ADJUSTMENT OF AN INVERSE MATRIX CORRESPONDING TO A CHANGE IN ONE ELEMENT OF A GIVEN MATRIX

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1. Introduction. Many methods have been published in recent years for carrying out the numerical computation of the inverse of a matrix [1], [2]. In all these methods, the amount of computation increases rapidly with increase in order of the matrix.

The utility of a computational method for obtaining the inverse of a matrix would be increased considerably if the inverse could be transformed in a simple manner, corresponding to some specified change in the original matrix, thus eliminating the necessity of computing the new inverse from the beginning. The problem that is considered in the present paper is one of changing one element in the original matrix, and of computing the resulting changes in the elements of the new inverse directly from those of the old inverse.

2. Computational method. Let

a_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ denote the elements of an nth order square matrix $\boldsymbol{a}$;

b_{ij}, denote the elements of $\boldsymbol{b}$, the inverse of $\boldsymbol{a}$;

A_{ij}, denote the elements of $\boldsymbol{A}$ which differs from $\boldsymbol{a}$ only in one element, say A_{RS};

B_{ij}, denote the elements of $\boldsymbol{B}$, the inverse $\boldsymbol{A}$.

Let

$$A_{RS} = a_{RS} + \Delta a_{RS}.$$

The set of equations by means of which $\boldsymbol{B}$ may be computed from Δa_{RS} and $\boldsymbol{b}$ is

$$B_{rj} = b_{rj} - \frac{b_{rR}\, b_{Sj}\, \Delta\, a_{RS}}{1 + b_{SR}\, \Delta\, a_{RS}}, \qquad \begin{matrix} r = 1, 2, \cdots, n, \\ j = 1, 2, \cdots, n, \end{matrix} \tag{1}$$

provided that $1 + b_{SR} \Delta a_{RS} \neq 0$.

The validity of equation (1) may be demonstrated by multiplying through by A_{ir}, $(r = 1, 2, \cdots, n)$ and adding the results:

$$\sum_{r=1}^{n} A_{ir} B_{rj} = \sum_{r=1}^{n} A_{ir} b_{rj} - \frac{b_{Sj}\, \Delta\, a_{RS}}{1 + b_{SR}\, \Delta\, a_{RS}} \sum_{r=1}^{n} A_{ir} b_{rR}, \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n). \tag{2}$$

Consider separately the equations for which $i \neq R$, and for which $i = R$.

Case I. $i \neq R$. By hypothesis, $A_{ir} = a_{ir}$ for $i \neq R$. Hence equations (2) become

$$\sum_{r=1}^{n} A_{ir} B_{rj} = \sum_{r=1}^{n} a_{ir} b_{rj} - \frac{b_{Sj}\, \Delta\, a_{RS}}{1 + b_{SR}\, \Delta\, a_{RS}} \sum_{r=1}^{n} a_{ir} b_{rR}, \qquad (i = 1, 2, \cdots, R-1, R+1, \cdots, n;\ j = 1, 2, \cdots, n). \tag{3}$$

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The last sum vanishes because $\boldsymbol{a}$ and $\boldsymbol{b}$ are inverse matrices, and hence

$$\sum_{r=1}^{n} A_{ir} B_{rj} = \sum_{r=1}^{n} a_{ir} b_{rj} \qquad (i = 1, 2, \cdots, R-1, R+1, \cdots, n; j = 1, 2, \cdots, n). \tag{4}$$

Case II. $i = R$. Equation (2) becomes

$$\sum_{r=1}^{n} A_{Rr} B_{rj} = \sum_{r=1}^{n} A_{Rr} b_{rj} - \frac{b_{Sj} \Delta a_{RS}}{1 + b_{SR} \Delta a_{RS}} \sum_{r=1}^{n} A_{Rr} b_{rR} \quad (j = 1, 2, \cdots, n). \tag{5}$$

In each of the summations, there will be a term for which $r = S$, in which case $A_{RS} = a_{RS} + \Delta a_{RS}$. In all other cases, $A_{Rr} = a_{Rr}$. Hence (5) can be written as

$$\begin{aligned} \sum_{r=1}^{n} A_{Rr} B_{rj} &= \sum_{r=1}^{n} a_{Rr} b_{rj} + \Delta a_{RS} b_{Sj} \\ &\quad - \left(\frac{b_{Sj} \Delta a_{RS}}{1 + b_{SR} \Delta a_{RS}}\right)\left(\sum_{r=1}^{n} a_{Rr} b_{rR} + \Delta a_{RS} b_{SR}\right) \qquad (j = 1, 2, \cdots, n). \end{aligned} \tag{6}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are inverse matrices, the second summation on the right-hand side of (6) is equal to unity, and hence (6) becomes

$$\sum_{r=1}^{n} A_{Rr} B_{rj} = \sum_{r=1}^{n} a_{Rr} b_{rj} \qquad (j = 1, 2, \cdots, n). \tag{7}$$

The sets of equations (4) and (7) can be written as one set of equations:

$$\sum_{r=1}^{n} A_{ir} B_{rj} = \sum_{r=1}^{n} a_{ir} b_{rj} \quad (i = 1, 2, \cdots, n; j = 1, 2, \cdots, n), \tag{8}$$

and hence $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$.

