Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 6414960, 18 pages
https://doi.org/10.1155/2018/6414960

Research Article

Adiabatic Invariants for Generalized Fractional Birkhoffian Mechanics and Their Applications

C. J. Song

School of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou 215009, China

Correspondence should be addressed to C. J. Song; songchuanjingsun@126.com

Received 7 August 2018; Accepted 30 October 2018; Published 13 November 2018

Academic Editor: Isabel S. Jesus

Perturbation to Noether symmetry and adiabatic invariants are investigated for the generalized fractional Birkhoffian system with the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative, respectively. Firstly, differential equations of motion for the generalized fractional Birkhoffian system are established. Secondly, Noether symmetry and conserved quantity are studied. Thirdly, perturbation to Noether symmetry and adiabatic invariants are presented for the generalized fractional Birkhoffian mechanics. And finally, several applications are discussed to illustrate the results and methods.

1. Introduction

Compared with the integer order model, the fractional order model can describe the mechanical and physical behavior of the complex system more accurately. Miller and Ross [1] once pointed out that almost every field of science and engineering involves fractional calculus. Fractional calculus has been widely used in mathematics, physics, chemistry, signal processing, engineering, and so on [2]. Riemann-Liouville fractional derivative and Caputo fractional derivative are in common use. Cresson [3] presented two more general fractional derivatives in 2006, i.e., the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative.

The study of the fractional calculus of variations was started in 1996, when Riewe [4, 5] was considering how to deal with the friction force and other forms of dissipative force in the classical mechanics and the quantum mechanics. Since then, Agrawal [6, 7], Atanacković [8], Almeida [9, 10], and some other scholars [11, 12] also studied fractional variational problems. However, all of the results were related to Lagrangian system or Hamiltonian system. As a matter of fact, there is a more general system called Birkhoffian system, which was introduced in 1927 [13]. Birkhoffian mechanics is a generalization of Hamiltonian mechanics, which is described in detail in [14]. Birkhoffian dynamics has gained significant headways [15, 16]. Recently, in [17], Luo first established Birkhoffian mechanics with the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative.

There is a set of unique integral theory in analytical mechanics, which is useful in solving differential equations of motion. Besides, symmetry and conserved quantity can help to reveal the intrinsic physical properties of the dynamic system. The commonly used symmetry methods [18] are Noether symmetry method, Lie symmetry method, and Mei symmetry method. Noether symmetry refers to the invariance of the Hamilton action under the infinitesimal transformations [19]. Lie symmetry refers to the invariance of the differential equations of motion under the infinitesimal transformations [20]. Mei symmetry means the invariance of the forms of the differential equations of motion when the dynamical functions, such as the kinetic energy, the potential energy, the generalized forces, the Lagrangian, the Hamiltonian, and the Birkhoffian, are replaced by the transformed functions under the infinitesimal transformations. Many important research results have been achieved in terms of symmetry and conserved quantity of the constrained mechanical systems [21].

The fractional symmetry and conserved quantity were first studied by Frederico and Torres [22]. Based on the Riemann-Liouville fractional derivative, they established the fractional Noether theorem. Bourdin et al. [23] presented a new expression of the fractional conserved quantity and took the fractional harmonic oscillator as an example. Based on the Caputo fractional derivative, Muslih [24] extended the fractional Noether theorem of finite degree of freedom to the fractional field theory and presented the conserved quantity of the fractional Dirac field. In recent years, Zhang [25, 26], Jia [27], and Zhang [28] studied the Noether symmetry and conserved quantity of the fractional Birkhoffian system. Fu [29] investigated the Lie symmetry of the fractional nonholonomic Hamiltonian system and the corresponding inverse problem. Luo [30] presented the Mei symmetry and conserved quantity of the generalized fractional Hamiltonian system.

When the system suffers small disturbance, the original conserved quantity will change. Perturbation to symmetry and adiabatic invariants are closely related to the integrability of the system, and they have been widely used in many fields, such as mathematics, physics, and mechanics. The classical adiabatic invariant refers to a slower physical quantity [31] when a parameter changes slowly in the system. In 1981, Djukić [32] discussed the disturbed Hamiltonian system and gave the corresponding adiabatic invariant. In 1996, Zhao and Mei [33] pointed out that adiabatic invariant was not just the product of the Hamiltonian system. Thereafter, perturbation to symmetry and adiabatic invariants were studied in many constrained mechanical systems such as the Lagrangian system [34], the Hamiltonian system [35], the nonholonomic system [36], and the Birkhoffian system [37], as well as the fractional constrained mechanical system [38, 39]. It is worth mentioning that [39] presented conserved quantity and adiabatic invariant for the fractional generalized Birkhoffian system on the basis of Riemann-Liouville fractional derivative, where Riemann-Liouville fractional derivative is one of the special cases of the combined Riemann-Liouville fractional derivative.

Because the generalized fractional Birkhoffian system and the combined fractional derivatives are general, generalized fractional Birkhoff equation, Noether symmetry and conserved quantity, and perturbation to Noether symmetry and adiabatic invariant with the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative will be studied here. Then several special cases of this paper will be discussed. This paper is organized as follows. In Section 2, fractional derivatives and their properties are reviewed. In Section 3, generalized fractional Birkhoff equation is established through the generalized fractional Pfaff-Birkhoff-d'Alembert principle. Noether symmetry and conserved quantity for the generalized fractional Birkhoffian system are presented in Section 4. Perturbation to Noether symmetry and adiabatic invariant are investigated in Section 5. And in Section 6, several applications are discussed.

2. Some Preliminaries on Fractional Derivatives

Some fractional derivatives and their properties are listed in this section [17, 40].

Let $f(t)$ be continuous and integrable, then the left and the right Riemann-Liouville fractional derivatives are

$$ {}_{t_1}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t} (t-\xi)^{n-\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad (1)$$

$$ {}_{t}^{RL}D_{t_2}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t}^{t_2} (\xi-t)^{n-\beta-1} f(\xi)\,\mathrm{d}\xi, \quad (2)$$

the left and the right Caputo fractional derivatives are

$$ {}_{t_1}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_1}^{t} (t-\xi)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n f(\xi)\,\mathrm{d}\xi, \quad (3)$$

$$ {}_{t}^{C}D_{t_2}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_{t}^{t_2} (\xi-t)^{n-\beta-1} \left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n f(\xi)\,\mathrm{d}\xi, \quad (4)$$

and the Riesz-Riemann-Liouville fractional derivative and the Riesz-Caputo fractional derivative are

$$ {}_{t_1}^{R}D_{t_2}^{\alpha} f(t) = \frac{1}{2\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t_2} |t-\xi|^{n-\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad (5)$$

$$ {}_{t_1}^{RC}D_{t_2}^{\alpha} f(t) = \frac{1}{2\Gamma(n-\alpha)} \int_{t_1}^{t_2} |t-\xi|^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n f(\xi)\,\mathrm{d}\xi, \quad (6)$$

where α and β are the orders of the fractional derivatives with $n-1 \le \alpha, \beta < n$.

The combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative are defined as

$$ {}^{RL}D_{\gamma}^{\alpha,\beta} f(t) = \gamma\, {}_{t_1}^{RL}D_t^{\alpha} f(t) + (-1)^n (1-\gamma)\, {}_{t}^{RL}D_{t_2}^{\beta} f(t), \quad (7)$$

$$ {}^{C}D_{\gamma}^{\alpha,\beta} f(t) = \gamma\, {}_{t_1}^{C}D_t^{\alpha} f(t) + (-1)^n (1-\gamma)\, {}_{t}^{C}D_{t_2}^{\beta} f(t), \quad (8)$$

where ${}_{t_1}D_t^{\alpha}$ and ${}_{t}D_{t_2}^{\beta}$ can help deal with dynamical systems exhibiting the arrow of time and γ determines the different quantity of information from the past and the future to keep track of the past and the future of the dynamics.

When $\beta = \alpha$, $\gamma = 1/2$, formulae (7) and (8) reduce to formulae (5) and (6), respectively. That is, the Riesz-Riemann-Liouville fractional derivative and the Riesz-Caputo fractional derivative are the special cases of the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative.

Besides, when $\alpha, \beta \longrightarrow 1$, we have

$$ {}_{t_1}D_t^{1} = \frac{\mathrm{d}}{\mathrm{d}t}, \quad {}_{t}D_{t_2}^{1} = -\frac{\mathrm{d}}{\mathrm{d}t}, \quad D_{\gamma}^{\alpha,\beta} = \gamma\, {}_{t_1}D_t^{1} + (-1)^n (1-\gamma)\, {}_{t}D_{t_2}^{1} = \frac{\mathrm{d}}{\mathrm{d}t}. \quad (9)$$

The formulae for fractional integration by parts are

$$\begin{aligned}&\int_{t_1}^{t_2}[*]\,{}_{t_1}^{RL}D_t^{\alpha}\eta\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\eta\,{}_{t}^{C}D_{t_2}^{\alpha}[*]\,\mathrm{d}t\\&\quad-\sum_{j=0}^{n-1}(-1)^{n+j}\,{}_{t_1}^{RL}D_t^{\alpha+j-n}\eta(t)\,D^{n-1-j}[*]\Big|_{t_1}^{t_2},\end{aligned}\tag{10}$$

$$\begin{aligned}&\int_{t_1}^{t_2}[*]\,{}_{t}^{RL}D_{t_2}^{\beta}\eta\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\eta\,{}_{t_1}^{C}D_t^{\beta}[*]\,\mathrm{d}t\\&\quad-\sum_{j=0}^{n-1}{}_{t}^{RL}D_{t_2}^{\beta+j-n}[*]\,D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2},\end{aligned}\tag{11}$$

$$\begin{aligned}&\int_{t_1}^{t_2}[*]\,{}_{t_1}^{C}D_t^{\alpha}\eta\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\eta\,{}_{t}^{RL}D_{t_2}^{\alpha}[*]\,\mathrm{d}t\\&\quad+\sum_{j=0}^{n-1}{}_{t}^{RL}D_{t_2}^{\alpha+j-n}[*]\,D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2},\end{aligned}\tag{12}$$

$$\begin{aligned}&\int_{t_1}^{t_2}[*]\,{}_{t}^{C}D_{t_2}^{\beta}\eta\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\eta\,{}_{t_1}^{RL}D_t^{\beta}[*]\,\mathrm{d}t\\&\quad+\sum_{j=0}^{n-1}(-1)^{n+j}\,{}_{t_1}^{RL}D_t^{\beta+j-n}[*]\,D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2},\end{aligned}\tag{13}$$

$$\begin{aligned}&\int_{t_1}^{t_2}[*]\,{}_{t_1}^{R}D_{t_2}^{\alpha}\eta\,\mathrm{d}t\\&=(-1)^n\int_{t_1}^{t_2}\eta\,{}_{t_1}^{RC}D_{t_2}^{\alpha}[*]\,\mathrm{d}t\\&\quad+\sum_{j=0}^{n-1}(-1)^{n+j}\,{}_{t_1}^{R}D_{t_2}^{\alpha+j-n}\eta(t)\,D^{n-1-j}[*]\Big|_{t_1}^{t_2},\end{aligned}\tag{14}$$

$$\begin{aligned}&\int_{t_1}^{t_2}[*]\,{}_{t_1}^{RL}D_{t_2}^{\alpha}\eta\,\mathrm{d}t\\&=(-1)^n\int_{t_1}^{t_2}\eta\,{}_{t_1}^{R}D_{t_2}^{\alpha}[*]\,\mathrm{d}t\\&\quad+\sum_{j=0}^{n-1}(-1)^{j}\,{}_{t_1}^{R}D_{t_2}^{\alpha+j-n}[*]\,D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2}.\end{aligned}\tag{15}$$

3. Generalized Fractional Birkhoffian System

Generalized fractional Birkhoff equations in terms of the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative are established in this section.

Case 1. Based on the combined Riemann-Liouville fractional derivative, the fractional Pfaff action and the generalized fractional Pfaff-Birkhoff principle have their forms as follows:

$$\begin{aligned}&S_{RL}(a^{\nu}(\cdot))\\&\quad=\int_{t_1}^{t_2}\left(R_{\mu}(t,a^{\nu}(t))\,{}^{RL}D_{\gamma}^{\alpha,\beta}a^{\mu}(t)-B(t,a^{\nu}(t))\right)\mathrm{d}t,\end{aligned}\tag{16}$$

$$\begin{aligned}&\int_{t_1}^{t_2}\left[\delta\left(R_{\mu}\,{}^{RL}D_{\gamma}^{\alpha,\beta}a^{\mu}-B\right)+\delta' W\right]\mathrm{d}t=0,\\&{}_{t_1}^{RL}D_t^{\alpha}\delta a^{\mu}=\delta\,{}_{t_1}^{RL}D_t^{\alpha}a^{\mu},\\&{}_{t}^{RL}D_{t_2}^{\beta}\delta a^{\mu}=\delta\,{}_{t}^{RL}D_{t_2}^{\beta}a^{\mu},\\&\delta a^{\mu}\big|_{t=t_1}=\delta a^{\mu}\big|_{t=t_2}=0,\end{aligned}\tag{17}$$

where $a^{\mu}=a^{\mu}(t)$ is the Birkhoff's variable, $B=B(t,a^{\nu})$ is the Birkhoffian, $R_{\mu}=R_{\mu}(t,a^{\nu})$ is the Birkhoff's function, $\delta' W=\Lambda_{\mu}(t,a^{\nu})\delta a^{\mu}$, $\mu,\nu=1,2,\cdots,2n$, δ is the isochronous variation, $n-1\le\alpha,\beta<n$, and Einstein summation convention is used in this text.

