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Additive Runge-Kutta Methods for Stiff Ordinary Differential Equations

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Abstract. Certain pairs of Runge-Kutta methods may be used additively to solve a system of n differential equations $x' = J(t)x + g(t, x)$. Pairs of methods, of order $p \leq 4$, where one method is semiexplicit and A-stable and the other method is explicit, are obtained. These methods require the LU factorization of one $n \times n$ matrix, and p evaluations of g, in each step. It is shown that such methods have a stability property which is similar to a stability property of perturbed linear differential equations.

1. Introduction. In a recent article [2] the authors showed that certain pairs of methods may be used in an additive fashion to solve an initial value problem for a system of n differential equations

$$x' = f(t, x), \qquad x(a) = x_0, \qquad a \leq t \leq b,$$

where, for a particular step length h, a given additive method is associated with a sequence of decompositions

$$\{f = f_1^{(m)} + f_2^{(m)}\}.$$

In this article we consider the case where $\{f_1^{(m)}\}$ is a sequence of linear mappings so that

$$f(t, x) = J^{(m)}(t)x + g^{(m)}(t, x), \qquad m = 1, 2, 3, \ldots, \tag{1.1}$$

and, in particular, it is assumed that, for some norm on $\mathbf{R}^n$,

$$\|g^{(m)}(t, u) - g^{(m)}(t, v)\| \leq L\|u - v\| \quad \forall u, v \in \mathbf{R}^n, t \in I,$$

for $m = 1, 2, 3, \ldots$, where $[a, b]$ is contained in the open interval I. It is also supposed that each element of $\{J^{(m)}\}$ and $\{g^{(m)}\}$ is continuous on I. Other assumptions, which are needed to obtain order conditions for additive methods, are detailed in the previous article [2].

The aim is to obtain additive methods suitable for solving stiff systems of differential equations. Although f may be given directly in the form (1.1), it is necessary to choose the sequence of decompositions so that the Lipschitz constant L is small. Usually $\{J^{(m)}\}$ is chosen as an approximation to the Jacobian of f evaluated at some sequence of computed values. The elements of $\{J^{(m)}\}$ are often chosen to be independent of t. We consider pairs of Runge-Kutta methods where one method, which is A-stable and semiexplicit, is applied to the linear (stiff) part of the decompositions. The other method, which is explicit, is applied to the nonlinear part.

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In the previous article [2] the authors gave a few examples of low order additive methods of this type.

An additive method consists of a pair of methods, an (A, B_1) method and an (A, B_2) method, of the type described by Butcher [1], and is represented by the triple of real $s \times s$ matrices (A, B_1, B_2). In this article we consider only methods of Runge-Kutta type where

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & \vdots \\ 0 & & 0 & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Such methods are represented by an array $\mathbf{p} \mid B_1 \mid B_2 \mid \mathbf{c}$ or

$$\begin{array}{c|cccc|cccc|c} p_1 & b_{11} & b_{12} & \cdots & b_{1s} & \beta_{11} & \beta_{12} & \cdots & \beta_{1s} & c_1 \\ p_2 & b_{21} & b_{22} & & b_{2s} & \beta_{21} & \beta_{22} & & \beta_{2s} & c_2 \\ \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \vdots \\ p_s & b_{s1} & b_{s2} & \cdots & b_{ss} & \beta_{s1} & \beta_{s2} & \cdots & \beta_{ss} & c_s \end{array}$$

and it is assumed that

$$(1.2) \qquad c_i = \sum_{j=1}^{s} b_{ij} = \sum_{j=1}^{s} \beta_{ij}, \qquad i = 1, 2, \ldots, s.$$

A method consists of a sequence of steps, with step length h, where each step contains s stages,

$$(1.3) \qquad y_i^{(m)} = y_s^{(m-1)} + h \sum_{j=1}^{s} b_{ij} J^{(m)}\big(t_{m-1} + hc_j\big) y_j^{(m)} + h \sum_{j=1}^{s} \beta_{ij} g^{(m)}\big(t_{m-1} + hc_j,\, y_j^{(m)}\big),$$

for $i = 1, 2, \ldots, s$ and $m = 1, 2, 3, \ldots$. The consistency vector $\mathbf{c}$ defines the points at which the method gives approximations to the solution of the initial value problem, and the order vector $\mathbf{p}$ gives the order of convergence of each stage. That is, suppose the numerical integration is over the finite interval $[a, b]$, and let $t_m = a + mh$, $m = 0, 1, \ldots, M$, where $t_M = b$. Then there are constants K, C and H such that, for $h \leqslant H$,

$$\big\| y_i^{(m)} - x\big(t_{m-1} + hc_i\big) \big\| \leqslant K h^{p_i}, \qquad i = 1, 2, \ldots, s,\ m = 1, 2, \ldots, M,$$

provided that $\| y_s^{(0)} - x_0 \| \leqslant C h^{p_s}$. It is supposed that $c_s = 1$ and the (scalar) order is defined to be $p = p_s$, which corresponds with the conventional definition of order of a Runge-Kutta method.

