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ADJACENCY PRESERVING MAPPINGS OF INVARIANT SUBSPACES OF A NULL SYSTEM

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ABSTRACT. In the space I_r of invariant r-dimensional subspaces of a null system in $(2r+1)$-dimensional projective space, W.L. Chow characterized the basic group of transformations as all the bijections $\varphi : I_r \to I_r$, for which both φ and φ^{-1} preserve adjacency. In the present paper we show that the two conditions $\varphi : I_r \to I_r$ is a surjection and φ preserves adjacency are sufficient to characterize the basic group. At the end of this paper we give an application to Lie geometry.

1. Introduction

Let n, r be positive integers, $3 \leq n = 2r+1$. Let Π be an arbitrary n-dimensional Pappian projective space. A *null system* δ on Π is a polarity on Π which satisfies $x \in x^\delta$ for every point x of Π. The space of the r-dimensional subspaces of Π which are invariant under a fixed null system δ will be denoted by $I_r := \{a \in [r] \mid a^\delta = a\}$, where $[k]$, $-1 \leq k \leq n$, is the set of all k-dimensional subspaces of Π.

The *basic group of transformations* in the space I_r (also called the group of *semi-symplectic* transformations) consists of the transformations induced by all the collineations f of Π which satisfy $\delta f = f\delta$. Two invariant r-dimensional subspaces a, b are *at distance* d, if their intersection is $(r-d)$-dimensional. If $d = 1$, then they are called *adjacent*.

W.L. Chow [4] has shown that any bijection $\varphi : I_r \to I_r$ for which both φ and φ^{-1} preserve adjacency is induced by a collineation of Π. Observably, any collineation φ of Π with $\delta\varphi = \varphi\delta$ preserves adjacency in both directions. From a different point of view, L.K. Hua [5], [6] proved the fundamental theorem in the geometry of symmetric matrices under further hypotheses. For a brief history of the development of this problem see Wan [9], [10]. We may consider the theorem of Chow as a Beckman-Quarles type theorem [8] on distance preserving mappings of the space I_r. Thus Chow's theorem may be seen as an early result in the discipline *characterizations of geometrical mappings under mild hypotheses* [3].

In the present paper we characterize the basic group under mild hypotheses:

Theorem. *Let $r, n \in \mathbb{N}$, $3 \leq n = 2r+1$. Let I_r be the space of all invariant r-dimensional subspaces of a null system δ in an n-dimensional Pappian projective*

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space Π*. Let* $\varphi : I_r \to I_r$ *be a surjection satisfying*

$$\text{if } a, b \text{ are adjacent, then } a^\varphi,\ b^\varphi \text{ are adjacent} \tag{1.1}$$

for all $a, b \in I_r$*. Then* φ *is a transformation of the basic group of* I_r*.*

2. Preliminaries

In this paper, by dimension, intersection, and subspace we understand *projective* dimension, intersection, and subspace.

Let n, r be integers, $3 \le n = 2r+1$. Let Π be an arbitrary n-dimensional Pappian projective space, and let δ be a null system on Π. For any subspaces a, b of Π, we have the following well-known properties: $\dim a + \dim a^\delta = 2r$, $(a+b)^\delta = a^\delta \cap b^\delta$, $(a \cap b)^\delta = a^\delta + b^\delta$, and $a \subset b$ implies $b^\delta \subset a^\delta$. For $-1 \le k \le n$ let $[k]$ be the set of all k-dimensional subspaces of Π. For each $a \in [k]$, we call $\overline{a} := a^\delta \in [2r-k]$ the *conjugate* of a. An element $a \in [k]$ is called *invariant*, if $a \subset \overline{a}$ or $\overline{a} \subset a$. Let

$$I_k := \{a \in [k] \mid a \text{ is invariant }\}.$$

Two elements $a, b \in I_r$ are called *adjacent* if their intersection has dimension $r-1$. Let $a, b \in I_r$. The *distance* between a and b is defined to be $d(a,b) := r - \dim(a \cap b)$. If $a \ne b$, then $d = d(a,b)$ is the smallest positive integer with the property that there exists a sequence of $d+1$ invariant and consecutively adjacent subspaces $a_1 = a, a_2, \dots, a_{d+1} = b \in I_r$ (see [4]). From this property we obtain the triangle inequality $d(a,c) \le d(a,b) + d(b,c)$ for all $a, b, c \in I_r$. Let $P \in I_{r-1}$, $a \in I_r$. We define the distance between P and a by

$$d(P,a) := \min\{d(a,b) \mid b \in P^*\} = r - \dim(P \cap a) - 1$$

where $P^* := \{l \in I_r \mid P \subset l\}$.

A subset M of I_r is called a *maximal set* of adjacent elements in I_r if any two distinct elements of M are adjacent and if there are no other elements of I_r which are adjacent to each element of M.