When $0<\alpha,\beta<1$, it follows from

$$\begin{aligned}\int_{t_1}^{t_2}\gamma R_{\mu}\cdot\delta\,{}_{t_1}^{RL}D_t^{\alpha}a^{\mu}\,\mathrm{d}t&=\int_{t_1}^{t_2}\gamma R_{\mu}\cdot{}_{t_1}^{RL}D_t^{\alpha}\delta a^{\mu}\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\gamma\delta a^{\mu}\,{}_{t}^{C}D_{t_2}^{\alpha}R_{\mu}\,\mathrm{d}t\end{aligned}\tag{18}$$

and

$$\begin{aligned}&\int_{t_1}^{t_2}(1-\gamma)R_{\mu}\cdot\delta\,{}_{t}^{RL}D_{t_2}^{\beta}a^{\mu}\,\mathrm{d}t\\&\quad=\int_{t_1}^{t_2}(1-\gamma)R_{\mu}\cdot{}_{t}^{RL}D_{t_2}^{\beta}\delta a^{\mu}\,\mathrm{d}t\\&\quad=\int_{t_1}^{t_2}(1-\gamma)\delta a^{\mu}\,{}_{t_1}^{C}D_t^{\beta}R_{\mu}\,\mathrm{d}t\end{aligned}\tag{19}$$

that

$$\begin{aligned}&\int_{t_1}^{t_2}\left[\delta\left(R_{\mu}\,{}^{RL}D_{\gamma}^{\alpha,\beta}a^{\mu}-B\right)+\delta' W\right]\mathrm{d}t=\int_{t_1}^{t_2}\Big\{\delta R_{\mu}\\&\quad\cdot{}^{RL}D_{\gamma}^{\alpha,\beta}a^{\mu}+R_{\mu}\cdot\left[\gamma\delta\,{}_{t_1}^{RL}D_t^{\alpha}a^{\mu}\right.\\&\quad\left.-(1-\gamma)\,\delta\,{}_{t}^{RL}D_{t_2}^{\beta}a^{\mu}\right]-\delta B+\delta' W\Big\}\,\mathrm{d}t\\&\quad=\int_{t_1}^{t_2}\Big\{\Big[\frac{\partial R_{\nu}}{\partial a^{\mu}}\,{}^{RL}D_{\gamma}^{\alpha,\beta}a^{\nu}+\gamma\,{}_{t}^{C}D_{t_2}^{\alpha}R_{\mu}\\&\quad-(1-\gamma)\,{}_{t_1}^{C}D_t^{\beta}R_{\mu}-\frac{\partial B}{\partial a^{\mu}}+\Lambda_{\mu}\Big]\delta a^{\mu}\Big\}\,\mathrm{d}t\end{aligned}$$

$$= \int_{t_1}^{t_2} \left[\left(\frac{\partial R_\nu}{\partial a^\mu} {}^{RL}D_\gamma^{\alpha,\beta} a^\nu - {}^{C}D_{1-\gamma}^{\beta,\alpha} R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu \right) \cdot \delta a^\mu \right] \mathrm{d}t = 0. \tag{20}$$

From the arbitrariness of the integration interval $[t_1, t_2]$, we have

$$\left(\frac{\partial R_\nu}{\partial a^\mu} {}^{RL}D_\gamma^{\alpha,\beta} a^\nu - {}^{C}D_{1-\gamma}^{\beta,\alpha} R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu \right) \delta a^\mu = 0. \tag{21}$$

Formula (21) is called generalized fractional Pfaff-Birkhoff-d'Alembert principle with the combined Riemann-Liouville fractional derivative.

Due to the independence of δa^μ, we get

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{RL}D_\gamma^{\alpha,\beta} a^\nu - {}^{C}D_{1-\gamma}^{\beta,\alpha} R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = 0. \tag{22}$$

Equation (22) is called generalized fractional Birkhoff equation with the combined Riemann-Liouville fractional derivative.

Case 2. Based on the combined Caputo fractional derivative, the fractional Pfaff action and the generalized fractional Pfaff-Birkhoff principle have their forms as follows:

$$S_C\left(a^\nu(\cdot)\right) = \int_{t_1}^{t_2} \left(R_\mu\left(t, a^\nu(t)\right) {}^{C}D_\gamma^{\alpha,\beta} a^\mu(t) - B\left(t, a^\nu(t)\right) \right) \mathrm{d}t, \tag{23}$$

$$\begin{gathered} \int_{t_1}^{t_2} \left[\delta\left(R_\mu {}^{C}D_\gamma^{\alpha,\beta} a^\mu - B \right) + \delta' W \right] \mathrm{d}t = 0, \\ {}_{t_1}^{C}D_t^\alpha \delta a^\mu = \delta {}_{t_1}^{C}D_t^\alpha a^\mu, \\ {}_{t}^{C}D_{t_2}^\beta \delta a^\mu = \delta {}_{t}^{C}D_{t_2}^\beta a^\mu, \\ \delta a^\mu\big|_{t=t_1} = \delta a^\mu\big|_{t=t_2} = 0. \end{gathered} \tag{24}$$

Similarly, when $0 < \alpha, \beta < 1$, we get

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{C}D_\gamma^{\alpha,\beta} a^\nu - {}^{RL}D_{1-\gamma}^{\beta,\alpha} R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = 0. \tag{25}$$

Equation (25) is called generalized fractional Birkhoff equation with the combined Caputo fractional derivative.

Remark 1. When $\beta = \alpha$, $\gamma = 1/2$, from (22) and (25), we have

$$\frac{\partial R_\nu}{\partial a^\mu} {}_{t_1}^{R}D_{t_2}^\alpha a^\nu - {}_{t_1}^{RC}D_{t_2}^\alpha R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = 0 \tag{26}$$

and

$$\frac{\partial R_\nu}{\partial a^\mu} {}_{t_1}^{RC}D_{t_2}^\alpha a^\nu - {}_{t_1}^{R}D_{t_2}^\alpha R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = 0. \tag{27}$$

Equations (26) and (27) are called generalized fractional Birkhoff equations with the Riesz-Riemann-Liouville fractional derivative and the Riesz-Caputo fractional derivative, respectively.

Remark 2. When $\Lambda_\mu = 0$, $\mu = 1, 2, \cdots, 2n$, from (22), (25), (26), and (27), we have

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{RL}D_\gamma^{\alpha,\beta} a^\nu - {}^{C}D_{1-\gamma}^{\beta,\alpha} R_\mu - \frac{\partial B}{\partial a^\mu} = 0, \tag{28}$$

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{C}D_\gamma^{\alpha,\beta} a^\nu - {}^{RL}D_{1-\gamma}^{\beta,\alpha} R_\mu - \frac{\partial B}{\partial a^\mu} = 0, \tag{29}$$

$$\frac{\partial R_\nu}{\partial a^\mu} {}_{t_1}^{R}D_{t_2}^\alpha a^\nu - {}_{t_1}^{RC}D_{t_2}^\alpha R_\mu - \frac{\partial B}{\partial a^\mu} = 0, \tag{30}$$

and

$$\frac{\partial R_\nu}{\partial a^\mu} {}_{t_1}^{RC}D_{t_2}^\alpha a^\nu - {}_{t_1}^{R}D_{t_2}^\alpha R_\mu - \frac{\partial B}{\partial a^\mu} = 0. \tag{31}$$

Equations (28)-(31) are consistent with the results in [17].

Remark 3. When $0 < \alpha, \beta < 1$, the relationships between the Riemann-Liouville fractional derivative and the Caputo fractional derivative can be deduced from their definitions

$$\begin{aligned} {}_{t_1}^{RL}D_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_1}^{t} (t-\xi)^{-\alpha} f(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} -\alpha (t-\xi)^{-\alpha-1} f(\xi)\,\mathrm{d}\xi = \frac{1}{\Gamma(1-\alpha)} \cdot \int_{t_1}^{t} \left\{ f(\xi) \frac{\mathrm{d}}{\mathrm{d}\xi} \left[-(t-\xi)^{-\alpha} \right] \right\} \mathrm{d}\xi = \frac{1}{\Gamma(1-\alpha)} \\ &\cdot \int_{t_1}^{t} \left\{ \frac{\mathrm{d}}{\mathrm{d}\xi} \left[-(t-\xi)^{-\alpha} f(\xi) \right] - \left[-(t-\xi)^{-\alpha} \right] \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) \right\} \mathrm{d}\xi = \frac{1}{\Gamma(1-\alpha)} \\ &\cdot \int_{t_1}^{t} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[-(t-\xi)^{-\alpha} f(\xi) \right] \mathrm{d}\xi + \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} (t-\xi)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[-(t-\xi)^{-\alpha} f(\xi) \right] \mathrm{d}\xi + \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} (t-\xi)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi)\,\mathrm{d}\xi = {}_{t_1}^{C}D_t^\alpha f(t) \\ &- \frac{1}{\Gamma(1-\alpha)} \frac{f(t_1)}{(t-t_1)^\alpha}. \end{aligned} \tag{32}$$

That is,

$${}_{t_1}^{RL}D_t^\alpha f(t) = {}_{t_1}^{C}D_t^\alpha f(t) - \frac{1}{\Gamma(1-\alpha)} \frac{f(t_1)}{(t-t_1)^\alpha}. \tag{33}$$

Similarly, we have

$${}_{t}^{RL}D_{t_2}^\beta f(t) = {}_{t}^{C}D_{t_2}^\beta f(t) + \frac{1}{\Gamma(1-\beta)} \frac{f(t_2)}{(t_2-t)^\beta}. \tag{34}$$

Substituting formulae (33) and (34) into formulae (18) and (19), we get

$$\begin{aligned}\int_{t_1}^{t_2}\gamma R_\mu\cdot\delta\,{}^{RL}_{t_1}D^{\alpha}_{t}a^\mu\,\mathrm{d}t&=\int_{t_1}^{t_2}\gamma R_\mu\cdot{}^{RL}_{t_1}D^{\alpha}_{t}\delta a^\mu\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\gamma R_\mu\cdot\left[{}^{C}_{t_1}D^{\alpha}_{t}\delta a^\mu-\frac{1}{\Gamma(1-\alpha)}\frac{\delta a^\mu(t_1)}{(t-t_1)^\alpha}\right]\mathrm{d}t\\&=\int_{t_1}^{t_2}\gamma R_\mu\cdot{}^{C}_{t_1}D^{\alpha}_{t}\delta a^\mu\,\mathrm{d}t=\int_{t_1}^{t_2}\gamma\delta a^\mu\cdot{}^{RL}_{t}D^{\alpha}_{t_2}R_\mu\,\mathrm{d}t\end{aligned}\tag{35}$$

and

$$\begin{aligned}&\int_{t_1}^{t_2}(1-\gamma)R_\mu\cdot\delta\,{}^{RL}_{t}D^{\beta}_{t_2}a^\mu\,\mathrm{d}t\\&\quad=\int_{t_1}^{t_2}(1-\gamma)\delta a^\mu\,{}^{RL}_{t_1}D^{\beta}_{t}R_\mu\,\mathrm{d}t.\end{aligned}\tag{36}$$

Using formulae (35) and (36), another form of the generalized fractional Birkhoff equation with the combined Riemann-Liouville fractional derivative can be achieved

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{RL}D^{\alpha,\beta}_{\gamma}a^\nu-{}^{RL}D^{\beta,\alpha}_{1-\gamma}R_\mu-\frac{\partial B}{\partial a^\mu}+\Lambda_\mu=0.\tag{37}$$

Similarly, another form of the generalized fractional Birkhoff equation with the combined Caputo fractional derivative can be achieved

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{C}D^{\alpha,\beta}_{\gamma}a^\nu-{}^{C}D^{\beta,\alpha}_{1-\gamma}R_\mu-\frac{\partial B}{\partial a^\mu}+\Lambda_\mu=0.\tag{38}$$

Remark 4. When $\beta=\alpha$, $\gamma=1/2$, from (37) and (38), we can get other forms of the generalized fractional Birkhoff equations with the Riesz-Riemann-Liouville fractional derivative and the Riesz-Caputo fractional derivative

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{R}_{t_1}D^{\alpha}_{t_2}a^\nu-{}^{R}_{t_1}D^{\alpha}_{t_2}R_\mu-\frac{\partial B}{\partial a^\mu}+\Lambda_\mu=0,\tag{39}$$

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{RC}_{t_1}D^{\alpha}_{t_2}a^\nu-{}^{RC}_{t_1}D^{\alpha}_{t_2}R_\mu-\frac{\partial B}{\partial a^\mu}+\Lambda_\mu=0.\tag{40}$$

Remark 5. When $\gamma=1$, it follows from (37) that

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{RL}_{t_1}D^{\alpha}_{t}a^\nu+{}^{RL}_{t}D^{\alpha}_{t_2}R_\mu-\frac{\partial B}{\partial a^\mu}+\Lambda_\mu=0.\tag{41}$$

Equation (41) is the generalized fractional Birkhoff equation with the Riemann-Liouville fractional derivative [39].

Remark 6. When $\gamma=1$, $\Lambda_\mu=0$, $\mu=1,2,\cdots,2n$, it follows from (37) that

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{RL}_{t_1}D^{\alpha}_{t}a^\nu+{}^{RL}_{t}D^{\alpha}_{t_2}R_\mu-\frac{\partial B}{\partial a^\mu}=0.\tag{42}$$

Equation (42) is the fractional Birkhoff equation with the Riemann-Liouville fractional derivative [41].

Remark 7. When $\alpha,\beta\longrightarrow 1$, (22), (25), (26), and (27) all reduce to the classical generalized Birkhoff equation [21]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\Lambda_\mu=0.\tag{43}$$

The following discussions are all carried out under the condition $0<\alpha,\beta<1$.

4. Noether Symmetry and Conserved Quantity

Assuming the infinitesimal transformations are

$$\begin{aligned}\bar t&=t+\Delta t,\\\bar a^\nu(\bar t)&=a^\nu(t)+\Delta a^\nu,\end{aligned}\tag{44}$$

and their expansions are

$$\begin{aligned}\bar t&=t+\varepsilon\xi^0_0(t,a^\mu)+o(\varepsilon),\\\bar a^\nu(\bar t)&=a^\nu(t)+\varepsilon\xi^0_\nu(t,a^\mu)+o(\varepsilon),\end{aligned}\tag{45}$$

where ε is an infinitesimal parameter and ξ^0_0 and ξ^0_ν are infinitesimal generators, $o(\varepsilon)$ will be ignored in the following calculation.