We are concerned with *linearly implicit* methods where B_1 is a lower triangular matrix and B_2 is a strictly lower triangular matrix. That is, the (A, B_1) method is a semiexplicit Runge-Kutta method, and the (A, B_2) method is explicit. Since the (A, B_1) method is to be A-stable, at least one diagonal element of B_1 must be nonzero and, in particular, the possibility $b_{ss} \neq 0$ is allowed. On the other hand, (1.2) implies that $b_{11} = 0$. For a linearly implicit method $y_1^{(m)} = y_s^{(m-1)}$ and

$$y_i^{(m)} = y_s^{(m-1)} + h\sum_{j=1}^{i} b_{ij}J^{(m)}\big(t_{m-1} + hc_j\big)y_j^{(m)} + h\sum_{j=1}^{i-1} \beta_{ij}g^{(m)}\big(t_{m-1} + hc_j,\, y_j^{(m)}\big),$$

for $i = 2, 3, \ldots, s$ and $m = 1, 2, 3, \ldots$. At most $s - 1$ evaluations of $g^{(m)}$ are needed in step m. Suppose that the nonzero diagonal elements of B_1 are equal and that the elements of $\mathbf{c}$, which correspond to these nonzero elements, are equal also. Then each step requires the LU factorization of one $n \times n$ matrix of the form

$$I - hbJ^{(m)}(t_{m-1} + hc),$$

and it is not necessary to evaluate $J^{(m)}$ at other points since

$$J^{(m)}(t)x = f(t, x) - g^{(m)}(t, x), \qquad m = 1, 2, 3, \ldots.$$

Nevertheless, it is usually more efficient to employ decompositions where the linear terms are independent of t because then there is a gain in efficiency when the same decomposition is used in successive steps.

In the next section we give a number of linearly implicit (A, B_1, B_2) methods of Runge-Kutta type, where the (A, B_1) method is A-stable. Such methods can be obtained with $p_s = s - 1$ where $s \leqslant 4$. When $p_s = 4$, it is necessary to choose $s = 6$ but only four evaluations of $\{g^{(m)}\}$ are required.

In the third section we establish a stability result for such methods applied to perturbed linear systems of differential equations. Consider the initial value problem

$$x' = Jx + g(t, x), \qquad x(a) = x_0, \qquad t \geqslant a,$$

where the eigenvalues of J have negative real parts and where $\|g(t, u)\| = o(\|u\|)$. That is, it is assumed that

$$\|g(t, u)\| \leqslant \phi(\|u\|)\|u\| \quad \forall u \in \mathbf{R}^n,\ t \geqslant a,$$

where ϕ is continuous and $\phi(0) = 0$. It is known [5, p. 274] that there is an $\varepsilon > 0$ such that if $\|x_0\| \leqslant \varepsilon$, then $\|x(t)\|$ has limit zero. Now consider a linear implicit (A, B_1, B_2) method of Runge-Kutta type where the (A, B_1) method is A-stable. Suppose that this method is used, with a fixed step length h, to integrate the initial value problem on $[a, \infty)$, where the given decomposition is used throughout the numerical integration. For an arbitrary $y_s^{(0)}$ the method gives a sequence $\{y_s^{(m)}\}$, and it is shown that there is a $\delta > 0$ such that if $\|y_s^{(0)}\| \leqslant \delta$, then the sequence $\{\|y_s^{(m)}\|\}$ has limit zero.

One problem with this result is that it is difficult to assess the effect of a perturbation. Another problem is that the result applies to a single decomposition where the linear part remains constant for the entire numerical integration. Numerical results indicate that the methods are satisfactory for much more general sequences of decompositions.

2. The Conditions for Order and A-Stability. In the article [2] the authors obtained order conditions for a general additive method. For additive Runge-Kutta methods the order of the last stage $p_s = p$ is of principal interest and the conditions given below refer to this stage alone. The order of convergence of other stages may be determined after the method has been obtained.

