ADMISSIBILITY FOR ESTIMATION WITH QUADRATIC LOSS[1]

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0. Introduction. In dealing with estimation of a single unknown parameter, the criteria most commonly employed in evaluating the worth of given estimates is to make comparisons of the expected square deviation of the estimates from the true value. Suppose on the basis of an observation x (or series of observations) on a distribution $P(x, \omega)$ of the form $\int_{-\infty}^{x} p(\xi, \omega)\, d\mu(\xi)$ depending on an unknown parameter ω it is desired to estimate some function $h(\omega)$. The quantity $p(x, \omega)$ may be regarded as the density of $P(x, \omega)$ with respect to the completely additive measure μ. A non-randomized estimate of $h(\omega)$ is described by a function of the observations $a(x)$, and when the error of an estimate is evaluated in terms of quadratic loss, the expected risk for the estimate $a(x)$ when the true parameter value is ω is calculated by means of the formula

$$\rho(\omega, a) = \int (a(x) - h(\omega))^2 p(x, \omega)\, d\mu(x). \tag{1}$$

The object is to select the estimate a which minimizes (1) in some sense. The fact that the statistician may restrict attention only to non-randomized estimates is due to the convexity property of the loss function ([1], p. 294; [2], p. 4.3). The justification of the quadratic loss as a measure of the discrepancy of an estimate derives from the following two characteristics: (i) in the case where the $a(x)$ represents an unbiased estimate of $h(\omega)$, (1) may be interpreted as the variance of $a(x)$ and, of course, fluctuation as measured by variance is very traditional in the domain of classical estimation; (ii) from a technical and mathematical viewpoint square error lends itself most easily to manipulation and computations.

Principles used to determine a particular estimate which accomplishes appropriate optimizations are related to the minimax criteria, Bayes procedures, unbiased uniformly minimum variance estimates, etc. However, one prerequisite universally acceptable as desirable for statistical procedures is the property of admissibility. An estimate a is said to be admissible if there exists no other estimate a^* such that $\rho(\omega, a^*) \leqq \rho(\omega, a)$ with inequality for some ω. In other words, an estimating procedure is admissible if it cannot be uniformly improved upon in terms of risk by any other procedure. Certainly, no estimate should be used if we can do better by a different estimate—whatever the true state of nature. It would, therefore, be of considerable interest to establish the admissibility of some of the standard estimates employed in practice.

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A more ambitious undertaking would be to try to characterize all possible admissible estimates for the case of square error. This appears to be an almost insurmountable task. On the other hand, it is relatively easy to determine complete classes of procedures for many parametric problems. In fact, whenever the density $p(x, \omega)$ possesses a monotone likelihood ratio, all possible monotone functions $a(x)$ constitute an essentially complete class of estimating procedures [3]. Nevertheless, for any multi-action problem, which includes in particular estimation, it is known that many of the members of a complete class need not be admissible [3], [4]. Furthermore, we have found that admissibility is tied very closely to the order of growth of the loss functions. Square error falls into the category which admits many monotone inadmissible estimates. For absolute error, in contrast, the likelihood that one of the usual estimates is admissible seems to be greater.

Since the general question of resolving admissibility of all estimates measured with respect to quadratic loss function is intrinsically difficult, it seems worth while to concentrate on the investigation of whether some of the most commonly employed classical estimates are admissible.

In this paper we study the problem of admissibility of the usual estimates for three important classes of distributions.

