ADMISSIBILITY AND BAYES ESTIMATION IN SAMPLING FINITE POPULATIONS. II

By V. M. Joshi[1]

University of North Carolina

1. Introduction. In Part I of this paper the admissibility was investigated primarily for the class of unbiased estimates of the population total. In particular the Horvitz-Thomson estimate was shown to be admissible in the class of all unbiased estimates, (cf. Theorem 4.1 of Part I). In the following, the investigation is extended by removing the restriction of unbiasedness, with the corresponding modification of the definition of admissibility: Now some other estimate is shown to remain admissible for *all* sampling designs. The result appears to have implications concerning the basic logic of sampling with varying probabilities. These however are not discussed here.

2. Notation. The notation used here is the same as that formulated in the Section 2 of the Part I of this paper and is not restated here. The definitions and preliminaries, as given in that section, also apply in the following discussion. In addition for convenience of discussion, here we assume that the units u of the population U are numbered, that is $U = (u_1, \cdots, u_N)$, N being the total number of units u in U. As a result a sample s (Definition 2.2, Part I) can now be specified by the set of integers namely the serial numbers of the units $u \varepsilon s$. Thus for $u_r \varepsilon s$ now we write $r \varepsilon s$. Further, the variate value $x(u_r)$ associated with the unit u_r would be denoted simply by x_r, $r = 1, \cdots, N$. And we have $x = (x_1, \cdots, x_N)$, a point in Euclidean N-space R_N. Now the problem is to find an estimate (Definition 2.6, Part I), of the population total

$$T(x) = \sum_{r=1}^{N} x_r \tag{1}$$

by observing those x_r for which $r \varepsilon s$, the sample s being drawn according to a given sampling design (Definition 2.3, Part I). We extend the Definition 2.8, in Part I, of an admissible estimate by removing the restriction of unbiasedness as follows:

Definition. Given a sampling design $d = (S, p)$, an estimate $e(s, x)$ is said to be admissible for T in (1), if and only if there does not exist any other estimate $e'(s, x)$ such that

$$\sum_{s \varepsilon S} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s \varepsilon S} p(s)e(s, x) - T(x))^2 \tag{2}$$

for all $x \varepsilon R_N$, strict inequality holding true for at least one x.

3. Admissibility of an estimate. We now prove the following

Theorem. *The estimate* $e^*(s, x)$ *given by*

$$e^*(s, x) = (N/n(s)) \sum_{r \varepsilon s} x_r \tag{3}$$

Received 25 March 1964; revised 28 May 1965.

[1] On leave from Maharashtra Government, Bombay.

where $n(s)$ is the sample size (Definition 2.4, Part I), is admissible for T according to the Definition in the preceding section, for any sampling design.

REMARK. $e^*(s, x)$ can also be shown to be admissible on any subset of R_N given by $x: c_1 \leqq x_r \leqq c_2, r = 1, \cdots, N, c_1, c_2$ being some arbitrary constants with a slight obvious modification of the proof below.

PROOF. If e^* in (3), is not admissible, then by (2) there exists an estimate $e'(s, x)$ such that, for all $x \varepsilon R_N$,

$$\sum_{s \varepsilon S} p(s)(e'(s, x) - T(x))^2 \leqq \sum_{s \varepsilon S} p(s)(e^*(s, x) - T(x))^2. \tag{4}$$

We put

$$\begin{aligned} g(s, x) &= (N - n(s))^{-1}(e'(s, x) - \textstyle\sum_{r \varepsilon S} x_r), \\ g^*(s, x) &= (N - n(s))^{-1}(e^*(s, x) - \textstyle\sum_{r \varepsilon s} x_r), \end{aligned} \tag{5}$$

$n(s)$ being the sample size (Definition 2.4, Part I) of s. Now assuming $n(s) = N \rightarrow p(s) = 0$, and putting for such s, $g = g^* = c$ in (5), we have from (4)

$$\begin{aligned} &\sum_{s \varepsilon S} p(s)[(N - n(s))g(s, x) - \textstyle\sum_{r \notin s} x_r]^2 \\ &\qquad \leqq \sum_{s \varepsilon S} p(s)[(N - n(s))g^*(s, x) - \textstyle\sum_{r \notin s} x_r]^2. \end{aligned} \tag{6}$$