It is convenient to express the order conditions in terms of

$$b_i(\sigma) = c_i^\sigma - \sigma \sum_{j=1}^{s} b_{ij} c_j^{\sigma-1}, \qquad \beta_i(\sigma) = c_i^\sigma - \sigma \sum_{j=1}^{s} \beta_{ij} c_j^{\sigma-1}, \qquad i = 1, 2, \ldots, s,\ \sigma = 1, 2, 3, \ldots,$$

where the assumption (1.2) gives

(2.1) $$b_i(1) = \beta_i(1) = 0, \qquad i = 1, 2, \ldots, s.$$

Subject to this assumption, an additive Runge-Kutta method is of order $p \leqslant 4$ if and only if

(2.2.1) $b_s(\sigma) = 0, \quad \sigma \leqslant p,$

(2.2.2) $\beta_s(\sigma) = 0, \quad \sigma \leqslant p,$

(2.3.1) $\sum b_{si} c_i^{\tau-1} b_i(\sigma) = 0, \quad \sigma + \tau \leqslant p,$

(2.3.2) $\sum b_{si} c_i^{\tau-1} \beta_i(\sigma) = 0, \quad \sigma + \tau \leqslant p,$

(2.3.3) $\sum \beta_{si} c_i^{\tau-1} b_i(\sigma) = 0, \quad \sigma + \tau \leqslant p,$

(2.3.4) $\sum \beta_{si} c_i^{\tau-1} \beta_i(\sigma) = 0, \quad \sigma + \tau \leqslant p,$

(2.4.1) $\sum b_{si} \sum b_{ij} b_j(2) = 0, \quad p = 4,$

(2.4.2) $\sum b_{si} \sum b_{ij} \beta_j(2) = 0, \quad p = 4,$

(2.4.3) $\sum b_{si} \sum \beta_{ij} b_j(2) = 0, \quad p = 4,$

(2.4.4) $\sum b_{si} \sum \beta_{ij} \beta_j(2) = 0, \quad p = 4,$

(2.4.5) $\sum \beta_{si} \sum b_{ij} b_j(2) = 0, \quad p = 4,$

(2.4.6) $\sum \beta_{si} \sum b_{ij} \beta_j(2) = 0, \quad p = 4,$

(2.4.7) $\sum \beta_{si} \sum \beta_{ij} b_j(2) = 0, \quad p = 4,$

(2.4.8) $\sum \beta_{si} \sum \beta_{ij} \beta_j(2) = 0, \quad p = 4,$

where σ and τ take all possible positive integer values, and where each summation is from 1 to s. These conditions simplify greatly when $\beta_{si} = b_{si}$, $i = 1, 2, \ldots, s$.

The aim is to obtain linearly implicit (A, B_1, B_2) methods of Runge-Kutta type, where the (A, B_1) method is A-stable. Since the (A, B_2) method is a conventional $s - 1$ stage explicit Runge-Kutta method, the order of such an additive method cannot exceed $s - 1$. If $s > 5$, the other cannot exceed $s - 2$. The conditions for the additive method to be of order $p \leqslant s - 1$ must be satisfied together with conditions for A-stability.

Necessary and sufficient conditions for a semiexplicit Runge-Kutta method to be A-stable have been given by the authors [3] and by Norsett [6]. We give these conditions for an (A, B_1) semiexplicit Runge-Kutta method, where at least one diagonal element of B_1 is zero, in terms of parameters α_0, α_1, $\alpha_2, \ldots,$ and β_0, β_1,

$\beta_2, \ldots$. Let β_r, $r = 0, 1, 2, \ldots$, be defined by

$$\prod_{r=1}^{s} (1 - \tau b_{rr}) = \beta_0 - \tau\beta_1 + \tau^2\beta_2 - \cdots,$$

so that $\beta_0 = 1$ and $\beta_s = \beta_{s+1} = \cdots = 0$. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_s$ be the natural basis for $\mathbf{R}^s$ and let $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_s$ be the vector with unit elements. The terms

$$\mathbf{e}_s^T B_1^r \mathbf{e}, \qquad r = 1, 2, 3, \ldots,$$

are the sums of the elements in row s of B_1, B_1^2, $B_1^3, \ldots$, and for a method of order p it is known that

$$\mathbf{e}_s^T B^r \mathbf{e} = \frac{1}{r!}, \qquad r = 1, 2, \ldots, p. \tag{2.5}$$