The first class of distributions comprises the exponential family where $p(x, \omega) = \beta(\omega)e^{x\omega}$. The family $a_\gamma(x) = \gamma x$ is considered as possible estimators for $h(\omega) = \frac{-\beta'(\omega)}{\beta(\omega)} = E_\omega(x) = \beta(\omega)\int xe^{x\omega}\, d\mu(x)$. Usually x represents a sufficient statistic based on several observations coming from an exponential distribution. The problem examined in general is whether γx is an admissible estimate of $E_\omega(x)$ measured in terms of quadratic loss. The parameter ω is taken to vary over its natural range Ω consisting of all ω for which $\int e^{x\omega}\, d\mu(x) < \infty$. It is well known that the natural range Ω is an interval which may be finite or infinite. In the case where $\Omega = (-\infty, \infty)$, it has been shown that $a_1(x) = x$ is admissible (see [4] and [5]). The method of proof in both references rests heavily upon the use of the Cramér-Rao inequality and associated differential inequalities. The fact that x is an unbiased estimate of $E_\omega(x)$ seems also to play a fundamental role in this proof. It seems difficult to perceive the meaning behind the analysis and the reasons why things work. In Section 1 we develop a direct proof of this fact. Our methods yield the further interesting and possibly surprising result that γx for any γ satisfying $0 < \gamma \leqq 1$ is an admissible estimate of $E_\omega(x)$ whenever μ possesses positive measure in the regions $x \geqq 0$ and $x \leqq 0$ and $\Omega = (-\infty, \infty)$. On the other hand, for any $\gamma > 1$, γx is not admissible. In view of the fact that any contraction of x (γx, $0 < \gamma \leqq 1$) is admissible it seems surprising that in practice one always uses the extreme estimate of this kind. The criteria of unbiasedness traditionally has dominated the choice of an estimate. Yet we find in several types of estimation problems that this feature of biasing the estimate by scaling it downward is necessary to achieve admissibility. We shall elaborate later on this phenomenon.

If the natural range Ω of ω is not the full infinite interval, then the full determination of the problem of admissibility of γx appears to be complicated. For the special case where

$$p(x, \omega) = \begin{cases} \dfrac{\omega^\alpha x^{\alpha-1} e^{-\omega x}}{\Gamma(\alpha)}, & x \geqq 0, \\ 0, & x < 0, \end{cases}$$

for which $\Omega = (0, \infty)$ we find that of all estimates of the form γx there exists a single admissible member in this class, namely $\gamma = [\alpha/(\alpha + 1)]$, which is a biased estimate of $E_\omega(x) = \dfrac{\alpha}{\omega}$ (see [4]).

When ω naturally ranges over a finite interval, the problem of admissibility is even more difficult. The analysis seems to depend on the rate at which $\beta(\omega)$ tends to zero as ω approaches its boundary. For example, it is shown later that, if $p(x, \omega) = \beta(\omega)e^{x\omega}(e^{-|x|}/2)$ for which $\Omega = (-1, 1)$ and $\beta(\omega) = 1 - \omega^2$, then all estimates γx $(0 < \gamma \leqq \frac{1}{2})$ are admissible estimates of $E_\omega(x)$ while for any other $\gamma > \frac{1}{2}, \gamma x$ may be uniformly improved upon in terms of risk. In general, the possible values of γ for which γx is admissible appears to be very sensitive to the explicit measure $d\mu(x)$ of the exponential family and generally consists of a subinterval of the unit interval.

The following general result concerning admissibility of γx is the assertion of Theorem 1 of Section 1: if $\beta^{-\lambda}(\omega)$ is not integrable in the neighborhood of both boundaries of Ω, then $[1/(\lambda + 1)]x$ is an admissible estimate of $E_\omega(x)$. This includes as special cases all previously known results in this direction.

Admissibility is next investigated for the class of distributions where

$$p(x, \omega) = \left\{\begin{matrix} q(\omega)r(x), & 0 \leqq x \leqq \omega, \\ 0, & x > \omega \text{ or } x < 0, \end{matrix}\right\} \quad \text{and} \quad d\mu(x) = dx.$$