(Even without this assumption, the proof needs only a slight modification. For, obviously it is enough to consider in (4) estimates e' such that $e' = T$, for sample s for which $n(s) = N$.) Now taking the expectations of both sides of (6) wrt a probability distribution of R_N such that $x_1, \cdots, X_N$ are *independently* and *identically* distributed, with a common finite discrete frequency function w, common mean $\theta(w)$ and common variance $\sigma^2(w)$, we have

$$\begin{aligned} &\sum_{s \varepsilon S} p(s)(N - n(s))^2 E_w[(g(s, x) - \theta(w)) + (\theta(w) \\ &\qquad - (N - n(s))^{-1} \textstyle\sum_{r \notin s} x_r)]^2 \leqq \sum_{s \varepsilon S} p(s)(N - n(s))^2 E_w \\ &\qquad \cdot[(g^*(s, x) - \theta(w)) + (\theta(w) - (N - n(s))^{-1} \textstyle\sum_{r \notin s} x_r)]^2. \end{aligned} \tag{7}$$

The existence of E_w in (7) follows from the finite discreteness of the frequency function w. Now noting that the expectations of the product terms on both sides of (7) vanish due to the independence of $x_1, \cdots, x_N$ and cancelling out the common term $\sum_{s \varepsilon S} p(s)(N - n(s))^2 \sigma^2(w)$ on both sides of (7), we get

$$\begin{aligned} &\sum_{s \varepsilon S} p(s)(N - n(s))^2 E_w(g(s, x) - \theta(w))^2 \\ &\qquad \leqq \sum_{s \varepsilon S} p(s)(N - n(s))^2 E_w(g^*(s, x) - \theta(w))^2. \end{aligned} \tag{8}$$

Since $x_r, r = 1, \cdots, N$ are distributed independently and identically we replace in $h(s, x)$ and $h^*(s, x)$ in (8) the variates $x_r, r \varepsilon s$, in some order by $x_1, x_2, \cdots, x_m$ respectively, and let

(9) $h(s, x)$ and $h^*(s, x)$ denote the resulting values of $g(s, x)$ and $g^*(s, x)$, respectively.

Next putting in (7),

$$\sum_{s \varepsilon S_m} p(s)h(s, x) = P_m \phi_m(x) \tag{10}$$

where S_m is the set of all samples s with fixed size m, i.e. $n(s) = m$ and $P_m = \sum_{s \varepsilon S} p(s)$, we have

$$\begin{aligned} &\sum_{s \varepsilon S} p(s)(N - n(s))^2 E_w(h(s, x) - \theta(w))^2 \\ &\qquad = \sum_{m=1}^{N} (N - m)^2 \sum_{s \varepsilon S_m} p(s) E_w(h(s, x) - \phi_m(x))^2 \\ &\qquad\quad + \sum_{m=1}^{N} (N - m)^2 P_m E_w(\phi_m(x) - \theta(w))^2. \end{aligned} \tag{11}$$

Now if in (10) $h(s, x)$ is replaced by $h^*(s, x)$ in (5) and $\phi_m(x)$ by $\phi_m{}^*(x)$, then from (3), we get

$$h^*(s, x) = \phi_m{}^*(x) = \sum_{r=1}^{n(s)} x_r/n(s). \tag{12}$$

Hence from (11) and (12)

$$\begin{aligned} &\sum_{s \varepsilon S} p(s)(N - n(s))^2 E_w(h^*(s, x) - \theta(w))^2 \\ &\qquad = \sum_{m=1}^{N} P_m (N - m)^2 (\phi_m{}^*(x) - \theta(w))^2. \end{aligned} \tag{13}$$