Now define $\alpha_s = \alpha_{s+1} = \cdots = 0$ and

$$\alpha_r = \beta_r - \beta_{r-1}\mathbf{e}_s^T B_1 \mathbf{e} + \cdots + (-1)^r \beta_0 \mathbf{e}_s^T B^r \mathbf{e}, \qquad r = 0, 1, \ldots, s-1, \tag{2.6}$$

so that $\alpha_0 = 1$. Then a method of order p is A-stable if and only if $b_{rr} \geqslant 0$ for $r = 1, 2, \ldots, s$ and

$$\sum_{r=\pi}^{s-1} y^r \sum_{j=0}^{r}{}^{*} (-1)^{r+j}\left(\beta_{2r-j}\beta_j - \alpha_{2r-j}\alpha_j\right) \geqslant 0 \quad \forall y \geqslant 0,$$

where π is the integral part of $p/2 + 1$ and the asterisk denotes that the terms with $j = r$ are halved.

Some low order cases are considered now. In these cases $p = s - 1$ so that $\alpha_0, \alpha_1, \alpha_2, \ldots$ are completely determined by the diagonal elements of B_1. These elements are chosen so that the A-stability conditions are satisfied. The remaining elements of B_1 and B_2 are obtained by satisfying the order conditions.

For $p \leqslant 2$ the A-stability conditions can be satisfied when B_1 has just one nonzero diagonal element b, and for $p = 1$ it suffices that $b \geqslant 1/2$. The order conditions give the methods represented by the array

$$\begin{array}{c|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1-b & b & 1 & 0 & 1 \end{array} \qquad b \geqslant \frac{1}{2}.$$

When $p = 2$ it is necessary to choose $b = 1/2$. The order conditions give, in particular, the methods

$$\begin{array}{c|ccc|ccc|c} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & \frac{\mu}{2} & 0 & 0 & \frac{\mu}{2} & 0 & 0 & \frac{\mu}{2} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\mu-1}{\mu} & \frac{1}{\mu} & 0 & 1 \end{array} \qquad \mu \neq 0.$$

When $p = 3$ the A-stability conditions imply that B_1 must have at least two nonzero diagonal elements. Suppose that these two elements are equal to b. Then the stability conditions are satisfied only if b is the larger root of $6b^2 - 6b + 1$, and the order conditions may be solved to give, in particular, the methods

$$\begin{array}{c|cccc|cccc|c}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & \frac{2-3b}{\mu} & b & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} \\
2 & \frac{8\mu - 3 + 9b - 12b\mu}{12\mu} & \frac{1-3b}{4\mu} & b & 0 & \frac{8\mu - 3}{12\mu} & \frac{1}{4\mu} & 0 & 0 & \frac{2}{3} \\
3 & \frac{1}{4} & \frac{3-4\mu}{4} & \mu & 0 & \frac{1}{4} & \frac{3-4\mu}{4} & \mu & 0 & 1
\end{array}$$

where $b = (3 + \surd 3)/6$ and $\mu \neq 0$.

Now consider the case where $p = 4$ and $s \geqslant 5$. Suppose that B_1 has at most two nonzero diagonal elements so that $\beta_3 = \beta_4 = \cdots = 0$. Then the conditions for A-stability can be satisfied only if $\alpha_3 = \alpha_4 = \cdots = 0$. But it follows from (2.5) and (2.6) that

$$\alpha_3 = -\beta_2 + \frac{1}{2}\beta_1 - \frac{1}{6}, \qquad \alpha_4 = \frac{1}{2}\beta_2 - \frac{1}{6}\beta_1 + \frac{1}{24},$$

and the stability conditions cannot be satisfied because the diagonal elements of B_1 are real. Now suppose that $s = 5$ and that B_1 has at least three nonzero diagonal elements. Then it may be shown that the order conditions cannot be satisfied. (The details are not given here but are available on request.) That is, when $p = 4$ and $s = 5$, there is no linearly implicit (A, B_1, B_2) method of Runge-Kutta type where the (A, B_1) method is A-stable.