r is a positive function of x and $q(\omega)$ represents a normalizing constant. This includes, in particular, extremal distributions arising from the uniform density; e.g., $r(x) = nx^{n-1}$, $n \geqq 1$, and $q(\omega) = 1/\omega^n$. We assume in what follows that $r(x)$ is such that the integral $\int_0^\infty r(x)\,dx$ diverges. This requires that the normalizing factor $q(\omega)$ approach zero as ω increases to infinity. In dealing with the estimation problem it is convenient to consider estimates of $1/[q^\alpha(\omega)]$, $\alpha > 0$, a strictly monotone increasing function of ω. Again we limit attention to estimates which are functions of a single observation x. This in fact is justifiable in every sense whenever the observation x summarizes a sufficient statistic. For example, if $x_1, \cdots, x_n$ represent independent observations from a uniform density spread on the interval $(0, \omega)$, then $\max_{1 \leqq r \leqq n}(x_r) = y$ possesses a density of the form described above, where $r(y) = ny^{n-1}$, and the justification of basing estimates of ω solely on y is manifestly clear.

Although an unbiased estimate of $h(\omega) = 1/2q(\omega)$ is $a(x) = 1/q(x)$, the only admissible estimate of the form $\gamma[1/2q(x)]$, γ a constant, is obtained for the unique value $\gamma = \frac{3}{2}$. Thus, the characteristic phenomenon appears once again to the effect that admissible estimates are obtained provided the estimate is biased by scaling downward. The same is true when treating the problem of estimating the function $h(\omega) = [1/q(\omega)]^\alpha$ with $\alpha > 0$. Analogous results are also valid for the class of distributions

$$p(x, \omega) = \begin{cases} q(\omega)r(x), & x > \omega' \\ 0, & x < \omega, \end{cases}$$

of which

$$p(x, \omega) = \begin{cases} e^{-(x-\omega)}, & x > \omega, \\ 0, & x < \omega, \end{cases}$$

is a typical example.

A possible source of explanation for the excessive uses of the principle of unbiasedness as a basis for selecting one estimate in preference to another may be due to the following considerations: First, a familiar theorem due to Blackwell states that within the collection of all unbiased estimates there exists a uniformly minimum variance unbiased estimate [6]. [This is the case if the family of densities generated by the various parameters is large enough in the sense of forming a "complete family" ([2], p. 3.6.)] This certainly lends importance and some cognizance to the consideration of unbiased estimates. Second, in considering asymptotic or large sample theory, it is found that consistent estimators are, for large sample size, nearly unbiased. For these two reasons, a tradition demanding an estimate be unbiased regardless of the sample size has become acceptable practice. From the point of view of admissibility this is almost universally the wrong estimate to use. We find the desire and need to bias an estimate to insure admissibility.

The third group of distributions studied from the point of view of estimation is related to the important translation parameter problem. The underlying density is assumed known except for a location parameter; that is, $p(x, \omega) = p(x - \omega)$ and we wish to estimate ω. In order for the problem to possess the proper invariance structure we further suppose that $d\mu(x) = dx$, $\int \xi p(\xi)\, d\xi = 0$, and $\int \xi^2 p(\xi)\, d\xi < \infty$. Consequently, we readily observe that for the case of a single observation, x is an unbiased estimate of ω. In the present context the relevance and justification of using the estimate rests primarily on its characteristics of invariance with respect to translations and only incidentally on the property of unbiasedness. With further slight conditions we establish that x is an admissible estimate of ω.

For the situation of several independent observations $x_1, x_2, \cdots, x_n$ the minimum variance invariant estimate is the familiar Pitman estimate

$$(2) \quad a^*(x) = x_1 - \frac{\int \theta p(\theta)p(x_2 - x_1 + \theta)p(x_3 - x_1 + \theta)\cdots p(x_n - x_1 + \theta)\, d\theta}{\int p(\theta)p(x_2 - x_1 + \theta)p(x_3 - x_1 + \theta)\cdots p(x_n - x_1 + \theta)\, d\theta}$$

which represents the multi-observation analogue of the estimate x [5]. If the density $p(\xi)$ is assumed to possess a sufficient number of moments, the expression for $a^*(x)$ is well-defined. Again, subject to sufficient smoothness requirements, we will show that $a^*(x)$ is an admissible estimate of ω. A special case of this result where the parameter and observation both traverse the set of integers was discussed by Blackwell [7]. He demonstrated in this case that $a^*(x)$ is admissible whenever $p(\xi)$ vanishes outside a finite interval. He also showed that without some limitations on the nature of the density p the admissibility of $a^*(x)$ is not generally valid. In connection with the translation parameter problem, a notion of local admissibility is also examined; this notion may possess a greater degree of applicability than indicated in the present context.