And further from (8), (11) and (13) we get

$$\begin{aligned} &\sum_{m=1}^{N} (N - m)^2 \sum_{s \varepsilon S_m} p(s) E_w(h(s, x) - \phi_m(x))^2 \\ &\qquad + \sum_{m=1}^{N} (N - m)^2 P_m E_w(\phi_m(x) - \theta(w))^2 \\ &\qquad \leqq \sum_{m=1}^{N} (N - m)^2 P_m E_w(\phi_m{}^*(x) - \theta(w))^2. \end{aligned} \tag{14}$$

That is

$$\begin{aligned} &\sum_{m=1}^{N} (N - m)^2 P_m E_w(\phi_m(x) - \theta(w))^2 \\ &\qquad \leqq \sum_{m=1}^{N} (N - m)^2 P_m E_w(\phi_m{}^*(x) - \theta(w))^2. \end{aligned} \tag{15}$$

Now from (15) and Lemma 1 in the next section we get if $P_m \neq 0$,

$$\phi_m(x) = \phi_m{}^*(x) \tag{16}$$

for all $x \varepsilon R_N$. Further, substituting (16) in (14) we have

$$E_w(h(s, x) - \phi_m{}^*(x))^2 = 0 \tag{17}$$

for all samples s having $p(s) \neq 0$. Next from (17) and Lemma 2, in the next section, we have

$$h(s, x) = \phi_m{}^*(x) \tag{18}$$

for all s having $p(s) \neq 0$ and all x. Further from (5), (12), (18) and (19) follows the result

$$e'(s, x) = e^*(s, x). \tag{19}$$

Now (4) and (19) imply the Theorem stated at the beginning of this section.

It is interesting to note that using a result due to Hodges and Lehmann (1951) establishing the admissibility of sample mean, wrt squared error as loss, for the mean of a normal population with unit variance, we can from (15) straightaway deduce, that a.e. *in* R_m,

$$(20) \qquad \phi_m(x) = \phi_m^*(x)$$

for a fixed sample size design (i.e. $p(s) = 0$ *if* $n(s) \neq m$). Note here we have not used Lemma 1. Apart from the restriction of fixed sample size design in (20), it is important that $\phi_m(x) = \phi_m^*(x)$ in (20) is established for *almost all* points in R_m; while what we need for establishing our ultimate result is $\phi_m(x) = \phi_m^*(x)$ for *all* points in R_m, which is achieved in (16) with the help of Lemma 1.

It is also worth while to note that Aggarwal (1959) has already investigated the minimaxity of the estimate $e^*(s, x)$ in (3), on a certain subset of R_N. However he restricts himself to simple random sampling without replacement with fixed number of draws. In contrast, we establish the admissibility of the estimate e^* for any sampling design (Definition 2.3, Part I) what so ever. Further the subset of R_N considered by Aggarwal is given by $x = (x_1, \cdots, x_N)$: $\sum_{r=1}^N (x_r - T(x)/N)^2 \leqq$ const. while our Remark following the Theorem in this section establishes the admissibility of $e^*(s, x)$ on a practically much more realistic subset of R_N as explained in Section 3 of Part I of this paper.

4. Lemmas. Now we would prove the lemmas referred to in the last section.

LEMMA 1. *If*

(a) $x_1, x_2, \cdots, x_N$ *are independently and identically distributed real random variates,*

(b) *for every* $m = 1, \cdots, N$, $\phi_m(x)$ *is a real function of* $x_1, x_2, \cdots, x_m$,

(c) *for every* $m = 1, \cdots, N$, $\bar{x}_m = (1/m)\sum_{i=1}^m x_i$,

(d) *for every common finite discrete frequency function* w *of* $x_1, \cdots, x_N$,

$$\sum_{m=1}^N A_m^2 E_w(\phi_m(x) - \theta(w))^2 \leqq \sum^N A_m^2 E_w(\bar{x}_m - \theta(w))^2,$$

E_w *denoting the expectation,* $\theta(w)$ *the common mean of* $x_1, \cdots, x_N$ *and* A_m, $m = 1, \cdots, N$ *being arbitrary real constants,* then *for every* $x = (x_1, x_2, \cdots, x_N) \varepsilon R_N$, $\phi_m(x) = \bar{x}_m$ *for all* m, $m = 1, \cdots, N$ *for which* $A_m \neq 0$.