Suppose that $p = 4$ and $s = 6$, and suppose that B_1 has exactly three nonzero diagonal elements so that $\beta_4 = \beta_5 = \cdots = 0$. Then the (A, B_1) method is A-stable if and only if the diagonal elements of B_1 are nonnegative, $\alpha_4 = \alpha_5 = \cdots = 0$, and $\beta_3^2 \geqslant \alpha_3^2$. Let the three nonzero diagonal elements be equal to b. Then the condition $\alpha_4 = 0$ gives $24b^3 - 36b^2 + 12b - 1 = 0$, but $\beta_3^2 \geqslant \alpha_3^2$ only when b is the largest root $b = 1.06857902130\ldots$. By definition $\alpha_s = \alpha_{s+1} = \cdots = 0$, so that the only stability condition still to be satisfied is $\alpha_5 = 0$. It follows from (2.5) and (2.6) that this is equivalent to the condition

$$\mathbf{e}_6^T B_1^5 \mathbf{e} = \frac{1}{2}\beta_3 - \frac{1}{6}\beta_2 + \frac{1}{24}\beta_1,$$

which must be satisfied together with the order conditions. Although a variety of methods can be obtained, the arrays

$$\begin{array}{c|cccccc|cccccc|c}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & \frac{1-2b}{2} & b & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
3 & \frac{1}{4} & \frac{1-4b}{4} & b & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
3 & \frac{1}{4} & \frac{b}{2} & \frac{1-6b}{4} & b & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
3 & 0 & -2b & \frac{1-6b-8b^2}{1-4b} & \frac{4b}{1-4b} & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 1 \\
4 & \frac{1}{6} & 0 & 0 & \frac{2}{3} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1
\end{array}$$

4	0	0	0	0	0	0	0	0	0	0	0	0	0
2	$\frac{1-2b}{2}$	b	0	0	0	0	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$
2	$\frac{1-6b+8b^2}{2}$	$2b(1-2b)$	b	0	0	0	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$
2	b	$\frac{1-2b}{4}$	$\frac{1-6b}{4}$	b	0	0	0	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$
3	0	$\frac{1-2b}{2}$	$\frac{6b-1}{2}$	$1-2b$	0	0	0	0	0	1	0	0	1
4	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{6}$	0	1

where $b = 1.06857902130\ldots$, give two methods which are particularly efficient. These methods require just four evaluations of $\{g^{(m)}\}$ and, in this respect, are comparable with explicit Runge-Kutta methods of order four. Each step requires the LU factorization of one $n \times n$ matrix.

3. A Stability Property. In this section we establish a stability result for linearly implicit (A, B_1, B_2) methods of Runge-Kutta type, where the (A, B_1) method is A-stable. It is likely that the result holds also when the methods are not linearly implicit.

The result deals with the behavior of a method when applied to a stable linear perturbed system of differential equations. Let x be the particular solution of the system

$$x' = Jx + g(t, x), \qquad t \geqslant a,$$

which has the initial value $x(a) = x_0$. The solution $u = 0$ of the linear system $u' = Ju$ is exponentially stable [5, p. 113] if and only if $\operatorname{Re} \lambda < 0$ for all $\lambda \in \lambda[J]$, where $\lambda[J]$ is the spectrum of J. Now suppose that the trivial solution of $u' = Ju$ is exponentially stable *and* that $\|g(t, u)\| = o(\|u\|)$. Then it is known [5, p. 274] that the trivial solution of $u' = Ju + g(t, u)$ is also exponentially stable. In particular, this implies that there is an $\varepsilon > 0$ such that if $\|x_0\| \leqslant \varepsilon$, then $\|x(t)\|$ has limit zero.

Suppose that an (A, B_1, B_2) Runge-Kutta method is used to integrate this stable perturbed differential system on $[a, \infty)$. The method gives, for a fixed positive step length h,

$$y_i^{(m)} = y_s^{(m-1)} + h \sum_{j=1}^{s} b_{ij} J y_j^{(m)} + h \sum_{j=1}^{s} \beta_{ij} g\left(t_{m-1} + hc_j, y_j^{(m)}\right),$$

for $i = 1, 2, \ldots, s$, and $m = 1, 2, 3, \ldots$. Suppose that the method is linearly implicit and that the (A, B_1) method is A-stable. It will be shown that there is a $\delta > 0$ such that if $\|y_s^{(0)}\| \leqslant \delta$, then the sequence $\{\|y_s^{(m)}\|\}$ has limit zero.