The fractional Pfaff actions (16) and (23) will change under the infinitesimal transformations (44); we denote them as $\Delta S_{RL}=S_{RL}(\bar a^\nu(\bar t))-S_{RL}(a^\nu(t))$ and $\Delta S_C=S_C(\bar a^\nu(\bar t))-S_C(a^\nu(t))$, respectively.

Definition 8. The infinitesimal transformations (44) are called Noether symmetric transformations if and only if

$$\begin{aligned}&\Delta S_{RL}=-\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)+\Lambda_\mu\delta a^\mu\right]\mathrm{d}t\\&\qquad\left(\text{resp. }\Delta S_C=-\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)+\Lambda_\mu\delta a^\mu\right]\mathrm{d}t\right)\end{aligned}\tag{46}$$

holds, where $\Delta G=\varepsilon G^0(t,a^\nu)$, G^0 is called a gauge function.

Noether symmetry can be verified from Noether symmetric transformations.

It follows from formula (46) and

$$\begin{aligned}\Delta S_{RL}&=S_{RL}(\bar a^\nu(\bar t))-S_{RL}(a^\nu(t))=\int_{\bar t_1}^{\bar t_2}\Big(R_\mu(\bar t,\bar a^\nu(\bar t))\\&\quad\cdot\left[\gamma\,{}^{RL}_{\bar t_1}D^{\alpha}_{\bar t}-(1-\gamma)\,{}^{RL}_{\bar t}D^{\beta}_{\bar t_2}\right]{}^{RL}D^{\alpha,\beta}_{\gamma}\bar a^\mu(\bar t)\\&\quad-B(\bar t,\bar a^\nu(\bar t))\Big)\mathrm{d}\bar t-\int_{t_1}^{t_2}\Big(R_\mu(t,a^\nu(t))\\&\quad\cdot{}^{RL}D^{\alpha,\beta}_{\gamma}a^\mu(t)-B(t,a^\nu(t))\Big)\mathrm{d}t\\&=\int_{t_1}^{t_2}\Big\{R_\nu\,{}^{\mathrm{RL}}D^{\alpha,\beta}_{\gamma}\delta a^\nu\\&\quad+\left(R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D^{\alpha,\beta}_{\gamma}a^\nu+\frac{\partial R_\nu}{\partial t}\,{}^{\mathrm{RL}}D^{\alpha,\beta}_{\gamma}a^\nu-\frac{\partial B}{\partial t}\right)\Delta t\end{aligned}$$

$$+\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)\Delta a^\mu$$
$$-\gamma\left[R_\nu\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\Delta t_1\right]$$
$$+\left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t+(1-\gamma)$$
$$\cdot\left[R_\nu\frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^\nu(t_2)\Delta t_2\right]\Bigg\}\,\mathrm{d}t \quad (47)$$

that

$$R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^0-\dot a^\nu\xi_0^0\right)$$
$$+\left(R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu+\frac{\partial R_\nu}{\partial t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)\xi_0^0$$
$$+\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu^0-\gamma\frac{R_\nu}{\Gamma(1-\alpha)}$$
$$\cdot\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\xi_0^0(t_1,a^\mu(t_1)) \quad (48)$$
$$+\left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)\dot\xi_0^0+(1-\gamma)\frac{R_\nu}{\Gamma(1-\beta)}$$
$$\cdot\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^\nu(t_2)\xi_0^0(t_2,a^\mu(t_2))+\dot G^0$$
$$+\Lambda_\mu\left(\xi_\mu^0-\dot a^\mu\xi_0^0\right)=0.$$

Formula (48) is called Noether identity with the combined Riemann-Liouville fractional derivative.

Similarly, we have

$$R_\nu\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^0-\dot a^\nu\xi_0^0\right)$$
$$+\left(R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu+\frac{\partial R_\nu}{\partial t}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial t}\right)\xi_0^0$$
$$+\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu^0$$
$$-R_\nu\frac{\gamma}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot a^\nu(t_1)\xi_0^0(t_1,a^\mu(t_1)) \quad (49)$$
$$+\left(R_\nu\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^\nu-B\right)\dot\xi_0^0$$
$$+(1-\gamma)\frac{R_\nu}{\Gamma(1-\beta)}(t_2-t)^{-\beta}\dot a^\nu(t_2)\xi_0^0(t_2,a^\mu(t_2))$$
$$+\dot G^0+\Lambda_\mu\left(\xi_\mu^0-\dot a^\mu\xi_0^0\right)=0.$$

Formula (49) is called Noether identity with the combined Caputo fractional derivative.

Remark 9. When $\beta=\alpha$, $\gamma=1/2$, from formulae (48) and (49), we get

$$R_\nu\,{}_{t_1}^{R}D_{t_2}^{\alpha}\left(\xi_\nu^0-\dot a^\nu\xi_0^0\right)$$
$$+\left(R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{R}D_{t_2}^{\alpha}a^\nu+\frac{\partial R_\nu}{\partial t}\,{}_{t_1}^{R}D_{t_2}^{\alpha}a^\nu-\frac{\partial B}{\partial t}\right)\xi_0^0$$
$$+\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_{t_1}^{R}D_{t_2}^{\alpha}a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu^0$$
$$+\frac{R_\nu}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_2\right|^{-\alpha}a^\nu(t_2)\xi_0^0(t_2,a^\mu(t_2)) \quad (50)$$
$$+\left(R_\nu\,{}_{t_1}^{R}D_{t_2}^{\alpha}a^\nu-B\right)\dot\xi_0^0$$
$$-\frac{R_\nu}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha}a^\nu(t_1)\xi_0^0(t_1,a^\mu(t_1))$$
$$+\dot G^0+\Lambda_\mu\left(\xi_\mu^0-\dot a^\mu\xi_0^0\right)=0$$

and

$$R_\nu\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\left(\xi_\nu^0-\dot a^\nu\xi_0^0\right)+\left(R_\nu\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^\nu\right.$$
$$\left.+\frac{\partial R_\nu}{\partial t}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^\nu-\frac{\partial B}{\partial t}\right)\xi_0^0$$
$$+\frac{R_\nu}{2\Gamma(1-\alpha)}\left[\left|t-t_2\right|^{-\alpha}\dot a^\nu(t_2)\xi_0^0(t_2,a^\mu(t_2))\right. \quad (51)$$
$$\left.-\left|t-t_1\right|^{-\alpha}\dot a^\nu(t_1)\xi_0^0(t_1,a^\mu(t_1))\right]+\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_{t_1}^{RC}D_{t_2}^{\alpha}a^\nu\right.$$
$$\left.-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu^0+\left(R_\nu\,{}_{t_1}^{RC}D_{t_2}^{\alpha}a^\nu-B\right)\dot\xi_0^0+\dot G^0+\Lambda_\mu\left(\xi_\mu^0\right.$$
$$\left.-\dot a^\mu\xi_0^0\right)=0.$$

Formula (50) and formula (51) are called Noether identities with the Riesz-Riemann-Liouville fractional derivative and the Riesz-Caputo fractional derivative, respectively.

Definition 10. A quantity I is called a conserved quantity if and only if $(\mathrm{d}/\mathrm{d}t)I=0$.

Therefore, we have the following.

Theorem 11. *If there exists a gauge function G^0 such that ξ_0^0 and ξ_ν^0 satisfy the Noether identity (48) for the generalized fractional Birkhoffian system (22), then this system has a conserved quantity*

$$I_{\mathrm{RL}0}=\left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu-B\right)\xi_0^0+\int_{t_1}^{t}\left[R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^0\right.\right.$$
$$\left.\left.-\dot a^\nu\xi_0^0\right)+\left(\xi_\nu^0-\dot a^\nu\xi_0^0\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\nu\right]\mathrm{d}\tau$$
$$-\int_{t_1}^{t}\left[\frac{R_\nu}{\Gamma(1-\alpha)}\gamma\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t_1)^{-\alpha}a^\nu(t_1)\right.$$

$$\cdot \xi_0^0 (t_1, a^\mu (t_1)) - \frac{R_\nu}{\Gamma(1-\beta)} (1 - \gamma) \frac{\mathrm{d}}{\mathrm{d}\tau} (t_2 - \tau)^{-\beta} a^\nu (t_2) \xi_0^0 (t_2, a^\mu (t_2)) \Big] \mathrm{d}\tau + G^0 = \text{const.}. \tag{52}$$

Theorem 12. *If there exists a gauge function G^0 such that ξ_0^0 and ξ_ν^0 satisfy the Noether identity (49) for the generalized fractional Birkhoffian system (25), then this system has a conserved quantity*

$$\begin{aligned} I_{\mathrm{C}0} &= \left(R_\nu {}^{\mathrm{C}}D_\gamma^{\alpha,\beta} a^\nu - B\right) \xi_0^0 + \int_{t_1}^{t} \left[R_\nu {}^{\mathrm{C}}D_\gamma^{\alpha,\beta} \left(\xi_\nu^0 - \dot{a}^\nu \xi_0^0\right) + \left(\xi_\nu^0 - \dot{a}^\nu \xi_0^0\right) \cdot {}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha} R_\nu\right] \mathrm{d}\tau - \int_{t_1}^{t} \left[\frac{R_\nu}{\Gamma(1-\alpha)} \gamma (\tau - t_1)^{-\alpha} \right. \\ &\cdot \dot{a}^\nu (t_1) \xi_0^0 (t_1, a^\mu (t_1)) - \frac{R_\nu}{\Gamma(1-\beta)} (1-\gamma) \cdot \left. (t_2 - \tau)^{-\beta} \dot{a}^\nu (t_2) \xi_0^0 (t_2, a^\mu (t_2)) \right] \mathrm{d}\tau + G^0 = \text{const.}. \end{aligned} \tag{53}$$

Theorem 13. *If there exists a gauge function G^0 such that ξ_0^0 and ξ_ν^0 satisfy the Noether identity (50) for the generalized fractional Birkhoffian system (26), then this system has a conserved quantity*

$$\begin{aligned} I_{\mathrm{R}0} &= \left(R_\nu {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu - B\right) \xi_0^0 + \int_{t_1}^{t} \left[R_\nu {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} \left(\xi_\nu^0 - \dot{a}^\nu \xi_0^0\right) + \left(\xi_\nu^0 - \dot{a}^\nu \xi_0^0\right) {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} R_\nu\right] \mathrm{d}\tau \\ &+ \int_{t_1}^{t} \frac{R_\nu}{2\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\left|\tau - t_2\right|^{-\alpha} a^\nu (t_2) \cdot \xi_0^0 (t_2, a^\mu (t_2)) - \left|\tau - t_1\right|^{-\alpha} a^\nu (t_1) \cdot \xi_0^0 (t_1, a^\mu (t_1))\right] \mathrm{d}\tau + G^0 = \text{const.}. \end{aligned} \tag{54}$$

Theorem 14. *If there exists a gauge function G^0 such that ξ_0^0 and ξ_ν^0 satisfy the Noether identity (51) for the generalized fractional Birkhoffian system (27), then this system has a conserved quantity*

$$\begin{aligned} I_{\mathrm{RC}0} &= \left(R_\nu {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} a^\nu - B\right) \xi_0^0 + \int_{t_1}^{t} \left[R_\nu {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} \left(\xi_\nu^0 - \dot{a}^\nu \xi_0^0\right) + \left(\xi_\nu^0 - \dot{a}^\nu \xi_0^0\right) \cdot {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} R_\nu\right] \mathrm{d}\tau + \int_{t_1}^{t} \frac{R_\nu}{2\Gamma(1-\alpha)} \left[\left|\tau - t_2\right|^{-\alpha} \right. \\ &\cdot \dot{a}^\nu (t_2) \xi_0^0 (t_2, a^\mu (t_2)) - \left|\tau - t_1\right|^{-\alpha} \dot{a}^\nu (t_1) \cdot \left. \xi_0^0 (t_1, a^\mu (t_1))\right] \mathrm{d}\tau + G^0 = \text{const.}. \end{aligned} \tag{55}$$

When $\alpha, \beta \longrightarrow 1$, we can get the classical conserved quantity [21].

Theorem 15. *If there exists a gauge function G^0 such that ξ_0^0 and ξ_ν^0 satisfy the Noether identity*

$$R_\nu \dot{\xi}_\nu^0 - B\dot{\xi}_0^0 + \left(\frac{\partial R_\nu}{\partial t} \dot{a}^\nu - \frac{\partial B}{\partial t}\right) \xi_0^0 + \left(\frac{\partial R_\nu}{\partial a^\mu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right) \xi_\mu^0 + \dot{G}^0 + \Lambda_\mu \left(\xi_\mu^0 - \dot{a}^\mu \xi_0^0\right) = 0 \tag{56}$$

for the classical generalized Birkhoffian system

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} + \Lambda_\mu = 0, \tag{57}$$

then this system has a conserved quantity

$$I = R_\nu \xi_\nu^0 - B\xi_0^0 + G^0. \tag{58}$$

Remark 16. Let $G^0 = 0$, $\Lambda_\mu = 0$, $\mu = 1, 2, \cdots, 2n$ in Theorems 11–14; then we can get the conserved quantity for the fractional Birkhoffian system with the combined Riemann-Liouville fractional derivative, the combined Caputo fractional derivative, the Riesz-Riemann-Liouville fractional derivative, and the Riesz-Caputo fractional derivative, respectively. And these results are consistent with the results in [26].

5. Perturbation to Noether Symmetry and Adiabatic Invariants

If a small disturbance is imposed on a mechanical system, then the conserved quantity for the original mechanical system may also change. In this section, we discuss the perturbation to Noether symmetry for the generalized fractional Birkhoffian system.

Definition 17. If I_z is a quantity with a small parameter ε whose highest power is z for a mechanical system, and $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}, then I_z is called a z-th order adiabatic invariant for this system.

For the generalized fractional Birkhoffian system, assume it is disturbed by the small force of εQ_μ ($\mu = 1, 2, \cdots, 2n$). Under the small disturbance, suppose the gauge function G and the infinitesimal generators ξ_0 and ξ_μ for the disturbed system are expressed as

$$\begin{aligned} G &= G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots, \\ \xi_0 &= \xi_0^0 + \varepsilon \xi_0^1 + \varepsilon^2 \xi_0^2 + \cdots, \\ \xi_\mu &= \xi_\mu^0 + \varepsilon \xi_\mu^1 + \varepsilon^2 \xi_\mu^2 + \cdots. \end{aligned} \tag{59}$$

Then we have the following.