Lemma 2.1. *A set* $M \subset I_r$ *is maximal iff there is a* $P \in I_{r-1}$ *with* $M = P^*$*.*

Proof. Assume there exists a "triangle" $a, b, c \in I_r$, such that a, b, resp. b, c, resp. c, a, are adjacent and $P := a \cap b \ne b \cap c =: Q$. Then $H := a + b = b + c = c + a \in [r+1]$. Since $P \subset a$, we have $a = a^\delta \subset P^\delta \in I_{r+1}$ and $H = a + b = P^\delta$. Similarly, $H = b + c = Q^\delta$, i.e. $P^\delta = Q^\delta$ in contradiction to that δ is a one-to-one correspondence. □

Let $\varphi : I_r \to I_r$ be a mapping which satisfies (1.1). Then $(P^*)^\varphi$ is contained in a unique maximal set Q^*. We define

$$P^\varphi := Q \qquad \text{if} \qquad (P^*)^\varphi \subset Q^*. \tag{2.1}$$

Furthermore, following the definition of distance, for any $a, b \in I_r$ we have

$$d(a,b) \ge d(a^\varphi, b^\varphi). \tag{2.2}$$

Lemma 2.2. *For any invariant subspaces* $x \in I_s$, $y \in I_t$, $t, s \le r$, *the subspace* $x + (\overline{x} \cap y)$ *is invariant and has dimension* $\le r$*. In the case* $r = t$, $x + (\overline{x} \cap y)$ *has dimension* r*.*

Lemma 2.3. *For any* $a \in I_r$ *there exists* $b \in I_r$ *with* $a \cap b = \emptyset$*, i.e.* $d(a,b) = r+1$*.*

For a proof of Lemma 2.2 and Lemma 2.3 see [4].

Lemma 2.4. *For any $a, b \in I_r$, $a \neq b$, there exists $c \in I_r$ with $d(a,c) = r+1$ and $d(b,c) < r+1$.*

Proof. Let $l \in I_r$ with $a \cap l = \emptyset$. If $b \cap l \neq \emptyset$, then take $c := l$. Suppose l satisfies $l \cap b = \emptyset$. Let $k := \dim(a \cap b) < r$. Let s be an $(r-k-1)$-dimensional subspace of b with $s + (a \cap b) = b$. Following Lemma 2.2, $c := s + (\overline{s} \cap l)$ is an invariant element with dimension r. It is clear that $b \cap c = s$ and $a \cap b \cap c = \emptyset$. Suppose $d(a,c) \neq r+1$. There is an $x \in a \cap c$. Hence $x \in c \subset \overline{s}$, and $s \subset \overline{x}$. On the other hand, $x \in a \subset \overline{x}$ implies $a \cap b \subset \overline{x}$, $b = (a \cap b) + s \subset \overline{x}$ and $x \in b$, a contradiction to $a \cap b \cap c = \emptyset$. □

Lemma 2.5. *Let $P \in I_{r-1}$, $a \in I_r$. Then $d(P,a) = k$ iff there is a uniquely determined $b \in P^*$ with $d(a,b) = k$ and $d(a,l) = k+1$ for any $l \in P^* \setminus \{b\}$.*

Proof. "$\Rightarrow$": Let $b := P + (a \cap \overline{P}) \in I_r$. Then $a \cap b = a \cap \overline{P} = \overline{a + P}$, hence $\dim(a \cap b) = 2r - \dim(a + P) = r - k$ and $d(a,b) = k$. For any $l \in P^* \setminus \{b\}$ we have $k \leq d(a,l) \leq d(a,b) + d(b,l) = k+1$. Let $l \in P^*$ with $d(a,l) = k$, i.e. $\dim(a \cap l) = r-k$. Since $a \cap l \subset a \cap \overline{P}$ and $\dim(a \cap l) = \dim(a \cap \overline{P})$, $a \cap \overline{P} = a \cap l \subset l$. Hence $l = P + (a \cap \overline{P}) = b$. "$\Leftarrow$" follows straightforwardly from the definition. □

Lemma 2.6. *Let $a, b \in I_r$ with $d(a,b) = k$. Then*

$$\min\{d(P,a) \mid P \subset b,\ P \in I_{r-1}\} = k-1.$$

Furthermore, for all $P \subset b$, $P \in I_{r-1}$ we have $d(P,a) = k-1$ iff $a \cap b \subset P$.