With this end in view, and to introduce a matrix notation for the methods, consider column vectors in $\mathbf{R}^N$, where $N = ns$, of the form

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_s \end{bmatrix}, \quad G(t, Y) = \begin{bmatrix} g(t + hc_1, y_1) \\ g(t + hc_2, y_2) \\ \vdots \\ g(t + hc_s, y_s) \end{bmatrix}, \quad Y_s = \begin{bmatrix} y_s \\ y_s \\ \vdots \\ y_s \end{bmatrix},$$

where $y_1, y_2, \ldots, y_s$ are column vectors in $\mathbf{R}^n$. For any r let I denote the $r \times r$ identity matrix. Let $B \times J$ be the tensor product of an arbitrary $s \times s$ matrix $B = \{b_{ij}\}$ and an arbitrary $n \times n$ matrix J,

$$B \times J = \begin{bmatrix} b_{11}J & b_{12}J & \cdots & b_{1s}J \\ b_{21}J & b_{22}J & & b_{2s}J \\ \vdots & & & \vdots \\ b_{s1}J & b_{s2}J & \cdots & b_{ss}J \end{bmatrix}.$$

Then an (A, B_1, B_2) Runge-Kutta method may be expressed in the form

$$Y^{(m)} = Y_s^{(m-1)} + hB_1 \times JY^{(m)} + hB_2 \times IG\big(t_{m-1}, Y^{(m)}\big),$$

where $Y_s^{(0)}$ is given and $Y_s^{(m)} = A \times IY^{(m)}$, $m = 1, 2, 3, \ldots$. Now suppose that the method is linearly implicit and that the (A, B_1) method is A-stable. It will be shown that for *some* norm on $\mathbf{R}^N$ there is a $\Delta > 0$ such that if $\|Y_s^{(0)}\| \leqslant \Delta$, then the sequence $\{\|Y_s^{(m)}\|\}$ is strictly decreasing and has limit zero. It follows that $\{\|y_s^{(m)}\|\}$ has limit zero.

THEOREM. *Suppose that the trivial solution of $u' = Ju$ is exponentially stable and let $\|g(t, u)\| = o(\|u\|)$. Suppose that the (A, B_1, B_2) Runge-Kutta method is linearly implicit and that the (A, B_1) method is A-stable. For any fixed positive h and arbitrary $y_s^{(0)}$, the method uniquely defines a sequence $\{y_s^{(m)}\}$ where*

$$y_i^{(m)} = y_s^{(m-1)} + h \sum_{j=1}^{i} b_{ij} J y_j^{(m)} + h \sum_{j=1}^{i-1} \beta_{ij} g\big(t_{m-1} + hc_j, y_j^{(m)}\big),$$

for $i = 1, 2, \ldots, s$, and $m = 1, 2, 3, \ldots$, and there is a $\delta > 0$ such that if $\|y_s^{(0)}\| \leqslant \delta$, then the sequence $\{\|y_s^{(m)}\|\}$ has limit zero.

Proof. (i) Since the diagonal elements of B_1 are nonnegative and since the eigenvalues of J have negative real parts, the matrices $I - hb_{ii}J$, $i = 1, 2, \ldots, s$, are nonsingular. Further, B_2 is strictly lower triangular, so that the method uniquely defines the sequence $\{Y^{(m)}\}$ where $Y_s^{(0)}$ is given and

$$Y^{(m)} = (I - hB_1 \times J)^{-1}\big[Y_s^{(m-1)} + hB_2 \times IG\big(t_{m-1}, Y^{(m)}\big)\big], \qquad m = 1, 2, 3, \ldots.$$

We are concerned with the vectors

$$(3.1) \qquad Y_s^{(m)} = A \times I(I - hB_1 \times J)^{-1}\big[Y_s^{(m-1)} + hB_2 \times IG\big(t_{m-1}, Y^{(m)}\big)\big], \quad m = 1, 2, 3, \ldots,$$

and, in particular, it has to be shown that $\|A \times I(I - hB_1 \times J)^{-1}\| \leqslant \alpha < 1$ for *some* norm. This is equivalent to showing that the spectral radius satisfies

$$\rho\big[A \times I(I - hB_1 \times J)^{-1}\big] < 1.$$

To this end, suppose that the (A, B_1) method is applied to the scalar initial value problem $x' = \lambda x$, $x(0) = 1$, where λ is a constant. Since $N = s$, the method gives

$$AY^{(m)} = A(I - h\lambda B_1)^{-1} AY^{(m-1)}, \qquad m = 1, 2, 3, \ldots,$$

for a fixed positive step length h. Since the method is A-stable, $\rho[A(I - h\lambda B_1)^{-1}] < 1$ for any λ with $\operatorname{Re} \lambda < 0$.