The method of analysis in all three cases revolves about an inversion process which we proceed to explain in formal terms. Suppose it is desired to establish that $a(x)$ is an admissible estimate of $h(\omega)$ with respect to the loss function measured by square deviation. Assume the contrary that $b(x)$ is an estimating procedure which improves upon $a(x)$. This states that the inequality

$$\int [b(x) - h(\omega)]^2 p(x, \omega)\, d\mu(x) \leqq \int [a(x) - h(\omega)]^2 p(x, \omega)\, d\mu(x)$$

must be true for all ω. Therefore

$$(3) \quad \int [b(x) - a(x)]^2 p(x, \omega)\, d\mu(x) \leqq 2 \int [a(x) - b(x)][a(x) - h(\omega)] p(x, \omega)\, d\mu(x)$$

also holds for all ω. In order to demonstrate that $a(x)$ is admissible, it is enough to show that the truth of (3) is only possible provided $b(x) = a(x)$ almost everywhere with respect to μ. Suppose it is possible to construct a monotone increasing function $F(\omega)$, not necessarily bounded, with the property that

$$\int h(\omega) p(x, \omega)\, dF(\omega) = a(x) \int p(x, \omega)\, dF(\omega).$$

Provided that all operations performed are legitimate, it follows that after integrating (3) with respect to dF and interchanging the order of integration

$$\int [b(x) - a(x)]^2 \left[\int p(x, \omega)\, dF(\omega)\right] d\mu(x) \leqq 0.$$

This implies, essentially, the desired result. Throughout what follows, we develop sufficient machinery to justify this formalism. The method may be applied to numerous other kinds of admissibility questions which are not studied in the present paper.

This formalism can also be related to the concept of the optimal Bayes procedure. If $F(\omega)$ represents a bona fide distribution and our objective is to obtain the Bayes estimate of $h(\omega)$ with respect to quadratic loss for $F(\omega)$, then it is a known fact that the best estimate is given by the expression

$$a(x) = \frac{\int h(\omega)p(x, \omega)\, dF(\omega)}{\int p(x, \omega)\, dF(\omega)}$$

(see [1], p. 299).

Unfortunately, in all cases we are concerned with the relevant $F(\omega)$ turns out to be a non-finite measure. One could then alternatively try to approach $F(\omega)$ by a sequence of distributions such that the corresponding estimates converge to the desired $a(x)$. Such a method of analysis for admissibility was proposed and exploited by Lehmann and Blyth ([2], Section 4.4; [8]). The present results might be viewed as a refinement of this idea.

The extensions of these results and method to the analogous sequential estimation problem will be published subsequently.

Finally, we wish to express our thanks to Mr. Rupert Miller for his help in the writing of this manuscript.

1. Exponential family. In this section the random variable X will be assumed to be distributed according to the probability density $dF_\omega(x) = \beta(\omega)e^{\omega x}\, d\mu(x)$. μ is a σ-finite measure defined on the real line, and ω, the unknown state of nature, belongs to the set $\Omega = \{\omega \mid \int_{-\infty}^{\infty} e^{\omega x}\, d\mu(x) < \infty\}$ which is an interval of the real line. Let $\bar{\omega}$ and $\underline{\omega}$ be the upper and lower endpoints of Ω, respectively. $\bar{\omega}$ and $\underline{\omega}$ may or may not belong to Ω, and in some cases $\bar{\omega} = +\infty$, $\underline{\omega} = -\infty$. The problem for consideration is the estimation of the quantity $\theta(\omega) = E_\omega(x) = -\beta'(\omega) / \beta(\omega)$ from a single observation x on X. There is no loss of generality in restricting our attention to the case of a single observation for, as is well-known, a sufficient statistic for n observations from an exponential distribution is the sum of the observations whose distribution is also a member of the exponential family ([1], p. 221).