PROOF. Let $B_k \subset R_N$ be such that if $x = (x_1, \cdots, x_r, \cdots, x_N) \varepsilon B_k$ then x_r, $r = 1, \cdots, N$ contain k or less distinct values. Now by the condition (d) of the Lemma 1, considering the discrete frequency function w which is zero every where except at one point, we have, for all $x \varepsilon B_1$,

$$(1^*) \qquad \phi_m(x) = \bar{x}_m \qquad \text{for all } m = 1, \cdots, N \text{ such that } A_m \neq 0.$$

Further in the next paragraph, we prove that *if* (1^*) holds for $x \varepsilon B_{k-1}$ *then* it also holds for all $x \varepsilon B_k$, which would mean (1^*) holds for all $x \varepsilon B_N = R_N$, proving the Lemma 1.

Let the common frequency function of $x_1, \cdots, x_N$, referred to in the condition (d) of the Lemma 1, be zero except at k specified distinct values namely,

$w(t_i) = p_i$, $p_i > 0$, $i = 1, \cdots, k$ and $\sum_{i=1}^k p_i = 1$. This frequency function clearly gives positive probability only to those points $x = (x_1, \cdots, x_r, \cdots, x_N)$ for which x_r, $r = 1, \cdots, N$ is one of the values $t_1, \cdots, t_k$. Let these points x constitute the set $B_k(t_1, \cdots, t_k)$. Then $B_k(t_1, \cdots, t_k) \subset B_k$ defined in the beginning of this proof.

Throughout the remainder of the proof, summations over all $x \varepsilon B_k(t_1, \cdots, t_k)$, $x(m) \varepsilon D_{mk}(t_1, \cdots, t_k)$ and $x(m) \varepsilon D'_{mk}(t_1, \cdots, t_k)$ will be indicated by $\sum_{B_k}$, $\sum_{D_{mk}}$ and $\sum_{D'_{mn}}$, respectively.

Now writing

$$(2^*) \qquad \phi_m(x) = \bar{x}_m + h_m(x),$$

we have from (d)

$$(3^*) \qquad \sum_{m=1}^N A_m^2 \sum_{B_k} h_m(x)(\bar{x}_m - \theta) \prod_{i=1}^k p_i^{g(t_i, x)} \leqq 0,$$

$g(t_i, x)$ denoting for each $x = (x_1, \cdots, x_r, \cdots, x_N)$ the total number of those x_r, $r = 1, \cdots, N$, which are equal to t_i. Note, for all $x \varepsilon B_k(t_1, \cdots, t_k)$, $\sum_{i=1}^k g(t_i, x) = N$, $g(t_i, x) \geqq 0$ and

$$(4^*) \qquad \theta = \sum_{i=1}^N p_i t_i .$$

Now let $D_{mk}(t_1, \cdots, t_k) \subset R_m$ the m-space of the points $x(m) = (x_1, \cdots, x_m)$, the first m coordinates of $x = (x_1, \cdots, x_N)$, such that

$$(5^*) \qquad x(m) \varepsilon D_{mk}(t_1, \cdots, t_k) \quad \text{if and only if} \quad x \varepsilon B_k(t_1, \cdots, t_k).$$

Since $h_m(x)$ and $\bar{x}_m$ are defined on R_m, by summing in (3*) for all $x \varepsilon B_k(t_1, \cdots, t_k)$ with a common $x(m)$, we have,

$$(6^*) \qquad \sum_{m=1}^N A_m^2 \sum_{D_{mk}} h_m(x)(\bar{x}_m - \theta) \prod_{i=1}^k p_i^{g(t_i, x(m))} \leqq 0,$$

where $g(t_i, x(m))$ is the total number of co-ordinates in $x(m) = (x_1, \cdots, x_m)$ which are equal to t_i, $i = 1, \cdots, k$. Note that for every $x(m) \varepsilon D_{mk}(t_1, \cdots, t_k)$, $g(t_i, x(m)) \geqq 0$, $i = 1, \cdots, k$, $\sum_{i=1}^k g(t_i, x(m)) = m$, and