The result is now obtained by transforming $A \times I(I - hB_1 \times J)^{-1}$. There is a permutation matrix P, depending only on s and n, such that $P^T B \times JP = J \times B$ for an arbitrary $s \times s$ matrix B and an arbitrary $n \times n$ matrix J. For given J there is a unitary matrix S such that $S^H JS = T$ a triangular matrix. Thus it has to be shown that $\rho[M] < 1$, where

$$M = P^T I \times S^H A \times I(I - hB_1 \times J)^{-1} I \times S\, P = I \times A(I - hT \times B_1)^{-1}.$$

Since M is block triangular and each diagonal block has the form $A(I - h\lambda B_1)^{-1}$ with $\lambda \in \lambda[J]$, it follows that $\rho[M] < 1$.

(ii) For fixed t let U be defined as a function of Y_s by

$$U = Y_s + hB_1 \times JU + hB_2 \times IG(t, U).$$

It has to be shown that $\|G(t, U)\| = o(\|Y_s\|)$ for *any* norm but, since norms on $\mathbf{R}^N$ are equivalent, it suffices to show this for the particular norm

$$\|U\| = \max_{1 \leqslant i \leqslant s} \|u_i\|.$$

For this norm it is clear that $\|G(t, U)\| = o(\|U\|)$, so that it is sufficient to show that there are positive constants E and K such that if $\|Y_s\| \leqslant E$, then $\|U\| \leqslant K\|Y_s\|$.

That is, it has to be shown that if $\|y_s\| \leqslant E$, then $\|u_i\| \leqslant K\|y_s\|$, where

$$u_i = (I - hb_{ii}J)^{-1}\left[y_s + h\sum_{j=1}^{i-1} b_{ij}Ju_j + h\sum_{j=1}^{i-1} \beta_{ij}g(t + hc_j, u_j)\right], \qquad i = 1, 2, \ldots, s.$$

Since $\|g(t, u)\| = o(\|u\|)$, for any positive L there is an $e > 0$ such that if $\|u\| \leqslant e$, then $\|g(t, u)\| \leqslant L\|u\|$. Let $K = c^{s-1}$ where

$$c = \max_{1 \leqslant i \leqslant s} \left\|(I - hb_{ii}J)^{-1}\right\|\left[1 + h\|J\| \sum_{j=1}^{i-1} |b_{ij}| + hL\sum_{j=1}^{i-1} |\beta_{ij}|\right]$$

and $c \geqslant 1$ because $b_{11} = 0$. Choose E so that $KE \leqslant e$. It follows by induction that $\|u_i\| \leqslant c^{i-1}\|y_s\|$, $i = 1, 2, \ldots, s$, and therefore $\|G(t, U)\| = o(\|Y_s\|)$.

(iii) It follows from (3.1) that for some norm

$$\left\|Y_s^{(m)}\right\| \leqslant \alpha\left[\left\|Y_s^{(m-1)}\right\| + \beta\left\|G\left(t_{m-1}, Y^{(m)}\right)\right\|\right], \qquad m = 1, 2, 3, \ldots,$$

where $\alpha = \|A \times I(I - hB_1 \times J)^{-1}\| < 1$ and $\beta = h\|B_2 \times I\|$. Choose a so that $\alpha < a < 1$ and consider some fixed value of m. Since $\|G(t_{m-1}, Y^{(m)})\| = o(\|Y_s^{(m-1)}\|)$, there is a $\Delta > 0$ such that if $\|Y_s^{(m-1)}\| \leqslant \Delta$, then

$$\left\|G\left(t_{m-1}, Y^{(m)}\right)\right\| \leqslant \frac{a - \alpha}{\alpha\beta}\left\|Y_s^{(m-1)}\right\| \qquad (a\beta \neq 0),$$

and this gives $\|Y_s^{(m)}\| \leqslant a\|Y_s^{(m-1)}\|$ so that $\|Y_s^{(m)}\| \leqslant \Delta$. It follows that if $\|Y_s^{(0)}\| \leqslant \Delta$, then the sequence $\{\|Y_s^{(m)}\|\}$ is strictly decreasing and has limit zero.

This result has no direct practical application in the sense that it cannot be used to measure the effect of a perturbation. This is so even though the condition $\|g(t, u)\| = o(\|u\|)$ may be replaced by the conditions $\|g(t, u)\| \leqslant L\|u\|$ and $h \leqslant H$. (This implies that the numerical solution may be bounded when the solution of the differential system is unbounded.) More importantly, the theorem assumes the use of a single decomposition where the linear part remains constant throughout. It seems to be difficult to obtain a similar result for a system of the form $x' = J(t)x + g(t, x)$.