Theorem 18. *For the disturbed generalized fractional Birkhoffian system with the combined Riemann-Liouville fractional derivative*

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = \varepsilon Q_\mu \quad (\mu,\nu = 1,2,\cdots,2n), \tag{60}$$

if there exists a gauge function G^j ($j = 0,1,2,\cdots,z$, z is an integer) such that the infinitesimal generators ξ_0^j and ξ_ν^j satisfy

$$\begin{aligned}
&R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^j - \dot a^\nu \xi_0^j\right) + \left(R_\nu \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu + \frac{\partial R_\nu}{\partial t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial t}\right)\xi_0^j \\
&+ \left(\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu^j - \gamma\frac{R_\nu}{\Gamma(1-\alpha)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\,\xi_0^j(t_1,a^\mu(t_1)) \\
&+ \left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - B\right)\dot\xi_0^j + (1-\gamma)\frac{R_\nu}{\Gamma(1-\beta)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^\nu(t_2)\,\xi_0^j(t_2,a^\mu(t_2)) + \dot G^j \\
&+ \Lambda_\mu\left(\xi_\mu^j - \dot a^\mu \xi_0^j\right) - Q_\mu\left(\xi_\mu^{j-1} - \dot a^\mu \xi_0^{j-1}\right) = 0,
\end{aligned} \tag{61}$$

where $\xi_\mu^{j-1} = \xi_0^{j-1} = 0$ when $j = 0$, then a z-th order adiabatic invariant exists as follows:

$$\begin{aligned}
I_{\mathrm{RLz}} = \sum_{j=0}^{z}\varepsilon^j \Bigg\{ &\left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - B\right)\xi_0^j + \int_{t_1}^{t}\left[R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^j - \dot a^\nu\xi_0^j\right) + \left(\xi_\nu^j - \dot a^\nu \xi_0^j\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\nu\right]\mathrm{d}\tau \\
&- \int_{t_1}^{t}\left[\frac{R_\nu}{\Gamma(1-\alpha)}\gamma\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t_1)^{-\alpha}a^\nu(t_1)\,\xi_0^j(t_1,a^\mu(t_1)) - \frac{R_\nu}{\Gamma(1-\beta)}(1-\gamma)\frac{\mathrm{d}}{\mathrm{d}\tau}(t_2-\tau)^{-\beta}a^\nu(t_2)\,\xi_0^j(t_2,a^\mu(t_2))\right]\mathrm{d}\tau \\
&+ G^j\Bigg\}.
\end{aligned} \tag{62}$$

Proof. From formulae (60) and (61), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{RLz}} = \sum_{j=0}^{z}\varepsilon^j \Bigg[&\left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - B\right)\dot\xi_0^j + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot a^\mu + \frac{\partial R_\nu}{\partial t}\right) \cdot \xi_0^j\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu + R_\nu \xi_0^j \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \left(\frac{\partial B}{\partial a^\mu}\dot a^\mu + \frac{\partial B}{\partial t}\right)\xi_0^j \\
&+ R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^j - \dot a^\nu\xi_0^j\right) + \left(\xi_\nu^j - \dot a^\nu\xi_0^j\right){}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\nu \\
&- \frac{R_\nu}{\Gamma(1-\alpha)}\gamma\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\,\xi_0^j(t_1,a^\mu(t_1)) \\
&+ \frac{R_\nu}{\Gamma(1-\beta)}(1-\gamma)\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^\nu(t_2)\,\xi_0^j(t_2,a^\mu(t_2)) \\
&+ \dot G^j\Bigg] \\
= \sum_{j=0}^{z}\varepsilon^j \Bigg[&-\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - {}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu\right)\left(\xi_\mu^j - \dot a^\mu\xi_0^j\right) + Q_\mu\left(\xi_\mu^{j-1} - \dot a^\mu\xi_0^{j-1}\right)\Bigg] = \sum_{j=0}^{z}\varepsilon^j\left[-\varepsilon Q_\mu\left(\xi_\mu^j - \dot a^\mu\xi_0^j\right) + Q_\mu\left(\xi_\mu^{j-1} - \dot a^\mu\xi_0^{j-1}\right)\right] = -\varepsilon^{z+1}Q_\mu\left(\xi_\mu^z - \dot a^\mu\xi_0^z\right).
\end{aligned} \tag{63}$$

This proof is completed. □

Because the generalized fractional Birkhoffian system with the combined Riemann-Liouville fractional derivative has another form (37), we can present another form of the adiabatic invariant when the system is disturbed.

Theorem 19. *For the disturbed generalized fractional Birkhoffian system with the combined Riemann-Liouville fractional derivative*

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - {}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = \varepsilon Q_\mu \quad (\mu,\nu = 1,2,\cdots,2n), \tag{64}$$

if there exists a gauge function G^j ($j = 0,1,2,\cdots,z$, z is an integer) such that the infinitesimal generators ξ_0^j and ξ_ν^j satisfy

$$\begin{aligned}
&R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_\nu^j - \dot a^\nu \xi_0^j\right) + \left(R_\nu \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu + \frac{\partial R_\nu}{\partial t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial t}\right)\xi_0^j \\
&+ \left(\frac{\partial R_\nu}{\partial a^\mu}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu^j - \gamma\frac{R_\nu}{\Gamma(1-\alpha)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^\nu(t_1)\,\xi_0^j(t_1,a^\mu(t_1)) \\
&+ \left(R_\nu\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^\nu - B\right)\dot\xi_0^j + (1-\gamma)\frac{R_\nu}{\Gamma(1-\beta)}
\end{aligned}$$

$$\cdot \frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta} a^{\nu}(t_2)\xi_0^j(t_2,a^{\mu}(t_2)) + \dot{G}^j + \Lambda_{\mu}(\xi_{\mu}^j - \dot{a}^{\mu}\xi_0^j) - Q_{\mu}(\xi_{\mu}^{j-1} - \dot{a}^{\mu}\xi_0^{j-1}) = 0, \quad (65)$$

where $\xi_{\mu}^{j-1} = \xi_0^{j-1} = 0$ when $j = 0$, then a z-th order adiabatic invariant exists as follows:

$$\begin{aligned} I_{\mathrm{RLz1}} = \sum_{j=0}^{z}\varepsilon^j \Big\{ & \left(R_{\nu}\, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^{\nu} - B\right)\xi_0^j + \int_{t_1}^{t}\Big[R_{\nu}\, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}(\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j) \\ & + (\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j)\, {}^{RL}D_{1-\gamma}^{\beta,\alpha}R_{\nu}\Big]\mathrm{d}\tau - \int_{t_1}^{t}\Big[\frac{R_{\nu}}{\Gamma(1-\alpha)} \\ & \cdot \gamma\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t_1)^{-\alpha}a^{\nu}(t_1)\xi_0^j(t_1,a^{\mu}(t_1)) \\ & - \frac{R_{\nu}}{\Gamma(1-\beta)}(1-\gamma)\frac{\mathrm{d}}{\mathrm{d}\tau}(t_2-\tau)^{-\beta}a^{\nu}(t_2)\xi_0^j(t_2,a^{\mu}(t_2))\Big]\mathrm{d}\tau \\ & + G^j\Big\}. \end{aligned} \quad (66)$$

When $\gamma = 1$, we can get special theorem from Theorem 19 as follows.

Theorem 20. *For the disturbed generalized fractional Birkhoffian system*

$$\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}^{RL}_{t_1}D_t^{\alpha}a^{\nu} + {}^{RL}_{t}D_{t_2}^{\alpha}R_{\mu} - \frac{\partial B}{\partial a^{\mu}} + \Lambda_{\mu} = \varepsilon Q_{\mu} \quad (\mu,\nu = 1,2,\cdots,2n), \quad (67)$$

if there exists a gauge function G^j ($j = 0,1,2,\cdots,z$, z is an integer) such that the infinitesimal generators ξ_0^j and ξ_{ν}^j satisfy

$$\begin{aligned} & R_{\nu}\, {}^{RL}_{t_1}D_t^{\alpha}(\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j) \\ & + \left(R_{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\, {}^{RL}_{t_1}D_t^{\alpha}a^{\nu} + \frac{\partial R_{\nu}}{\partial t}\, {}^{RL}_{t_1}D_t^{\alpha}a^{\nu} - \frac{\partial B}{\partial t}\right)\xi_0^j \\ & + \left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}^{RL}_{t_1}D_t^{\alpha}a^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right)\xi_{\mu}^j \\ & - \frac{R_{\nu}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^{\nu}(t_1)\xi_0^j(t_1,a^{\mu}(t_1)) \\ & + \left(R_{\nu}\, {}^{RL}_{t_1}D_t^{\alpha}a^{\nu} - B\right)\dot{\xi}_0^j + \dot{G}^j + \Lambda_{\mu}(\xi_{\mu}^j - \dot{a}^{\mu}\xi_0^j) \\ & - Q_{\mu}(\xi_{\mu}^{j-1} - \dot{a}^{\mu}\xi_0^{j-1}) = 0, \end{aligned} \quad (68)$$

where $\xi_{\mu}^{j-1} = \xi_0^{j-1} = 0$ when $j = 0$, then a z-th order adiabatic invariant exists as follows:

$$\begin{aligned} I_{\mathrm{RLz11}} = \sum_{j=0}^{z}\varepsilon^j \Big\{ & \left(R_{\nu}\, {}^{RL}_{t_1}D_t^{\alpha}a^{\nu} - B\right)\xi_0^j \\ & + \int_{t_1}^{t}\Big[R_{\nu}\, {}^{RL}_{t_1}D_{\tau}^{\alpha}(\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j) \\ & + (\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j)\, {}^{RL}_{\tau}D_{t_2}^{\alpha}R_{\nu}\Big]\mathrm{d}\tau \\ & - \int_{t_1}^{t}\frac{R_{\nu}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t_1)^{-\alpha}a^{\nu}(t_1) \\ & \cdot \xi_0^j(t_1,a^{\mu}(t_1))\mathrm{d}\tau + G^j\Big\}. \end{aligned} \quad (69)$$

Remark 21. When $a^{\nu}(t_1) = 0$ ($\nu = 1,2,\cdots,2n$), Theorem 20 is consistent with the results in [39].

Theorem 22. *For the disturbed generalized fractional Birkhoffian system with the combined Caputo fractional derivative*

$$\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}^{C}D_{\gamma}^{\alpha,\beta}a^{\nu} - {}^{RL}D_{1-\gamma}^{\beta,\alpha}R_{\mu} - \frac{\partial B}{\partial a^{\mu}} + \Lambda_{\mu} = \varepsilon Q_{\mu} \quad (\mu,\nu = 1,2,\cdots,2n), \quad (70)$$

if there exists a gauge function G^j ($j = 0,1,2,\cdots,z$, z is an integer) such that the infinitesimal generators ξ_0^j and ξ_{ν}^j satisfy

$$\begin{aligned} & R_{\nu}\, {}^{C}D_{\gamma}^{\alpha,\beta}(\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j) \\ & + \left(R_{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\, {}^{C}D_{\gamma}^{\alpha,\beta}a^{\nu} + \frac{\partial R_{\nu}}{\partial t}\, {}^{C}D_{\gamma}^{\alpha,\beta}a^{\nu} - \frac{\partial B}{\partial t}\right)\xi_0^j \\ & + \left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}^{C}D_{\gamma}^{\alpha,\beta}a^{\nu} - \frac{\partial B}{\partial a^{\mu}}\right)\xi_{\mu}^j \\ & - \gamma\frac{R_{\nu}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{a}^{\nu}(t_1)\xi_0^j(t_1,a^{\mu}(t_1)) \\ & + \left(R_{\nu}\, {}^{C}D_{\gamma}^{\alpha,\beta}a^{\nu} - B\right)\dot{\xi}_0^j \\ & + (1-\gamma)\frac{R_{\nu}}{\Gamma(1-\beta)}(t_2-t)^{-\beta}\dot{a}^{\nu}(t_2)\xi_0^j(t_2,a^{\mu}(t_2)) \\ & + \dot{G}^j + \Lambda_{\mu}(\xi_{\mu}^j - \dot{a}^{\mu}\xi_0^j) - Q_{\mu}(\xi_{\mu}^{j-1} - \dot{a}^{\mu}\xi_0^{j-1}) = 0, \end{aligned} \quad (71)$$

where $\xi_{\mu}^{j-1} = \xi_0^{j-1} = 0$ when $j = 0$, then a z-th order adiabatic invariant exists as follows:

$$\begin{aligned} I_{\mathrm{Cz}} = \sum_{j=0}^{z}\varepsilon^j \Big\{ & \left(R_{\nu}\, {}^{C}D_{\gamma}^{\alpha,\beta}a^{\nu} - B\right)\xi_0^j \\ & + \int_{t_1}^{t}\Big[R_{\nu}\, {}^{C}D_{\gamma}^{\alpha,\beta}(\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j) + (\xi_{\nu}^j - \dot{a}^{\nu}\xi_0^j) \\ & \cdot {}^{RL}D_{1-\gamma}^{\beta,\alpha}R_{\nu}\Big]\mathrm{d}\tau - \int_{t_1}^{t}\Big[\frac{R_{\nu}}{\Gamma(1-\alpha)}\gamma(\tau-t_1)^{-\alpha} \\ & \cdot \dot{a}^{\nu}(t_1)\xi_0^j(t_1,a^{\mu}(t_1)) - \frac{R_{\nu}}{\Gamma(1-\beta)}(1-\gamma) \\ & \cdot (t_2-\tau)^{-\beta}\dot{a}^{\nu}(t_2)\xi_0^j(t_2,a^{\mu}(t_2))\Big]\mathrm{d}\tau + G^j\Big\}. \end{aligned} \quad (72)$$

Remark 23. The generalized fractional Birkhoffian system with the combined Caputo fractional derivative has another form (38), so we can also present another form of the adiabatic invariant when the system is disturbed just like Theorem 19.