$$(7^*) \qquad (1/m) \sum_{i=1}^k t_i g(t_i, x(m)) = \bar{x}_m .$$

Now in (6*) let

$$(8^*) \qquad D_{mk}(t_1, \cdots, t_k) = D'_{mk}(t_1, \cdots, t_k) + D^2_{mk}(t_1, \cdots, t_k),$$

where $x(m) = (x_1, \cdots, x_m) \varepsilon D'_{mk}(t_1, \cdots, t_k)$ if and only if $x_1, \cdots, x_m$ contain *all* the distinct values $t_1, \cdots, t_k$. Now we assume that (1*) holds for $x \varepsilon B_{k-1}$. Since this assumption obviously means $A_m \neq 0 \Rightarrow h_m(x) = 0$ if the coordinates of $x(m)$ contain *less* than k distinct values, we have for $m = 1, \cdots, N$,

$$(9^*) \qquad \text{if} \quad A_m \neq 0 \quad \text{in (8*) for all} \quad x(m) \varepsilon D^2_{mk}(t_1, \cdots, t_k), \quad h_m(x) = 0.$$

From (6*) and (9*)

$$(10^*) \qquad \sum_{m=1}^N A_m^2 \sum_{D'_{mk}} h_m(x)(\bar{x}_m - \theta) \prod_{i=1}^k p_i^{g(t_i, x(m))} \leqq 0.$$

We note that in (10*),

$$(11^*) \qquad g(t_i, x(m)) \geqq 1, \qquad i = 1, \cdots, k.$$

Next we substitute (4*) and (7*) in the left hand side of (10*) and multiply it by $1/\prod_{i=1}^k p_i$. The resulting expression (note here (11*)) is further integrated over the domain

$$Q = [p_1, \cdots, p_k : p_i > 0, i = 1, \cdots, k \text{ and } \textstyle\sum_{i=1}^k p_i = 1].$$

We then have

$$(12^*) \qquad \begin{aligned} &\textstyle\sum_{m=1}^N A_m^2 \sum_{D'_{mk}} \int_Q h_m(x)(\bar{x}_m - \theta) \prod_{i=1}^k p_i^{g(t_i, x(m))-1} \prod_{i=1}^{k-1} dp_i \\ &= \textstyle\sum_{m=1}^N A_m^2 \sum_{D'_{mk}} h_m(x) \int_Q (\sum_{j=1}^k (g(t_j, x(m))/m - p_j)t_j) \\ &\quad \cdot \textstyle\prod_{i=1}^k p_i^{g(t_i, x(m))-1} \prod_{i=1}^{k-1} dp_i \\ &= 0, \end{aligned}$$

as for every j,

$$\int_Q t_j(g(t_j, x(m))/m - p_j) \prod_{i=1}^k p_i^{g(t_i, x(m))-1} \prod_{i=1}^{k-1} dp_i = 0.$$

[Note that:

$$\int_Q \prod_{i=1}^k p_i^{n_i - 1} \prod_{i=1}^{k-1} dp_i = [\Gamma(\textstyle\sum_{i=1}^k n_i)]^{-1} \prod_{i=1}^k \Gamma(n_i) \quad \text{for } n_i \geqq 1, i = 1, \cdots, k$$

Now because of (10*) the integrand in (12*) $\leqq 0$ and is also continuous in $p = (p_1, \cdots, p_k)$ for all $p \in Q$. Therefore from (12*), we have

$$(13^*) \qquad \textstyle\sum_{m=1}^N A_m^2 \sum_{D'_{mk}} h_m(x)(\bar{x}_m - \theta) \prod_{i=1}^k p_i^{g(t_i, x(m))-1} = 0$$

for all $p \in Q$. Next the condition (d) of the Lemma also gives in place of (3*), the stronger relation

$$(14^*) \qquad \textstyle\sum_{m=1}^N A_m \sum_{B_k} [h_m^2(x) + 2h_m(x)(\bar{x}_m - \theta)] \prod_{i=1}^k p_i^{g(t_i, x)} \leqq 0.$$