Proof. From Lemma 2.5, for all $P \subset b$, $P \in I_{r-1}$,

$$k = d(a,b) \in \{d(P,a), d(P,a)+1\} = \{r-1-\dim(P \cap a), r - \dim(P \cap a)\}.$$

Thus $a \cap b \subset P$ is equivalent to $\dim(P \cap a) = \dim(a \cap b) = r-k$ and to $d(P,a) = k-1$. □

Lemma 2.7. *For any $a \in I_r$ and any projective point $x \in a$, there exists $b \in I_r$ with $d(a,b) = r$ and $a \cap b = \{x\}$.*

Proof. Let $l \in I_r$, $l \cap a = \emptyset$. Define $b := x + (\overline{x} \cap l) \in I_r$. □

Lemma 2.8. *Let $P \in I_{r-1}$, $a \in I_r$. Then for any mapping $\varphi : I_r \to I_r$ satisfying* (1.1) *we have $d(P,a) \geq d(P^\varphi, a^\varphi)$ where P^φ is defined in* (2.1).

Proof. Let $b \in P^*$ with $d(a,b) = d(P,a)$. Then $b^\varphi \in (P^\varphi)^*$ and

$$d(P,a) = d(a,b) \geq d(a^\varphi, b^\varphi) \geq d(P^\varphi, a^\varphi). \tag{2.3}$$

□

3. Proof of the theorem

1. If there are $a, b \in I_r$ with $d(a,b) = r+1$ and $d(a^\varphi, b^\varphi) = r$, then for any $P \in I_{r-1}$, $P \subset a$ we have $a^\varphi \cap b^\varphi \subset P^\varphi$.

Proof. For any $P \in I_{r-1}$, $P \subset a$ we have $P^\varphi \subset a^\varphi$. Let $c \in P^* \setminus \{a\}$ with $d(b,c) = r$. Then $d(b^\varphi, c^\varphi) \leq r$ and $c^\varphi \in (P^\varphi)^*$. Since a, c are adjacent, also a^φ, c^φ are adjacent, and we have $c^\varphi \neq a^\varphi$. From Lemma 2.5, $d(P^\varphi, b^\varphi) = r-1$ and $a^\varphi \cap b^\varphi \subset P^\varphi$. □

2. For any $a, b \in I_r$, $d(a^\varphi, b^\varphi) = r$ implies $d(a,b) = r$.

Proof. Suppose not; then $d(a,b)=r+1$. Choose $c_1,\dots,c_r \in I_r$ with $d(c_i^\varphi, a^\varphi)=r$ and such that the set $\{a^\varphi \cap c_i^\varphi \mid i=1,\dots,r\} \cup \{a^\varphi \cap b^\varphi\}$ is a projective basis of a^φ. Then $r \le d(c_i,a) \le r+1$ for all $i=1,\dots,r$. Choose $Q \subset a$, $Q \in I_{r-1}$ with $Q \cap c_i = a \cap c_i$ for all $i=1,\dots,r$. In the case $d(a,c_i)=r+1$, from 1., $a^\varphi \cap c_i^\varphi \subset Q^\varphi$. In the other case $d(a,c_i)=r$ we have $d(Q^\varphi, c_i^\varphi) \le d(Q,c_i)=r-1$, so $a^\varphi \cap c_i^\varphi \subset Q^\varphi$ for all $i=1,\dots,r$. Furthermore, $a^\varphi \cap b^\varphi \subset Q^\varphi$. This is a contradiction to that $\{a^\varphi \cap c_i^\varphi \mid i=1,\dots,r\} \cup \{a^\varphi \cap b^\varphi\}$ is a basis of a^φ.

3. For any $a \in I_r$, $Q \in I_{r-1}$, $Q \subset a^\varphi$, there exists $P \in I_{r-1}$, $P \subset a$ with $P^\varphi = Q$.

Proof. Choose $c_1,\dots,c_r \in I_r$ with $d(a^\varphi, c_i^\varphi)=r$ such that $\{a^\varphi \cap c_i^\varphi \mid i=1,\dots,r\}$ is a basis of Q. Then for any $P \in I_{r-1}$ with $P \subset a$, $a \cap c_i \subset P$ implies $a^\varphi \cap c_i^\varphi \subset P^\varphi$ and $P^\varphi = Q$.

4. For any $a,b \in I_r$ with $d(a,b)=r+1$, we have $d(a^\varphi,b^\varphi)=r+1$.

Proof. We prove 4. by induction. From 2., $d(a^\varphi,b^\varphi) \ne r$. Let $d(a^\varphi,b^\varphi) \ne r+1-k$ for some $k \in \{1,\dots,r\}$. Assume that $d(a^\varphi,b^\varphi)=r-k$. Let $Q \subset a^\varphi$ with $d(Q,b^\varphi)=r-k$. Let $P \subset a$ with $P^\varphi = Q$. Choose $l \in P^* \setminus \{a\}$ with $d(l,b)=r+1$. Then, by Lemma 2.5, $l^\varphi \in Q^* \setminus \{a^\varphi\}$ implies $d(l^\varphi,b^\varphi)=r+1-k$, a contradiction. Hence $d(a^\varphi,b^\varphi) \ne r-k$.