When $\beta = \alpha$, $\gamma = 1/2$, we can get the adiabatic invariants for the disturbed generalized fractional Birkhoffian system with the Riesz-Riemann-Liouville fractional derivative and the Riesz-Caputo fractional derivative from Theorems 18 and 22, respectively.

Theorem 24. *For the disturbed generalized fractional Birkhoffian system with the Riesz-Riemann-Liouville fractional derivative*

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{R}_{t_1}D^{\alpha}_{t_2} a^\nu - {}^{RC}_{t_1}D^{\alpha}_{t_2} R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = \varepsilon Q_\mu \quad (\mu, \nu = 1, 2, \cdots, 2n), \tag{73}$$

if there exists a gauge function G^j ($j = 0, 1, 2, \cdots, z$, z is an integer) such that the infinitesimal generators ξ_0^j and ξ_ν^j satisfy

$$\begin{aligned}
&R_\nu {}^{R}_{t_1}D^{\alpha}_{t_2} \left(\xi_\nu^j - \dot{a}^\nu \xi_0^j\right) \\
&+ \left(R_\nu \frac{d}{dt} {}^{R}_{t_1}D^{\alpha}_{t_2} a^\nu + \frac{\partial R_\nu}{\partial t} {}^{R}_{t_1}D^{\alpha}_{t_2} a^\nu - \frac{\partial B}{\partial t}\right) \xi_0^j \\
&+ \left(\frac{\partial R_\nu}{\partial a^\mu} {}^{R}_{t_1}D^{\alpha}_{t_2} a^\nu - \frac{\partial B}{\partial a^\mu}\right) \xi_\mu^j \\
&+ \frac{R_\nu}{2\Gamma(1-\alpha)} \frac{d}{dt} \left|t - t_2\right|^{-\alpha} a^\nu(t_2)\, \xi_0^j\left(t_2, a^\mu(t_2)\right) \\
&+ \left(R_\nu {}^{R}_{t_1}D^{\alpha}_{t_2} a^\nu - B\right) \dot{\xi}_0^j + \Lambda_\mu \left(\xi_\mu^j - \dot{a}^\mu \xi_0^j\right) \\
&- \frac{R_\nu}{2\Gamma(1-\alpha)} \frac{d}{dt} \left|t - t_1\right|^{-\alpha} a^\nu(t_1)\, \xi_0^j\left(t_1, a^\mu(t_1)\right) \\
&+ \dot{G}^j - Q_\mu \left(\xi_\mu^{j-1} - \dot{a}^\mu \xi_0^{j-1}\right) = 0,
\end{aligned} \tag{74}$$

where $\xi_\mu^{j-1} = \xi_0^{j-1} = 0$ when $j = 0$, then a z-th order adiabatic invariant exists as follows:

$$\begin{aligned}
I_{Rz} = \sum_{j=0}^{z} \varepsilon^j \Bigg\{ &\left(R_\nu {}^{R}_{t_1}D^{\alpha}_{t_2} a^\nu - B\right) \xi_0^j + \int_{t_1}^{t} \left[R_\nu {}^{R}_{t_1}D^{\alpha}_{t_2} \left(\xi_\nu^j - \dot{a}^\nu \xi_0^j\right) + \left(\xi_\nu^j - \dot{a}^\nu \xi_0^j\right) {}^{RC}_{t_1}D^{\alpha}_{t_2} R_\nu\right] d\tau + G^j \\
&+ \int_{t_1}^{t} \frac{R_\nu}{2\Gamma(1-\alpha)} \frac{d}{d\tau} \left[\left|\tau - t_2\right|^{-\alpha} a^\nu(t_2)\, \xi_0^j\left(t_2, a^\mu(t_2)\right) - \left|\tau - t_1\right|^{-\alpha} a^\nu(t_1)\, \xi_0^j\left(t_1, a^\mu(t_1)\right)\right] d\tau \Bigg\}.
\end{aligned} \tag{75}$$

Theorem 25. *For the disturbed generalized fractional Birkhoffian system with the Riesz-Caputo fractional derivative*

$$\frac{\partial R_\nu}{\partial a^\mu} {}^{RC}_{t_1}D^{\alpha}_{t_2} a^\nu - {}^{R}_{t_1}D^{\alpha}_{t_2} R_\mu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = \varepsilon Q_\mu \quad (\mu, \nu = 1, 2, \cdots, 2n), \tag{76}$$

if there exists a gauge function G^j ($j = 0, 1, 2, \cdots, z$, z is an integer) such that the infinitesimal generators ξ_0^j and ξ_ν^j satisfy

$$\begin{aligned}
&R_\nu {}^{RC}_{t_1}D^{\alpha}_{t_2} \left(\xi_\nu^j - \dot{a}^\nu \xi_0^j\right) + \left(R_\nu \frac{d}{dt} {}^{RC}_{t_1}D^{\alpha}_{t_2} a^\nu \right. \\
&\left. + \frac{\partial R_\nu}{\partial t} {}^{RC}_{t_1}D^{\alpha}_{t_2} a^\nu - \frac{\partial B}{\partial t}\right) \xi_0^j + \left(R_\nu {}^{RC}_{t_1}D^{\alpha}_{t_2} a^\nu - B\right) \dot{\xi}_0^j \\
&+ \frac{R_\nu}{2\Gamma(1-\alpha)} \left[\left|t - t_2\right|^{-\alpha} \dot{a}^\nu(t_2)\, \xi_0^j\left(t_2, a^\mu(t_2)\right) \right. \\
&\left. - \left|t - t_1\right|^{-\alpha} \dot{a}^\nu(t_1)\, \xi_0^j\left(t_1, a^\mu(t_1)\right)\right] \\
&+ \left(\frac{\partial R_\nu}{\partial a^\mu} {}^{RC}_{t_1}D^{\alpha}_{t_2} a^\nu - \frac{\partial B}{\partial a^\mu}\right) \xi_\mu^j + \Lambda_\mu \left(\xi_\mu^j - \dot{a}^\mu \xi_0^j\right) \\
&- Q_\mu \left(\xi_\mu^{j-1} - \dot{a}^\mu \xi_0^{j-1}\right) + \dot{G}^j = 0,
\end{aligned} \tag{77}$$

where $\xi_\mu^{j-1} = \xi_0^{j-1} = 0$ when $j = 0$, then a z-th order adiabatic invariant exists as follows:

$$\begin{aligned}
I_{RCz} = \sum_{j=0}^{z} \varepsilon^j \Bigg\{ &\left(R_\nu {}^{RC}_{t_1}D^{\alpha}_{t_2} a^\nu - B\right) \xi_0^j + \int_{t_1}^{t} \left[R_\nu {}^{RC}_{t_1}D^{\alpha}_{t_2} \left(\xi_\nu^j - \dot{a}^\nu \xi_0^j\right) + \left(\xi_\nu^j - \dot{a}^\nu \xi_0^j\right) {}^{R}_{t_1}D^{\alpha}_{t_2} R_\nu\right] d\tau \\
&+ \int_{t_1}^{t} \frac{R_\nu}{2\Gamma(1-\alpha)} \left[\left|\tau - t_2\right|^{-\alpha} \dot{a}^\nu(t_2)\, \xi_0^j\left(t_2, a^\mu(t_2)\right) - \left|\tau - t_1\right|^{-\alpha} \dot{a}^\nu(t_1)\, \xi_0^j\left(t_1, a^\mu(t_1)\right)\right] d\tau + G^j \Bigg\}.
\end{aligned} \tag{78}$$

Remark 26. When $\alpha, \beta \longrightarrow 1$, since $\Gamma(0) = \infty$, we can get the classical adiabatic invariant for the classical generalized Birkhoffian system [42].

6. Applications

Application 1. The fractional Lotka biochemical oscillator model in terms of combined Riemann-Liouville fractional derivative has the form [17]

$$\begin{aligned} &\frac{1}{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2 + \frac{1}{2}\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^2 - \alpha_2 - \beta_2 \exp a^1 = 0, \\ &\frac{1}{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \frac{1}{2}\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^2 - \alpha_2 - \beta_2 \exp a^1 = 0, \end{aligned} \tag{79}$$

where the Birkhoffian and Birkhoff functions are

$$\begin{aligned} B &= \alpha_2 a^1 - \alpha_1 a^2 - \beta_1 \exp a^2 + \beta_2 \exp a^1, \\ R_1 &= -\frac{1}{2}a^2, \\ R_2 &= \frac{1}{2}a^1. \end{aligned} \tag{80}$$

Formula (48) gives

$$\begin{aligned} &-\frac{1}{2}a^{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_1^0 - \dot{a}^1\xi_0^0\right) + \frac{1}{2}a^{1}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_2^0 - \dot{a}^2\xi_0^0\right) \\ &+\left(-\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \frac{1}{2}a^1 \times \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2\right)\xi_0^0 \\ &-\gamma\left[-\frac{1}{2}a^2\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^1(t_1)\right. \\ &\left.\cdot\,\xi_0^0(t_1, a^{\mu}(t_1))\right] - \gamma\left[\frac{1}{2}a^1 \times \frac{1}{\Gamma(1-\alpha)}\right. \\ &\left.\cdot\,\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^2(t_1)\,\xi_0^0(t_1, a^{\mu}(t_1))\right] + (1-\gamma) \\ &\cdot\left[-\frac{1}{2}a^2\frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}\times a^1(t_2)\right. \\ &\left.\cdot\,\xi_0^0(t_2, a^{\mu}(t_2))\right] + (1-\gamma)\left[\frac{1}{2}a^1\frac{1}{\Gamma(1-\beta)}\right. \\ &\left.\cdot\,\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^2(t_2)\,\xi_0^0(t_2, a^{\mu}(t_2))\right] + \left(-\frac{1}{2}\right. \\ &\cdot\,a^{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \frac{1}{2}a^{1}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2 - \alpha_2 a^1 + \alpha_1 a^2 + \beta_1 \\ &\left.\cdot\,\exp a^2 - \beta_2 \exp a^1\right)\dot{\xi}_0^0 + \xi_1^0\left(\frac{1}{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2 - \alpha_2\right. \\ &\left.-\beta_2 \exp a^1\right) + \xi_2^0\left(-\frac{1}{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \alpha_1 + \beta_1 \exp a^2\right) \\ &+\dot{G}^0 = 0, \end{aligned} \tag{81}$$

where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}_{t_1}D_t^{\alpha}a^{\nu} &= {}^{\mathrm{RL}}_{t_1}D_t^{\alpha}\dot{a}^{\nu} \\ &\quad + \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^{\nu}(t_1), \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}_{t}D_{t_2}^{\beta}a^{\nu} &= {}^{\mathrm{RL}}_{t}D_{t_2}^{\beta}\dot{a}^{\nu} \\ &\quad + \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^{\nu}(t_2). \end{aligned} \tag{83}$$

Taking calculation, we have

$$\begin{aligned} \xi_0^0 &= 1, \\ \xi_1^0 &= \xi_2^0 = 0, \\ G^0 &= 0. \end{aligned} \tag{84}$$

It follows from Theorem 11 that

$$\begin{aligned} I_{\mathrm{RL}0} &= -\frac{1}{2}a^{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \frac{1}{2}a^{1}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2 - \alpha_2 a^1 + \alpha_1 a^2 \\ &+ \beta_1 \exp a^2 - \beta_2 \exp a^1 + \int_{t_1}^{t}\left(\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1\right. \\ &- \frac{1}{2}a^1\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2 + \frac{1}{2}\dot{a}^{1}\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^2 \\ &\left.- \frac{1}{2}\dot{a}^{2}\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^1\right)\mathrm{d}\tau. \end{aligned} \tag{85}$$

Assume this system is disturbed by $\varepsilon Q_1 = \varepsilon a^2$, $\varepsilon Q_2 = \varepsilon a^1$; then from formula (61), we have

$$\begin{aligned} &-\frac{1}{2}a^{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_1^1 - \dot{a}^1\xi_0^1\right) + \frac{1}{2}a^{1}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_2^1 - \dot{a}^2\xi_0^1\right) \\ &+\left(-\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \frac{1}{2}a^1 \times \frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2\right)\xi_0^1 \\ &-\gamma\left[-\frac{1}{2}a^2\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^1(t_1)\right. \\ &\left.\cdot\,\xi_0^1(t_1, a^{\mu}(t_1))\right] - \gamma\left[\frac{1}{2}a^1 \times \frac{1}{\Gamma(1-\alpha)}\right. \\ &\left.\cdot\,\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}a^2(t_1)\,\xi_0^1(t_1, a^{\mu}(t_1))\right] + (1-\gamma) \\ &\cdot\left[-\frac{1}{2}a^2\frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}\times a^1(t_2)\right. \\ &\left.\cdot\,\xi_0^1(t_2, a^{\mu}(t_2))\right] + (1-\gamma)\left[\frac{1}{2}a^1\frac{1}{\Gamma(1-\beta)}\right. \\ &\left.\cdot\,\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}a^2(t_2)\,\xi_0^1(t_2, a^{\mu}(t_2))\right] + \left(-\frac{1}{2}\right. \\ &\cdot\,a^{2}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^1 + \frac{1}{2}a^{1}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}a^2 - \alpha_2 a^1 + \alpha_1 a^2 + \beta_1 \end{aligned}$$

$$\cdot \exp a^2 - \beta_2 \exp a^1\Big)\dot{\xi}_0^1 + \xi_1^1\Big(\frac{1}{2}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^2 - \alpha_2 - \beta_2\exp a^1\Big) + \xi_2^1\Big(-\frac{1}{2}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1 + \alpha_1 + \beta_1 \cdot \exp a^2\Big),$$
$$+ a^2\dot{a}^1 + a^1\dot{a}^2 + \dot{G}^1 = 0. \quad (86)$$