Then proceeding exactly as from (3*) to (10*) and lastly dividing by $\prod_{i=1}^k p_i$, from (14*), we have for all $p \in Q$,

$$(15^*) \qquad \textstyle\sum_{m=1}^N A_m^2 \sum_{D'_{mk}} [h_m^2(x) + 2h_m(x)(\bar{x}_m - \theta)] \prod_{i=1}^k p_i^{g(t_i, x(m))-1} \leqq 0.$$

Further from (13*) and (15*) we get

$$(16^*) \qquad \textstyle\sum_{m=1}^N A_m^2 \sum_{D'_{mk}} h_m^2(x) \prod_{i=1}^k p_i^{g(t_i, x(m))-1} \leqq 0$$

for all $p \in Q$. Next considering the inequality (16*) for a point $p = (p_1, \cdots, p_k) \in Q$, we have

$$(17^*) \qquad A_m \neq 0 \Rightarrow h_m(x) = 0 \qquad \text{for all } x(m) \in D'_{mk} \quad (t_1, \cdots, t_k).$$

Thus from (8*), (9*) and (17*) we have, for $m = 1, \cdots, N$,

$$(18^*) \qquad A_m \neq 0 \Rightarrow h_m(x) = 0 \qquad \text{for all } x(m) \in D_{mk}(t_1, \cdots, t_k).$$

But since $h_m(x)$ is a function of $x_1, \cdots, x_m$ we have from (5*), (18*)

$$(19^*) \qquad A_m \neq 0 \Rightarrow h_m(x) = 0 \qquad \text{for all } x \,\varepsilon\, B_k(t_1, \cdots, t_k).$$

Further since the set B_k as defined in the beginning of this proof satisfies $B_k = \bigcup_{t_1, \cdots, t_k} B_k(t_1, \cdots, t_k)$, we have from (19*), for $m = 1, \cdots, N$, $A_m \neq 0 \Rightarrow h_m(x) = 0$ for all $x \,\varepsilon\, B_k$, which along with (2*) means that, for $m = 1, \cdots, N$,

$$(20^*) \qquad A_m \neq 0 \Rightarrow \phi_m(x) = \bar{x}_m \qquad \text{for all } x \,\varepsilon\, B_k.$$

Thus as stated in the first paragraph of this proof, the Lemma 1 is proved by induction.

LEMMA 2. *If*

(a) $x_1, \cdots, x_m$ *are independently and identically distributed real random variates,*

(b) $G(x)$ *and* $H(x)$ *be real functions of* $x = (x_1, \cdots, x_m) \,\varepsilon\, R_m$,

(c) *for every common discrete frequency function* w *of* $x_1, \cdots, x_m$, $E_w(g(x) - H(x))^2 = 0$,

then $G(x) = H(x)$ for all $x = (x_1, \cdots, x_m) \,\varepsilon\, R_m$.

PROOF. Let the common frequency function w in the condition (c) of this Lemma be zero, except at m specified values, namely $w(t_i) = p_i$, $p_i > 0$, $i = 1, \cdots, m$ and $\sum_{i=1}^m p_i = 1$. This frequency function clearly gives *positive* probability say $P(x)$ only to those points $x = (x_1, \cdots, x_r, \cdots, x_m)$ for which x_r, $r = 1, \cdots, m$ is one of the values $t_1, \cdots, t_m$. Let these points x, constitute the set $B(t_1, \cdots, t_m)$. So that in condition (c) of this Lemma,

$$E_w(G(x) - H(x))^2 = \sum_{x \varepsilon B(t_1, \cdots, t_k)} P(x)(G(x) - H(x))^2 = 0,$$

which implies $G(x) = H(x)$ for all $x \,\varepsilon\, B(t_1, \cdots, t_m)$ and as $t_1, \cdots, t_m$ are arbitrary, the result $G(x) = H(x)$ for all $x \,\varepsilon\, R_m$ follows.

Acknowledgment. The author is indebted to Professor V. P. Godambe for guidance in this work, to Dr. J. Hájek for many helpful suggestions, to the referee for a thorough going scrutiny of this paper, and to Miss Patkar for doing the necessary typing very carefully.

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