{if } k = n. \end{cases}$$

Since the relation $\lim_{i\to\infty} P_ix = Px$ holds for all $x \in X$ the previous equation implies

$$PD(R_n)x_n = -\sum_{r\ne n} c_{rn}x_r.$$

It follows that the set $\{|c_{rn}|;\ r \in \mathbf{N}\}$ is bounded for all $n \in \mathbf{N}$. Let $M_n = \sup\{|c_{rn}|;\ r \in \mathbf{N}\}$.

Suppose now, that f is not identically equal to zero. Then one can find a sequence $(t_n) \subset \mathbf{F}$ having the properties

$$(4) \qquad |t_n| < 2^{-n}\min\{1, M_n^{-1}\},$$

$$(5) \qquad |f(t_n)| > n + |c_{11}| + |c_{nn}|.$$

We define $S \in \mathscr{B}(X)$ by $Sx_1 = \sum_{k=1}^{\infty} t_kx_k$, $Sx_k = 0$ for $k > 1$, and $S|_Y = 0$. Multiplying the relation

$$D(R_nSP_n) = R_nSD(P_n) + R_nD(S)P_n + D(R_n)SP_n$$

by R_n from the left side and by P_n from the right side we obtain

$$(6) \quad R_nD(S)P_n = R_nD(R_nSP_n)P_n - R_nSD(P_n)P_n - R_nD(R_n)SP_n.$$

The relation $P_n^2 = P_n$ implies $D(P_n) = P_n D(P_n) + D(P_n)P_n$. Multiplying from both sides by P_n we get $P_n D(P_n) P_n = 0$. Since $S = SP_n$ it follows that

$$R_n S D(P_n) P_n = 0. \tag{7}$$

The relation $R_n D(R_n S P_n) P_n x_1 = f(t_n) x_n + t_n (c_{11} - c_{nn}) x_n$ yields

$$\|R_n D(R_n S P_n) P_n x_1\| \geq |f(t_n)| - |t_n| (|c_{11}| + |c_{nn}|)$$

which gives us together with (5) that

$$\|R_n D(R_n S P_n) P_n x_1\| > n \tag{8}$$

holds for all positive integers n. Finally we have

$$R_n D(R_n) S P_n x_1 = R_n D(R_n) S x_1 = \sum_{k=1}^{\infty} t_k R_n D(R_n) x_k.$$

Using (3) we get

$$R_n D(R_n) S P_n x_1 = \left(\sum_{k \neq n} t_k c_{nk} \right) x_n.$$

This implies together with (4) the following inequalities

$$\|R_n D(R_n) S P_n x_1\| < 1. \tag{9}$$

Using (6), (7), (8) and (9) we see that

$$\|R_n D(S) P_n x_1\| \geq n - 1$$

is valid for all $n \in \mathbf{N}$ which is contradiction. Thus, we have $f(t) = 0$ for all $t \in \mathbf{F}$. As a consequence we have $P_\beta D(tA) P_\gamma = t P_\beta D(A) P_\gamma$ for all $t \in \mathbf{F}$. It follows that $D(tA) = tD(A)$ holds.

For an arbitrary finite rank operator A we have

$$D(tA) = D((t/2)(A + A^*) + (t/2)(A - A^*)) = (t/2) D(2A) = tD(A).$$

Using Theorem 1.1 we complete the proof.

COROLLARY 2.4. *Let $\mathscr{A}$ be a standard operator algebra on an infinite dimensional Hilbert space X. Then every additive derivation $D: \mathscr{A} \to \mathscr{B}(X)$ is of the form $D(A) = AT - TA$ for some $T \in \mathscr{B}(X)$.*

Proof. By Theorem 2.3 there exists $T \in \mathscr{B}(X)$ such that $D(A) = AT - TA$ holds for all $A \in \mathscr{F}(X)$. Now, let $A \in \mathscr{A}$ be arbitrary. Then for every $B \in \mathscr{F}(X)$ we have

$$BD(A) = D(BA) - D(B)A = BAT - TBA - BTA + TBA = B(AT - TA).$$

Accordingly, $D(A) - (AT - TA)$ annihilates $\mathscr{F}(X)$, and, therefore, $D(A) = AT - TA$.

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