Solving formula (86), we get

$$\xi_0^1 = 1,\quad \xi_1^1 = \xi_2^1 = 0,\quad G^1 = -a^1a^2. \quad (87)$$

From Theorem 18, we obtain

$$\begin{aligned}I_{\mathrm{RL1}} ={}& -\frac{1}{2}a^2\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1 + \frac{1}{2}a^1\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^2 - \alpha_2a^1 + \alpha_1a^2 + \beta_1\exp a^2 - \beta_2\exp a^1 + \int_{t_1}^{t}\Big(\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1 \\ &- \frac{1}{2}a^1\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^2 + \frac{1}{2}\dot{a}^1\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^2 - \frac{1}{2}\cdot\dot{a}^2\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^1\Big)\mathrm{d}\tau + \varepsilon\Big[-\frac{1}{2}a^2\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1 + \frac{1}{2}\cdot a^1\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^2 - \alpha_2a^1 + \alpha_1a^2 + \beta_1\exp a^2 - \beta_2 \\ &\cdot\exp a^1 - a^1a^2 + \int_{t_1}^{t}\Big(\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^1 - \frac{1}{2}a^1\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}a^2 + \frac{1}{2}\dot{a}^1\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^2 - \frac{1}{2}\dot{a}^2\,{}^{\mathrm{C}}D_{1-\gamma}^{\beta,\alpha}a^1\Big)\mathrm{d}\tau\Big].\end{aligned} \quad (88)$$

When $\alpha, \beta \longrightarrow 1$, we obtain the classical adiabatic invariant

$$\begin{aligned}I_{11} &= \alpha_2a^1 - \alpha_1a^2 - \beta_1\exp a^2 + \beta_2\exp a^1 \\ &\quad + \varepsilon\left(\alpha_2a^1 - \alpha_1a^2 - \beta_1\exp a^2 + \beta_2\exp a^1 - a^1a^2\right).\end{aligned} \quad (89)$$

Application 2. The fractional Whittaker model in terms of Riesz-Riemann-Liouville fractional derivative has the form [17]

$$\begin{aligned}&-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 - \frac{1}{2}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\left(a^1 - a^4\right) = 0,\\ &-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3 - \frac{1}{2}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3 - a^1 + a^4 = 0,\\ &\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^2 - a^4 = 0,\\ &-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 - \frac{1}{2}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4 + a^3 = 0,\end{aligned} \quad (90)$$

where the Birkhoffian and Birkhoff functions are

$$\begin{aligned}B &= \frac{1}{2}\left[2a^1a^4 - \left(a^3\right)^2 - \left(a^4\right)^2\right],\\ R_1 &= -\frac{1}{2}a^2,\\ R_2 &= \frac{1}{2}\left(a^1 - a^4\right),\\ R_3 &= \frac{1}{2}a^4,\\ R_4 &= \frac{1}{2}\left(a^2 - a^3\right).\end{aligned} \quad (91)$$

From formula (50), we have

$$\begin{aligned}&-\frac{1}{2}a^2\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_1^0 - \dot{a}^1\xi_0^0\right) + \frac{1}{2}\left(a^1 - a^4\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_2^0 - \dot{a}^2\xi_0^0\right) + \frac{1}{2}a^4\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_3^0 - \dot{a}^3\xi_0^0\right) + \frac{1}{2}\left(a^2 - a^3\right)\\ &\cdot{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_4^0 - \dot{a}^4\xi_0^0\right) + \Big[-\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\left(a^1 - a^4\right)\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}a^4\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3\\ &+ \frac{1}{2}\left(a^2 - a^3\right)\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4\Big]\xi_0^0 + \xi_1^0\Big(\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 - a^4\Big) + \xi_2^0\Big(-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4\Big)\\ &+ \xi_3^0\Big(-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 + a^3\Big) + \xi_4^0\Big(-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3 - a^1 + a^4\Big) - \frac{1}{2}a^2\frac{1}{2\Gamma(1-\alpha)}\times\frac{\mathrm{d}}{\mathrm{d}t}\left|t - t_2\right|^{-\alpha}a^1(t_2)\,\xi_0^0\left(t_2, a^\mu(t_2)\right) + \frac{1}{2}\left(a^1 - a^4\right)\\ &\cdot\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t - t_2\right|^{-\alpha}a^2(t_2)\times\xi_0^0\left(t_2, a^\mu(t_2)\right) + \frac{1}{2}a^4\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t - t_2\right|^{-\alpha}a^3(t_2)\,\xi_0^0\left(t_2, a^\mu(t_2)\right)\\ &+ \frac{1}{2}\left(a^2 - a^3\right)\times\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t - t_2\right|^{-\alpha}a^4(t_2)\cdot\xi_0^0\left(t_2, a^\mu(t_2)\right) + \Big[-\frac{1}{2}a^2\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\left(a^1 - a^4\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}a^4\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3\\ &+ \frac{1}{2}\left(a^2 - a^3\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 - a^1a^4 + \frac{1}{2}\left(a^3\right)^2 + \frac{1}{2}\left(a^4\right)^2\Big]\dot{\xi}_0^0 + \frac{1}{2}a^2\frac{1}{2\Gamma(1-\alpha)}\times\frac{\mathrm{d}}{\mathrm{d}t}\left|t - t_1\right|^{-\alpha}\end{aligned}$$

$$\cdot a^1(t_1)\,\xi_0^0(t_1,a^\mu(t_1)) - \frac{1}{2}\left(a^1 - a^4\right)\frac{1}{2\Gamma(1-\alpha)}$$
$$\cdot \frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha} a^2(t_1)\times \xi_0^0(t_1,a^\mu(t_1)) - \frac{1}{2}a^4$$
$$\cdot \frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha} a^\gamma(t_1)\,\xi_0^0(t_1,a^\mu(t_1))$$
$$-\frac{1}{2}\left(a^2-a^3\right)\times\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha} a^\gamma(t_1)$$
$$\cdot \xi_0^0(t_1,a^\mu(t_1)) + \dot{G}^0 = 0, \tag{92}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\gamma = {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\dot{a}^\gamma + \frac{1}{2\Gamma(1-\alpha)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\left|t-t_1\right|^{-\alpha}a^\gamma(t_1) - \left|t-t_2\right|^{-\alpha}a^\gamma(t_2)\right]. \tag{93}$$

Taking calculation, we have

$$\xi_0^0 = 1,$$
$$\xi_1^0 = \xi_2^0 = \xi_3^0 = \xi_4^0 = 0, \tag{94}$$
$$G^0 = 0.$$

Theorem 13 gives

$$I_{\mathrm{R}0} = -\frac{1}{2}a^2\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\left(a^1-a^4\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}$$
$$\cdot a^4\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3 + \frac{1}{2}\left(a^2-a^3\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 - \frac{1}{2}\left[2a^1a^4\right.$$
$$\left. -\left(a^3\right)^2 - \left(a^4\right)^2\right] + \int_{t_1}^{t}\left[\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1\right.$$
$$-\frac{1}{2}\left(a^1-a^4\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 - \frac{1}{2}a^4\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3 \tag{95}$$
$$-\frac{1}{2}\left(a^2-a^3\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 + \frac{1}{2}\dot{a}^1\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^2$$
$$-\frac{1}{2}\dot{a}^2\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\left(a^1-a^4\right) - \frac{1}{2}\dot{a}^3\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4$$
$$\left.-\frac{1}{2}\dot{a}^4\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\left(a^2-a^3\right)\right]\mathrm{d}\tau.$$

Assuming this system is disturbed by $\varepsilon Q_1 = 0$, $\varepsilon Q_2 = 0$, $\varepsilon Q_3 = \varepsilon a^4$, $\varepsilon Q_4 = \varepsilon a^3$, then formula (74) gives

$$-\frac{1}{2}a^2\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_1^1 - \dot{a}^1\xi_0^1\right) + \frac{1}{2}\left(a^1-a^4\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_2^1\right.$$
$$\left.-\dot{a}^2\xi_0^1\right) + \frac{1}{2}a^4\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_3^1 - \dot{a}^3\xi_0^1\right) + \frac{1}{2}\left(a^2-a^3\right)$$
$$\cdot\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\left(\xi_4^1 - \dot{a}^4\xi_0^1\right) + \left[-\frac{1}{2}a^2\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1\right.$$
$$+\frac{1}{2}\left(a^1-a^4\right)\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}a^4\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3$$
$$\left.+\frac{1}{2}\left(a^2-a^3\right)\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4\right]\xi_0^1 + \xi_1^1\left(\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2\right.$$
$$\left.-a^4\right) + \xi_2^1\left(-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4\right)$$
$$+\xi_3^1\left(-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 + a^3\right) + \xi_4^1\left(-\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2\right.$$
$$\left.+\frac{1}{2}\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3 - a^1 + a^4\right) - \frac{1}{2}a^2\frac{1}{2\Gamma(1-\alpha)}\times\frac{\mathrm{d}}{\mathrm{d}t}\left|t\right.$$
$$\left.-t_2\right|^{-\alpha}a^1(t_2)\,\xi_0^1(t_2,a^\mu(t_2)) + \frac{1}{2}\left(a^1-a^4\right)$$
$$\cdot\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_2\right|^{-\alpha}a^2(t_2)\times\xi_0^1(t_2,a^\mu(t_2))$$
$$+\frac{1}{2}a^4\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_2\right|^{-\alpha}a^3(t_2)\,\xi_0^1(t_2,a^\mu(t_2))$$
$$+\frac{1}{2}\left(a^2-a^3\right)\times\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_2\right|^{-\alpha}a^4(t_2)$$
$$\cdot\xi_0^1(t_2,a^\mu(t_2)) + \left[-\frac{1}{2}a^2\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1\right.$$
$$+\frac{1}{2}\left(a^1-a^4\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}a^4\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3$$
$$+\frac{1}{2}\left(a^2-a^3\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 - a^1a^4 + \frac{1}{2}\left(a^3\right)^2$$
$$\left.+\frac{1}{2}\left(a^4\right)^2\right]\dot{\xi}_0^1 + \frac{1}{2}a^2\frac{1}{2\Gamma(1-\alpha)}\times\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha}$$
$$\cdot a^1(t_1)\,\xi_0^1(t_1,a^\mu(t_1)) - \frac{1}{2}\left(a^1-a^4\right)\frac{1}{2\Gamma(1-\alpha)}$$
$$\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha}a^2(t_1)\times\xi_0^1(t_1,a^\mu(t_1)) - \frac{1}{2}a^4$$
$$\cdot\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha}a^\gamma(t_1)\,\xi_0^1(t_1,a^\mu(t_1))$$
$$-\frac{1}{2}\left(a^2-a^3\right)\times\frac{1}{2\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left|t-t_1\right|^{-\alpha}a^\gamma(t_1)$$
$$\cdot\xi_0^1(t_1,a^\mu(t_1)) + a^4\dot{a}^3 + a^3\dot{a}^4 + \dot{G}^1 = 0. \tag{96}$$

Taking calculation, we have

$$\xi_0^1 = 1,$$
$$\xi_1^1 = \xi_2^1 = \xi_3^1 = \xi_4^1 = 0, \tag{97}$$
$$G^1 = -a^3a^4.$$

From Theorem 24, we get

$$I_{\mathrm{R}1} = -\frac{1}{2}a^2\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1 + \frac{1}{2}\left(a^1-a^4\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2 + \frac{1}{2}$$
$$\cdot a^4\,{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^3 + \frac{1}{2}\left(a^2-a^3\right){}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4 - \frac{1}{2}\left[2a^1a^4\right.$$

$$- \left(a^3\right)^2 - \left(a^4\right)^2\Big] + \int_{t_1}^{t} \Big[\frac{1}{2}a^2 \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^1 - \frac{1}{2}\left(a^1 - a^4\right) \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^2 - \frac{1}{2}a^4 \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^3 - \frac{1}{2} \times \left(a^2 - a^3\right) \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^4 + \frac{1}{2}\dot{a}^1 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^2 - \frac{1}{2} \cdot \dot{a}^2 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}\left(a^1 - a^4\right) - \frac{1}{2}\dot{a}^3 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\dot{a}^4 \times {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}\left(a^2 - a^3\right)\Big] d\tau + \varepsilon \Big\{-\frac{1}{2}a^2 \, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^1 + \frac{1}{2}\left(a^1 - a^4\right) {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^2 + \frac{1}{2}a^4 \, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^3 + \frac{1}{2}\left(a^2 - a^3\right) \cdot {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\left[2a^1a^4 - \left(a^3\right)^2 - \left(a^4\right)^2\right] + \int_{t_1}^{t} \Big[\frac{1}{2}a^2 \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^1 - \frac{1}{2}\left(a^1 - a^4\right) \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^2 - \frac{1}{2}a^4 \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^3 - \frac{1}{2}\left(a^2 - a^3\right) \frac{d}{d\tau} {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^4 + \frac{1}{2}\dot{a}^1 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^2 - \frac{1}{2}\dot{a}^2 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}\left(a^1 - a^4\right) - \frac{1}{2}\dot{a}^3 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\dot{a}^4 \, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}\left(a^2 - a^3\right)\Big] d\tau - a^3a^4\Big\}. \tag{98}$$

When $\alpha \longrightarrow 1$, we obtain the classical adiabatic invariant

$$I_{21} = 2a^1a^4 - \left(a^3\right)^2 - \left(a^4\right)^2 + \varepsilon\left[2a^1a^4 - \left(a^3\right)^2 - \left(a^4\right)^2 - a^3a^4\right]. \tag{99}$$

Application 3. The fractional Hojman–Urrutia model in terms of combined Caputo derivative has the form [17]

$$\begin{aligned} {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma} a^2 + {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma} a^3 &= 0, \\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 - a^3 &= 0, \\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 - {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma} a^4 - \left(a^3 + a^2\right) &= 0, \\ {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^3 + a^4 &= 0, \end{aligned} \tag{100}$$

where the Birkhoffian and Birkhoff functions are

$$\begin{aligned} B &= \frac{1}{2}\left[\left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2\right], \\ R_1 &= a^2 + a^3, \\ R_2 &= 0, \\ R_3 &= a^4, \\ R_4 &= 0. \end{aligned} \tag{101}$$