3. Illustrative numerical example. In actual applications, equations (1) are conveniently subdivided into three groups, namely, those for which $r = S$, those for which $j = R$, and all others. In the first two cases, these reduce to

$$B_{Sj} = \frac{b_{Sj}}{1 + b_{SR} \Delta a_{RS}}, \qquad (j = 1, 2, \cdots, n), \tag{9}$$

$$B_{rR} = \frac{b_{rR}}{1 + b_{SR} \Delta a_{RS}}, \qquad (r = 1, 2, \cdots, n). \tag{10}$$

By utilizing (10), (1) becomes

$$\begin{aligned} &B_{rj} = b_{rj} - B_{rR} b_{Sj} \Delta a_{RS}, \\ &(r = 1, 2, \cdots, S-1, S+1, \cdots, n; \\ &j = 1, 2, \cdots, R-1, R+1, \cdots, n). \end{aligned} \tag{11}$$

Equations (10) and (11) show that the elements of $\boldsymbol{B}$ contained in the Sth row and Rth column are directly proportional to the corresponding elements of $\boldsymbol{b}$.

Consider

$$a = \begin{pmatrix} 2.384 & 1.238 & 0.861 & 2.413 \\ 0.648 & 1.113 & 0.761 & 0.137 \\ 1.119 & 0.643 & 3.172 & 1.139 \\ 0.745 & 2.137 & 1.268 & 0.542 \end{pmatrix}.$$

The inverse of b turns out to be

$$\mathbf{b} = \begin{pmatrix} 0.2220 & 2.5275 & -0.1012 & -1.4145 \\ -0.04806 & -0.2918 & -0.1999 & 0.7079 \\ -0.1692 & 0.01195 & 0.3656 & -0.01824 \\ 0.2801 & -2.3517 & 0.07209 & 1.0409 \end{pmatrix}.$$

Assume that a_{24} is increased by 0.4, so that

$$\mathbf{A} = \begin{pmatrix} 2.384 & 1.238 & 0.861 & 2.413 \\ 0.648 & 1.113 & 0.761 & 0.537 \\ 1.119 & 0.643 & 3.172 & 1.139 \\ 0.745 & 2.137 & 1.268 & 0.542 \end{pmatrix}.$$

Then (9), (10), and (11) become

$$B_{4j} = \frac{b_{4j}}{1 - 2.3517 \times 0.4} = 16.857\, b_{4j} \qquad (j = 1, 2, \cdots, n),$$

$$B_{r2} = 16.857\, b_{r2}\ (r = 1, 2, \cdots, n),$$

$$B_{rj} = b_{rj} - 0.4\, B_{r2} b_{4j}\ (r = 1, 2, \cdots, S-1, S+1, \cdots, n;$$
$$j = 1, 2, \cdots, R-1, R+1, \cdots, n).$$

Utilization of these equations gives

$$\mathbf{B} = \begin{pmatrix} -4.5518 & 42.608 & -1.3298 & -19.155 \\ 0.5031 & -4.9191 & -0.05805 & 2.7560 \\ -0.1919 & 0.2014 & 0.3598 & -0.1021 \\ 4.7218 & -39.644 & 1.2153 & 17.547 \end{pmatrix}.$$

4. Concluding remarks. It is seen from equation (1) that if $\Delta a_{RS} = -1/b_{SR}$, that is, if a_{RS} is increased by the negative of the reciprocal of the corresponding element in the transposed reciprocal matrix, then the denominator in the second term on the right-hand side of equation (1) becomes equal to zero, and B cannot be found by the present method. It is left to the reader to verify that under these conditions A is in fact singular.

In the illustrative numerical example, the denominator is only $1 - 2.3517 \times 0.4 = 0.05932$, which accounts for the large magnitude of some of the elements of B. If Δa_{24} were taken to be $1/2.3517 = 0.4252$ instead of 0.4, A would have become singular.

If two or more elements in the matrix a are to be changed, the new inverse can be found by successive applications of the method.

REFERENCES

[1] H. Hotelling, "Some new methods in matrix calculation," *Annals of Math Stat.*, Vol. 14 (1943), pp. 1–35.

[2] P. S. Dwyer, "The solution of simultaneous equations," *Psychometrika*, Vol. 6 (1941), p. 101.

A CLASS OF RANDOM VARIABLES WITH DISCRETE DISTRIBUTIONS

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1. General results. A large class of random variables with discrete probability distributions can be derived from certain power series. Let

$$f(z) = \sum_{x=0}^{\infty} a_x z^x, \quad a_x \text{ real}, \quad |z| < r.$$

We may have either non-negative coefficients a_x or we may have $(-1)^x a_x \geq 0$. In the first case take $0 < z < r$; and in the second case take $-r < z < 0$. Define a random variable with the distribution

$$P\{\xi = x\} = \frac{a_x z^x}{f(z)}; \qquad x = 0, 1, 2, \cdots. \tag{1}$$

The above conditions insure $P\{\xi = x\} \geq 0$ for all x; besides

$$\sum_x P\{\xi = x\} = \frac{1}{f(z)} \sum_x a_x z^x = 1.$$

The distribution of ξ may be called the power series distribution (p.s.d.). The mean of such a distribution is

$$E(\xi) = \sum_x xP\{\xi = x\} = \frac{1}{f(z)} \sum_x x a_x z^x.$$

Hence it follows that

$$E(\xi) = z\frac{f'(z)}{f(z)} = z\frac{d}{dz} \log f(z). \tag{2}$$

We have for the moments about the origin

$$\mu_r' = \sum_x x^r P\{\xi = x\} = \frac{1}{f(z)} \sum_x x^r a_x z^x,$$

and hence

$$z\frac{d\mu_r'}{dz} = \frac{1}{f(z)} \sum_x x^{r+1} a_x z^x - z\frac{f'(z)}{f(z)} \frac{1}{f(z)} \sum_x x^r a_x z^x.$$