5. φ is injective.

Proof. For any $a \ne b \in I_r$, from Lemma 2.4 there exists $c \in I_r$ with $d(a,c)=r+1$ and $d(b,c)<r+1$. Since $d(a^\varphi,c^\varphi)=r+1$ and $d(b^\varphi,c^\varphi) \le d(b,c) < r+1$, we have $a^\varphi \ne b^\varphi$.

6. $a,b \in I_r$ are adjacent if a^φ, b^φ are adjacent.

Proof. Choose $c \in I_r$ with $d(a^\varphi,c^\varphi)=r+1$ and $d(b^\varphi,c^\varphi)=r$. Denote $Q := a^\varphi \cap b^\varphi$. Let $P \subset a$, $P \in I_{r-1}$ with $P^\varphi = Q$. Let $l \in P^*$ with $d(l,c)=r$; then $l \ne a$. Since $l^\varphi \in Q^*$, $r = d(Q,c^\varphi) \le d(l^\varphi,c^\varphi) \le d(l,c) = r$. Following Lemma 2.5, we have $l^\varphi = b^\varphi$. φ is injective, hence $l=b$. So a and b are adjacent.

7. φ is a transformation of the basic group of I_r.

Proof. From 5. and 6., φ is a bijection of I_r, and φ and φ^{-1} preserve adjacency of pairs of elements of I_r. Chow's theorem completes the proof of the theorem. □

4. Application to Lie geometry

Let $\mathbf{Q}$ denote the Lie quadric

$$x_1x_2 + x_3x_4 + x_5^2 = 0$$

in the four-dimensional projective space $\Pi^4(K)$ over a commutative field K, $\operatorname{ch} K \ne 2$. We call two elements $X = K(x_1,\dots,x_5)$, $Y = K(y_1,\dots,y_5)$ of $\mathbf{Q}$ *conjugate* if

$$X \sim Y \quad :\Leftrightarrow \quad x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + 2x_5y_5 = 0.$$

The Lie transformations of $(\mathbf{Q},\sim)$ are defined as bijections α of $\mathbf{Q}$ satisfying $X \sim Y$ iff $X^\alpha \sim Y^\alpha$. Every Lie transformation is induced by a collineation of $\Pi^4(K)$ (see e.g. [2]).

On the three-dimensional projective space $\Pi^3(K)$ we define a null system δ by

$$P^\delta = \{Q \mid p_1q_3 - p_3q_1 + p_2q_4 - p_4q_2 = 0\}$$

where $P = K(p_1, \ldots, p_4)$, $Q = K(q_1, \ldots, q_4)$. There is a one-to-one correspondence γ between the space of invariant lines I_1 and the Lie quadric $\mathbf{Q}$ which satisfies

$$a, b \text{ are adjacent} \quad \Leftrightarrow \quad a^\gamma, b^\gamma \text{ are distinct and conjugate}$$

for all $a, b \in I_1$. This transformation γ is defined as follows. For every line a of $\Pi^3(K)$ consider the Plücker coordinates $K(a_{12}, a_{13}, a_{14}, a_{34}, a_{42}, a_{23})$ of a where $a_{ij} = p_i q_j - p_j q_i$ for any two distinct points $P, Q \in a$, $P = K(p_1, \ldots, p_4)$, $Q = K(q_1, \ldots, q_4)$. Then $a_{13} = a_{42}$ if, and only if, $a = a^\delta$, i.e. $a \in I_1$. Define $\gamma : I_1 \to \mathbf{Q}$ by $a \mapsto K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13})$. γ is a bijection. Furthermore, any distinct $a, b \in I_1$ with Plücker coordinates $K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13})$, $K(b_{12}, b_{34}, b_{14}, b_{23}, b_{13})$ are adjacent if, and only if,

$$\begin{aligned} & a_{12}b_{34} + a_{34}b_{12} + a_{13}b_{42} + a_{42}b_{13} + a_{14}b_{23} + a_{23}b_{14} = 0 \\ \Leftrightarrow \quad & a_{12}b_{34} + a_{34}b_{12} + a_{14}b_{23} + a_{23}b_{14} + 2a_{13}b_{13} = 0 \\ \Leftrightarrow \quad & a^\gamma, b^\gamma \text{ are conjugate.} \end{aligned}$$

In the case $(r, n) = (1, 3)$, the theorem implies the following corollary.

Corollary 4.1. *Let $\psi : \mathbf{Q} \to \mathbf{Q}$ be a surjective mapping which takes pairs of distinct conjugate points of $\mathbf{Q}$ to pairs of distinct conjugate points. Then ψ is a Lie transformation.*

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