As before, if the trivial solution of $u' = J(t)u$ is exponentially stable and $\| g(t, u)\| = o(\| u \|)$, then the trivial solution of $u' = J(t)u + g(t, u)$ is exponentially stable. However, the stability property can no longer be characterized by simple conditions on the spectrum of $J(t)$. Nevertheless, the theorem suggests that additive methods have a role to play in the solution of stiff problems.

4. Numerical Results. Some numerical results are given to illustrate that the additive methods, obtained in this article, are stable in quite general situations. The results indicate that these methods may give competitive procedures for solving stiff problems.

We give results for only one method, the additive Runge-Kutta method represented by the array

$$
\begin{array}{c|cccc|cccc|c}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & \frac{1-\sqrt{3}}{6} & \frac{3+\sqrt{3}}{6} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} \\
2 & \frac{5+\sqrt{3}}{12} & -\frac{1+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{2}{3} \\
3 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 1
\end{array}
$$

This method was applied to various systems $x' = f(x)$ using the sequence of decompositions given by

$$f(x) = J^{(m)}x + g^{(m)}(x), \qquad m = 1, 2, 3, \ldots,$$

where $\{J^{(m)}\}$ is the Jacobian of f evaluated at the sequence of computed values $\{y_4^{(m-1)}\}$.

Consider the stiff initial value problem, given by Gear [4],

$$
\begin{aligned}
x_1' &= -0.013x_1 - 1000x_1x_3, & x_1(0) &= 1,\\
x_2' &= -2500x_2x_3, & x_2(0) &= 1,\\
x_3' &= -0.013x_1 - 1000x_1x_3 - 2500x_2x_3, & x_3(0) &= 0,
\end{aligned}
$$

where the Jacbian has real eigenvalues. Table 1.1 gives some typical values of these eigenvalues. This problem was integrated using a step length $h = 0.1$, and Table 1.2 compares the numerical results obtained, at $t = 1$ and $t = 50$, with the (rounded) solution values.

TABLE 1.1
Eigenvalues of the Jacobian

$t = 0$	$t = 25$	$t = 50$
0	0	0
-0.0093	-0.0069	-0.0088
-3500	-3287	-4104

TABLE 1.2

Comparison of results ($h = 0.1$)

	$t = 1$		$t = 50$	
	Exact	Numerical	Exact	Numerical
x_1	0.990 731 92	0.990 731 89	0.597 654 70	0.597 654 66
x_2	1.009 264 41	1.009 264 50	1.402 343 41	1.402 343 44
x_3	−0.000 003 67	−0.000 003 61	−0.000 001 89	−0.000 001 89

We also give a second set of results, obtained with $h = 1$, for another system given by Gear [4],

$$\begin{aligned} x_1' &= -55x_1 + 65x_2 - x_1x_3, & x_1(0) &= 1, \\ x_2' &= 0.0785(x_1 - x_2), & x_2(0) &= 1, \\ x_3' &= 0.1x_1, & x_3(0) &= 0, \end{aligned}$$

where the Jacobian has complex eigenvalues. Typical values of the eigenvalues are given in Table 2.1. Table 2.2 compares numerical results obtained with the solution values. For both problems the results given are the rounded values obtained after computation with 12 significant digits. Results for other methods are similar.

TABLE 2.1

Eigenvalues for the second problem

$t = 0$	$t = 300$	$t = 500$
$0.0062 + 0.01\,i$	$0.0014 + 0.014\,i$	−0.015
$0.0062 - 0.01\,i$	$0.0014 - 0.014\,i$	−0.004
−55	−63.5	−81

TABLE 2.2

Results for the second problem ($h = 1$)

	$t = 10$		$t = 500$	
	Exact	Numerical	Exact	Numerical
x_1	1.360 591 81	1.356 753 78	88.926 078 46	88.925 900 60
x_2	1.152 321 04	1.152 322 69	87.276 035 35	87.275 999 91
x_3	0.036 059 18	0.035 675 38	8.792 607 85	8.792 590 06

A number of comparisons have been made with semiexplicit Runge-Kutta methods. These methods require the use of a modified Newton iteration where, in each step, the Jacobian is kept constant throughout the iteration. For both types of method, the Jacobian was evaluated at the start of each step. When only one iteration per step is used the semiexplicit methods require about the same amount of computation as the additive methods but are less accurate. When more iterations are used the methods seem to be slightly more accurate than the additive methods but require more computation.

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