From formula (49), we have

$$\left(a^2 + a^3\right) {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}\left(\xi^0_1 - \dot{a}^1\xi^0_0\right) + a^4 \, {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma}\left(\xi^0_3 - \dot{a}^3\xi^0_0\right) + \left[\left(a^2 + a^3\right) \frac{d}{dt} {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 + a^4 \frac{d}{dt} {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^3\right]\xi^0_0 + \xi^0_2\left({}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 - a^3\right) + \xi^0_3\left({}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 - a^3 - a^2\right) + \xi^0_4 \times \left({}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^3 + a^4\right) - \left(a^2 + a^3\right) \frac{\gamma}{\Gamma(1-\alpha)}\left(t - t_1\right)^{-\alpha} \dot{a}^1\left(t_1\right) \xi^0_0\left(t_1, a^{\mu}\left(t_1\right)\right) - a^4 \times \frac{\gamma}{\Gamma(1-\alpha)}\left(t - t_1\right)^{-\alpha} \dot{a}^3\left(t_1\right) \xi^0_0\left(t_1, a^{\mu}\left(t_1\right)\right) + \left[\left(a^2 + a^3\right) {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 + a^4 \, {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^3 - \frac{1}{2}\left(a^3\right)^2 - a^2a^3 + \frac{1}{2}\left(a^4\right)^2\right]\dot{\xi}^0_0 + \left(a^2 + a^3\right) \frac{1-\gamma}{\Gamma(1-\beta)}\left(t_2 - t\right)^{-\beta} \dot{a}^1\left(t_2\right) \times \xi^0_0\left(t_2, a^{\mu}\left(t_2\right)\right) + a^4 \frac{1-\gamma}{\Gamma(1-\beta)}\left(t_2 - t\right)^{-\beta} \dot{a}^3\left(t_2\right) \xi^0_0\left(t_2, a^{\mu}\left(t_2\right)\right) + \dot{G}^0 = 0, \tag{102}$$

where

$$\frac{d}{dt} {}^{\mathrm{C}}_{t_1}D^{\alpha}_{t} a^{\gamma} = {}^{\mathrm{C}}_{t_1}D^{\alpha}_{t} \dot{a}^{\gamma} + \frac{1}{\Gamma(1-\alpha)}\left(t - t_1\right)^{-\alpha} \dot{a}^{\gamma}\left(t_1\right), \tag{103}$$

$$\frac{d}{dt} {}^{\mathrm{C}}_{t}D^{\beta}_{t_2} a^{\gamma} = {}^{\mathrm{C}}_{t}D^{\beta}_{t_2} \dot{a}^{\gamma} + \frac{1}{\Gamma(1-\beta)}\left(t_2 - t\right)^{-\beta} \dot{a}^{\gamma}\left(t_2\right). \tag{104}$$

Taking calculation, we have

$$\begin{aligned} \xi^0_0 &= 1, \\ \xi^0_1 &= 0, \\ \xi^0_2 &= 1, \\ \xi^0_3 &= 0, \\ \xi^0_4 &= 1, \\ G^0 &= 0. \end{aligned} \tag{105}$$

It follows from Theorem 12 that

$$I_{\mathrm{C}0} = \left(a^2 + a^3\right) {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 + a^4 \, {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^3 - \frac{1}{2}\left(a^3\right)^2 - a^2a^3 + \frac{1}{2}\left(a^4\right)^2 - \int_{t_1}^{t}\Big[\left(a^2 + a^3\right) \times \frac{d}{d\tau} {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^1 + a^4 \frac{d}{d\tau} {}^{\mathrm{C}}D^{\alpha,\beta}_{\gamma} a^3 + \dot{a}^1 \, {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma}\left(a^2 + a^3\right) + \dot{a}^3 \, {}^{\mathrm{RL}}D^{\beta,\alpha}_{1-\gamma} a^4\Big] d\tau. \tag{106}$$

Assume this system is disturbed by $\varepsilon Q_1 = \varepsilon a^3$, , $\varepsilon Q_2 = 0$, $\varepsilon Q_3 = \varepsilon a^1$, $\varepsilon Q_4 = 0$.

From formula (71), we have

$$\begin{aligned}&\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(\xi_1^1-\dot a^1\xi_0^1\right)+a^4\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}\left(\xi_3^1-\dot a^3\xi_0^1\right)\\&+\left[\left(a^2+a^3\right)\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^4\times\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3\right]\xi_0^1\\&+\xi_2^1\left({}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1-a^3\right)+\xi_3^1\left({}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1-a^3-a^2\right)\\&+\xi_4^1\left({}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3+a^4\right)-\left(a^2+a^3\right)\frac{\gamma}{\Gamma(1-\alpha)}(t\\&-t_1)^{-\alpha}\dot a^1(t_1)\,\xi_0^1(t_1,a^\mu(t_1))-a^4\frac{\gamma}{\Gamma(1-\alpha)}(t\\&-t_1)^{-\alpha}\dot a^3(t_1)\times\xi_0^1(t_1,a^\mu(t_1))\\&+\left[\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^4\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3-\frac12\left(a^3\right)^2\right.\\&\left.-a^2a^3+\frac12\left(a^4\right)^2\right]\dot\xi_0^1+\left(a^2+a^3\right)\frac{1-\gamma}{\Gamma(1-\beta)}(t_2\\&-t)^{-\beta}\dot a^1(t_2)\,\xi_0^1(t_2,a^\mu(t_2))+a^4\frac{1-\gamma}{\Gamma(1-\beta)}(t_2\\&-t)^{-\beta}\dot a^3(t_2)\times\xi_0^1(t_2,a^\mu(t_2))+a^3\dot a^1+a^1\dot a^3+\dot G^1\\&=0.\end{aligned}\tag{107}$$

From formula (107), we obtain

$$\begin{aligned}\xi_0^1&=1,\\\xi_1^1&=\xi_2^1=\xi_3^1=\xi_4^1=0,\\G^1&=-a^1a^3.\end{aligned}\tag{108}$$

From Theorem 22, we achieve

$$\begin{aligned}I_{\mathrm{C}1}={}&\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^4\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3-\frac12\left(a^3\right)^2\\&-a^2a^3+\frac12\left(a^4\right)^2-\int_{t_1}^{t}\left[\left(a^2+a^3\right)\times\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1\right.\\&+a^4\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3+\dot a^1\,{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left(a^2+a^3\right)\\&\left.+\dot a^3\,{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}a^4\right]\mathrm{d}\tau+\varepsilon\left\{\left(a^2+a^3\right){}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1\right.\\&+a^4\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3-\frac12\left(a^3\right)^2-a^2a^3+\frac12\left(a^4\right)^2\\&-\int_{t_1}^{t}\left[\left(a^2+a^3\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^1+a^4\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{C}}D_\gamma^{\alpha,\beta}a^3\right.\\&\left.+\dot a^1\,{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left(a^2+a^3\right)+\dot a^3\,{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}a^4\right]\mathrm{d}\tau\\&\left.-a^1a^3\right\}.\end{aligned}\tag{109}$$

When $\alpha,\beta \longrightarrow 1$, we obtain the classical adiabatic invariant

$$\begin{aligned}I_{31}={}&\frac12\left(a^3\right)^2+a^2a^3-\frac12\left(a^4\right)^2\\&+\varepsilon\left[\frac12\left(a^3\right)^2+a^2a^3-\frac12\left(a^4\right)^2-a^1a^3\right].\end{aligned}\tag{110}$$

Application 4. The fractional Hénon–Heiles model in terms of Riesz-Caputo derivative has the form [17]

$$\begin{aligned}{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3+a^1+2a^1a^2&=0,\\{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4+a^2-\left(a^2\right)^2+\left(a^1\right)^2&=0,\\{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^1-a^3&=0,\\{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2-a^4&=0,\end{aligned}\tag{111}$$

where the Birkhoffian and Birkhoff functions are

$$\begin{aligned}B&=\frac12\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2\right.\\&\quad\left.-\frac23\left(a^2\right)^3\right],\\R_1&=R_2=0,\\R_3&=-a^1,\\R_4&=-a^2.\end{aligned}\tag{112}$$

Formula (51) gives

$$\begin{aligned}&-a^1\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\left(\xi_3^0-\dot a^3\xi_0^0\right)-a^2\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\left(\xi_4^0-\dot a^4\xi_0^0\right)\\&+\left[-a^1\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3-a^2\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4\right]\xi_0^0-a^1\\&\times\frac{1}{2\Gamma(1-\alpha)}\left[\left|t-t_2\right|^{-\alpha}\dot a^3(t_2)\,\xi_0^0(t_2,a^\mu(t_2))-\right|t\\&\left.-t_1\right|^{-\alpha}\dot a^3(t_1)\,\xi_0^0(t_1,a^\mu(t_1))\Big]-\frac{1}{2\Gamma(1-\alpha)}\times a^2\Big[\right|t\\&-t_2\left|^{-\alpha}\dot a^4(t_2)\,\xi_0^0(t_2,a^\mu(t_2))-\left|t-t_1\right|^{-\alpha}\dot a^4(t_1)\right.\\&\cdot\xi_0^0(t_1,a^\mu(t_1))\Big]-\xi_1^0\left({}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3+a^1+2a^1a^2\right)\\&-\xi_2^0\left[{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4+a^2+\left(a^1\right)^2-\left(a^2\right)^2\right]-\xi_3^0a^3\\&-\xi_4^0a^4+\left\{-a^1\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3-a^2\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^4-\frac12\left[\left(a^1\right)^2\right.\right.\\&\left.\left.+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2-\frac23\left(a^2\right)^3\right]\right\}\\&\cdot\dot\xi_0^0+\dot G^0=0,\end{aligned}\tag{113}$$

where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^\nu&={}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\dot a^\nu\\&\quad+\frac{1}{2\Gamma(1-\alpha)}\left[\left|t-t_1\right|^{-\alpha}\dot a^\nu(t_1)-\left|t-t_2\right|^{-\alpha}\dot a^\nu(t_2)\right].\end{aligned}\tag{114}$$

From formula (113), we have

$$\xi_0^0 = \xi_1^0 = \xi_2^0 = 1,$$
$$\xi_3^0 = \xi_4^0 = 0, \qquad (115)$$
$$G^0 = 0.$$

From Theorem 14, we obtain

$$\begin{aligned} I_{\mathrm{RC0}} = & -a^1\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 - a^2\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\Big[\left(a^1\right)^2 \\ & + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 + 2a^2\left(a^1\right)^2 - \frac{2}{3}\left(a^2\right)^3\Big] \\ & + \int_{t_1}^{t}\Big(a^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 + a^2 \frac{\mathrm{d}}{\mathrm{d}\tau}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 \\ & + \dot{a}^3\, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^1 + \dot{a}^4\, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^2\Big)\mathrm{d}\tau. \end{aligned} \qquad (116)$$

Assume this system is disturbed by $\varepsilon Q_1 = \varepsilon Q_2 = 0, \varepsilon Q_3 = \varepsilon a^4, \varepsilon Q_4 = \varepsilon a^3$.

From formula (77), we have

$$\begin{aligned} & - a^1\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}\left(\xi_3^1 - \dot{a}^3\xi_0^1\right) - a^2\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}\left(\xi_4^1 - \dot{a}^4\xi_0^1\right) \\ & + \left[-a^1 \frac{\mathrm{d}}{dt}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 - a^2 \frac{\mathrm{d}}{dt}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4\right]\xi_0^1 - a^1 \\ & \times \frac{1}{2\Gamma(1-\alpha)}\left[\left|t - t_2\right|^{-\alpha}\dot{a}^3(t_2)\,\xi_0^1\left(t_2, a^{\mu}(t_2)\right) - \left|t \right.\right. \\ & \left.\left. - t_1\right|^{-\alpha}\dot{a}^3(t_1)\,\xi_0^1\left(t_1, a^{\mu}(t_1)\right)\right] - \frac{1}{2\Gamma(1-\alpha)} \times a^2\left[\left|t \right.\right. \\ & \left. - t_2\right|^{-\alpha}\dot{a}^4(t_2)\,\xi_0^1\left(t_2, a^{\mu}(t_2)\right) - \left|t - t_1\right|^{-\alpha}\dot{a}^4(t_1) \\ & \left.\cdot\, \xi_0^1\left(t_1, a^{\mu}(t_1)\right)\right] - \xi_1^1\left({}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 + a^1 + 2a^1a^2\right) \\ & - \xi_2^1\left[{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 + a^2 + \left(a^1\right)^2 - \left(a^2\right)^2\right] - \xi_3^1 a^3 \\ & - \xi_4^1 a^4 + \Big\{-a^1\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 - a^2\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\Big[\left(a^1\right)^2 \\ & + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 + 2a^2\left(a^1\right)^2 - \frac{2}{3}\left(a^2\right)^3\Big]\Big\} \\ & \cdot\, \dot{\xi}_0^1 + a^4\dot{a}^3 + a^3\dot{a}^4 + \dot{G}^1 = 0. \end{aligned} \qquad (117)$$

Calculating formula (117), we obtain

$$\xi_0^1 = 1,$$
$$\xi_1^1 = \xi_2^1 = \xi_3^1 = \xi_4^1 = 0, \qquad (118)$$
$$G^1 = -a^3a^4.$$

From Theorem 25, we get

$$\begin{aligned} I_{\mathrm{RC1}} = & -a^1\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 - a^2\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\Big[\left(a^1\right)^2 \\ & + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 + 2a^2\left(a^1\right)^2 - \frac{2}{3}\left(a^2\right)^3\Big] \\ & + \int_{t_1}^{t}\Big(a^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 + a^2 \frac{\mathrm{d}}{\mathrm{d}\tau}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 \\ & + \dot{a}^3\, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^1 + \dot{a}^4\, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^2\Big)\mathrm{d}\tau \\ & + \varepsilon\Big\{-a^1\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 - a^2 \times {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 - \frac{1}{2}\Big[\left(a^1\right)^2 \\ & + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 + 2a^2\left(a^1\right)^2 - \frac{2}{3}\left(a^2\right)^3\Big] \\ & + \int_{t_1}^{t}\Big(a^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^3 + a^2 \frac{\mathrm{d}}{\mathrm{d}\tau}\, {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2} a^4 \\ & + \dot{a}^3\, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^1 + \dot{a}^4\, {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2} a^2\Big)\mathrm{d}\tau - a^3a^4\Big\}. \end{aligned} \qquad (119)$$

When $\alpha \longrightarrow 1$, we obtain the classical adiabatic invariant

$$\begin{aligned} I_{41} = & \left(a^1\right)^2 + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 + 2a^2\left(a^1\right)^2 \\ & - \frac{2}{3}\left(a^2\right)^3 + \varepsilon\Big[\left(a^1\right)^2 + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 \\ & + 2a^2\left(a^1\right)^2 - \frac{2}{3}\left(a^2\right)^3 - a^3a^4\Big]. \end{aligned} \qquad (120)$$

7. Conclusions

Differential equations of motion, Noether symmetry, conserved quantity, perturbation to Noether symmetry, and adiabatic invariants are investigated for the generalized fractional Birkhoffian system with the combined Riemann-Liouville fractional derivative, the combined Caputo fractional derivative, the Riesz-Riemann-Liouville fractional derivative, and the Riesz-Caputo fractional derivative, respectively. The classical conserved quantity and the classical adiabatic invariant are all discussed as special cases.

The fractional Lotka biochemical oscillator model with the combined Riemann-Liouville fractional derivative, the fractional Whittaker model with the Riesz-Riemann-Liouville fractional derivative, the fractional Hojman–Urrutia model with the combined Caputo fractional derivative, and the fractional Hénon–Heiles model with the Riesz-Caputo fractional derivative are discussed, and their first order adiabatic invariants are obtained.

The main results obtained in this paper can help understand the internal properties and dynamical behaviors of the systems. Since generalized fractional Birkhoffian mechanics plays an important role in many fields of modern science and engineering, more research is worth doing.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China [grant number 11802193, 11572212], the Natural Science Foundation of the Jiangsu Higher Education Institutions of China [grant number 18KJB130005], and the Science Research Foundation of Suzhou University of Science and Technology [grant number 331812137].

References

[1] K. S. Miller and B. Ross, *An Introduction to The Fractional Integrals and Derivatives-Theory and Applications*, John Wiley and Sons Inc, New York, NY, USA, 1993.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[3] J. Cresson, "Fractional embedding of differential operators and Lagrangian systems," *Journal of Mathematical Physics*, vol. 48, no. 3, Article ID 033504, 2007.

[4] F. Riewe, "Nonconservative Lagrangian and Hamiltonian mechanics," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 53, no. 2, pp. 1890–1899, 1996.

[5] F. Riewe, "Mechanics with fractional derivatives," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 55, no. 3, pp. 3581–3592, 1997.

[6] O. P. Agrawal, "Formulation of Euler-Lagrange equations for fractional variational problems," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 368–379, 2002.

[7] O. P. Agrawal, S. I. Muslih, and D. Baleanu, "Generalized variational calculus in terms of multi-parameters fractional derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 12, pp. 4756–4767, 2011.

[8] T. M. Atanacković, S. Konjik, L. Oparnica, and S. Pilipović, "Generalized Hamilton's principle with fractional derivatives," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 25, Article ID 255203, 2011.

[9] R. Almeida and D. F. M. Torres, "Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1490–1500, 2011.

[10] R. Almeida and D. F. M. Torres, "An expansion formula with high-order derivatives for fractional operators of variable order," *The Scientific World Journal*, vol. 2013, pp. 77–87, 2013.

[11] E. M. Rabei, I. M. Rawashdeh, S. Muslih, and D. Baleanu, "Hamilton-Jacobi formulation for systems in terms of Riesz's fractional derivatives," *International Journal of Theoretical Physics*, vol. 50, no. 5, pp. 1569–1576, 2011.

[12] R. A. El-Nabulsi, "Fractional variational problems from extended exponentially fractional integral," *Applied Mathematics and Computation*, vol. 217, no. 22, pp. 9492–9496, 2011.

[13] G. D. Birkhoff, *Dynamical Systems*, AMS College Publisher, Providence, RI, USA, 1927.

[14] R. M. Santilli, *Foundations of Theoretical Mechanics II*, Springer, New York, NY, USA, 1983.

[15] F. X. Mei, R. C. Shi, Y. F. Zhang, and H. B. Wu, *Dynamics of Birkhoff Systems*, Beijing University of Technology, Beijing, China, 1996.

[16] Y. M. Li and F. X. Mei, "Stability for mani-folds of equilibrium states of generalized Birkhoff system," *Chinese Physics B*, vol. 19, 2010.

[17] S.-K. Luo and Y.-L. Xu, "Fractional Birkhoffian mechanics," *Acta Mechanica*, vol. 226, no. 3, pp. 829–844, 2015.

[18] F. X. Mei, H. B. Wu, and Y. F. Zhang, "Symmetries and conserved quantities of constrained mechanical systems," *International Journal of Dynamics and Control*, vol. 2, no. 3, pp. 285–303, 2014.

[19] A. E. Noether, "Invariante variationsprobleme," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pp. 235–257, 1918.

[20] S. A. Hojman, "A new conservation law constructed without using either Lagrangians or Hamiltonians," *Journal of Physics A: Mathematical and General*, vol. 25, no. 7, pp. L291–L295, 1992.

[21] F. X. Mei, *Analytical Mechanics (II)*, Beijing Institute of Technology Press, Beijing, China, 2013.

[22] G. S. F. Frederico and D. F. Torres, "A formulation of Noether's theorem for fractional problems of the calculus of variations," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 834–846, 2007.

[23] L. Bourdin, J. Cresson, and I. Greff, "A continuous/discrete fractional Noether's theorem," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 4, pp. 878–887, 2013.

[24] S. I. Muslih, "A formulation of Noether's theorem for fractional classical fields," 2010, https://arxiv.org/abs/1003.0653.

[25] X.-H. Zhai and Y. Zhang, "Noether symmetries and conserved quantities for fractional Birkhoffian systems with time delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 36, pp. 81–97, 2016.

[26] C. J. Song and Y. Zhang, "Noether symmetry and conserved quantity for fractional Birkhoffian mechanics and its applications," *Fractional Calculus & Applied Analysis*, vol. 21, pp. 509–526, 2018.

[27] Q. Jia, H. Wu, and F. Mei, "Noether symmetries and conserved quantities for fractional forced Birkhoffian systems," *Journal of Mathematical Analysis and Applications*, vol. 442, no. 2, pp. 782–795, 2016.

[28] H.-B. Zhang and H.-B. Chen, "Noether's theorem of fractional Birkhoffian systems," *Journal of Mathematical Analysis and Applications*, vol. 456, no. 2, pp. 1442–1456, 2017.

[29] J.-L. Fu, L.-P. Fu, B.-Y. Chen, and Y. Sun, "Lie symmetries and their inverse problems of nonholonomic Hamilton systems with fractional derivatives," *Physics Letters A*, vol. 380, no. 1-2, pp. 15–21, 2016.

[30] S.-K. Luo, Y. Dai, X.-T. Zhang, and J.-M. He, "A new method of fractional dynamics, i.e., fractional mei symmetrical method for finding conserved quantity, and its applications to physics," *International Journal of Theoretical Physics*, vol. 55, no. 10, pp. 4298–4309, 2016.

[31] J. M. Burgers, "Die adiabatischen invarianten bedingt periodischer systems," *Annalen der Physik*, vol. 357, no. 2, pp. 195–202, 1917.

[32] D. S. Djukic, "Adiabatic invariants for dynamical systems with one degree of freedom," *International Journal of Non-Linear Mechanics*, vol. 16, no. 5-6, pp. 489–498, 1981.

[33] Y. Y. Zhao and F. X. Mei, "Exact invariants and adiabatic invariants of general dynamical systems," *Acta Mechanica Sinica*, vol. 28, no. 2, pp. 207–216, 1996 (Chinese).

[34] X.-W. Chen, Y.-M. Li, and Y.-H. Zhao, "Lie symmetries, perturbation to symmetries and adiabatic invariants of Lagrange system," *Physics Letters A*, vol. 337, no. 4–6, pp. 274–278, 2005.

[35] W. Jiang and S. Luo, "A new type of non-Noether exact invariants and adiabatic invariants of generalized Hamiltonian systems," *Nonlinear Dynamics*, vol. 67, no. 1, pp. 475–482, 2012.

[36] P. Wang, "Perturbation to symmetry and adiabatic invariants of discrete nonholonomic nonconservative mechanical system," *Nonlinear Dynamics*, vol. 68, no. 1-2, pp. 53–62, 2012.

[37] K. J. Zhang, J. H. Fang, and Y. Li, "Perturbation to Mei symmetry and Mei adiabatic invariants for discrete generalized Birkhoffian system," *Chinese Physics B*, vol. 20, 2010.

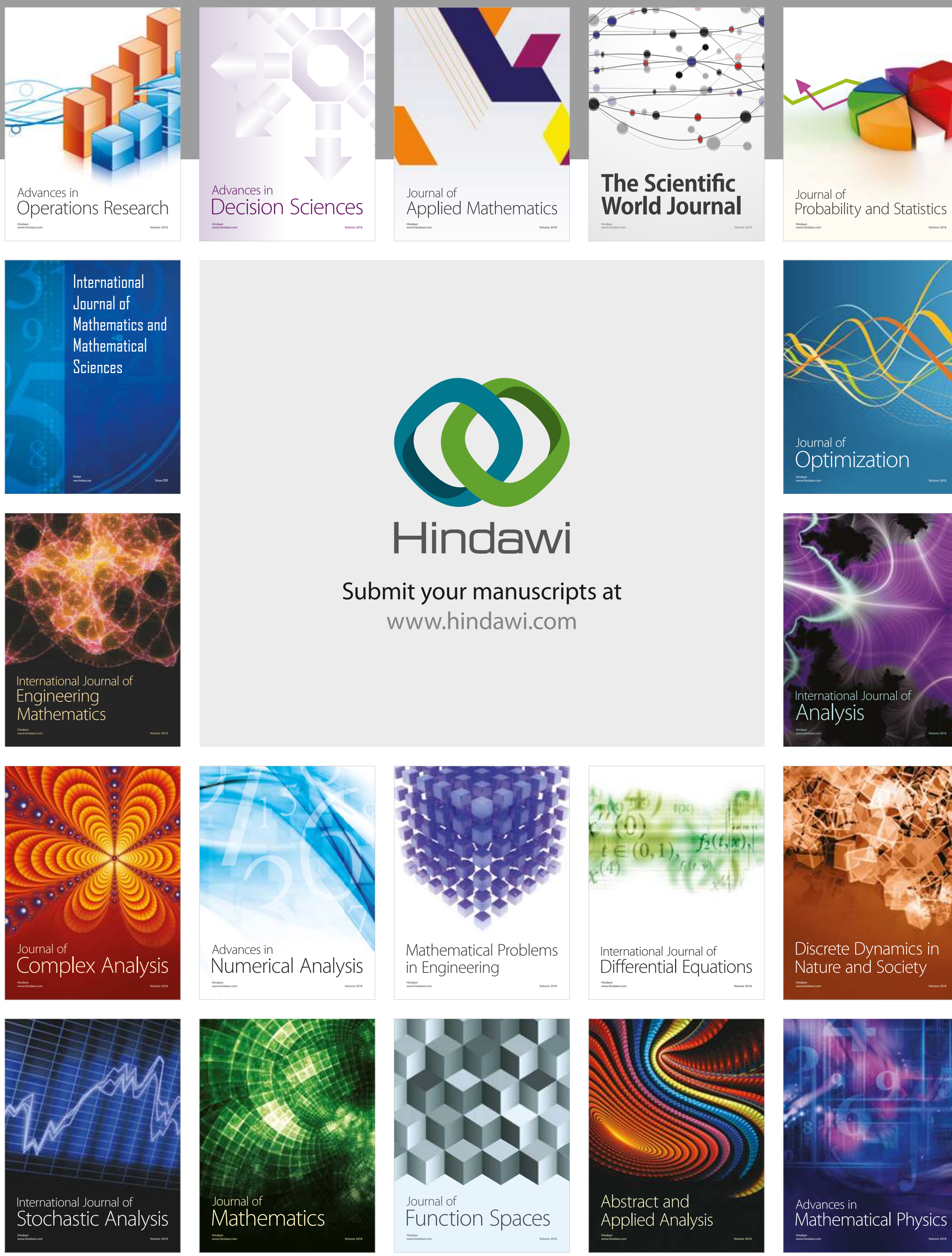
[38] J. Chen and Y. Zhang, "Perturbation to Noether symmetries and adiabatic invariants for disturbed Hamiltonian systems based on El-Nabulsi nonconservative dynamics model," *Nonlinear Dynamics*, vol. 77, no. 1-2, pp. 353–360, 2014.

[39] C.-J. Song and Y. Zhang, "Conserved quantities and adiabatic invariants for fractional generalized Birkhoffian systems," *International Journal of Non-Linear Mechanics*, vol. 90, pp. 32–38, 2017.

[40] O. P. Agrawal, "Fractional variational calculus in terms of Riesz fractional derivatives," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 24, pp. 6287–6303, 2007.

[41] Y. Zhang and X. H. Zhai, "Noether symmetries and conserved quantities for fractional Birkhoffian systems," *Nonlinear Dynamics*, vol. 81, pp. 469–480, 2015.

[42] Y. Zhang, "Perturbation to Noether symmetries and adiabatic invariants for generalized Birkhoffian systems," *Bulletin of Science and Technology*, vol. 26, pp. 477–481, 2010 (Chinese).


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
The Scientific World Journal
Journal of Probability and Statistics
International Journal of Mathematics and Mathematical Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of Optimization
International Journal of Engineering Mathematics
International Journal of Analysis
Journal of Complex Analysis
Advances in Numerical Analysis
Mathematical Problems in Engineering
International Journal of Differential Equations
Discrete Dynamics in Nature and Society
International Journal of Stochastic Analysis
Journal of Mathematics
Journal of Function Spaces
Abstract and Applied Analysis
Advances in Mathematical Physics