Admissible estimates of $\theta(\omega)$ will be derived for the different cases depending on the structure of Ω. We shall consider only classical type estimates of the form $\gamma x = a_\gamma(x)$ where γ is a positive constant. The value $\gamma = 1$ provides the unique unbiased estimate of $E_\omega(x)$ within this family $a_\gamma(x)$.

The only estimate ordinarily considered is $a_1(x) = x$ and this appears to be due to the influence the concept of unbiasedness has had on statistical theory and practice (see our discussion in the introduction). Square error as a measure of the value of an estimate has been tacitly associated also with the principle of unbiasedness. Nevertheless, we shall find that from the point of view of admissibility it is frequently preferred to bias the estimate. Hodges and Lehmann [4] demonstrated the admissibility of $a_1(x) = x$ for a few scattered examples. Girshick and Savage [5] showed that provided $\Omega = (-\infty, \infty)$, x is admissible. Our results cover a substantially larger subclass of the full exponential family for the whole set of estimates $a_\gamma(x)$.

In view of the relations

$$\beta(\omega) \int x^2 e^{\omega x} \, d\mu(x) = \frac{2[\beta'(\omega)]^2 - \beta(\omega)\beta''(\omega)}{\beta^2(\omega)}$$

$$\beta(\omega) \int x e^{\omega x} \, d\mu(x) = -\frac{\beta'(\omega)}{\beta(\omega)},$$

we obtain that

$$\beta(\omega) \int [\gamma x - \theta(\omega)]^2 e^{x\omega} \, d\mu(x) = \gamma^2 \left[\frac{2[\beta'(\omega)]^2 - \beta(\omega)\beta''(\omega)}{\beta^2(\omega)} \right] - 2\gamma \frac{[\beta'(\omega)]^2}{\beta^2(\omega)} + \frac{[\beta'(\omega)]^2}{\beta^2(\omega)}. \tag{4}$$

For each ω in Ω the minimum of the quadratic expression in γ is achieved uniquely for the value

$$\gamma_\omega = \frac{1}{1 + \dfrac{(\beta'(\omega))^2 - \beta(\omega)\beta''(\omega)}{(\beta'(\omega))^2}} = \frac{\left(\int x e^{x\omega} \, d\mu\right)^2}{\left(\int e^{x\omega} \, d\mu\right)\left(\int x^2 e^{x\omega} \, d\mu\right)} \tag{5}$$

But $(\beta'(\omega))^2 - \beta(\omega)\beta''(\omega) = \beta^2(\omega)\sigma_x^2 > 0$ (σ_x^2 = variance of x), so that $0 < \gamma_\omega \leqq 1$. This inequality satisfied by γ_ω can also be deduced as a consequence of the Schwarz inequality on inspection of the second formula for γ_ω. It follows for any $\gamma > 1$, $\rho(\omega, a_\gamma) = \beta(\omega)\int [\gamma x - \theta(\omega)]^2 e^{x\omega} \, d\mu(x)$ is strictly increasing in γ for all ω. Consequently, if γ' is chosen satisfying $1 \leqq \gamma' < \gamma$, then $\rho(\omega, a_{\gamma'}(x)) < \rho(\omega, a_\gamma(x))$ for all ω in Ω and therefore $a_\gamma(x)$ is not admissible. This argument can be extended as follows: Suppose $[(\beta'(\omega))^2 - \beta(\omega)\beta''(\omega)]/(\beta'(\omega))^2$ ranges between L and L' ($L < L'$) as ω traverses the interval $(\underline{\omega}, \bar{\omega})$. Then γ_ω lies in the range $(1/(1 + L'), 1/(1 + L)) = I$ and for any $\gamma > 1/(1 + L)$ the same reasoning shows that $a_\gamma(x)$ is not admissible. Whenever Ω is not the full infinite interval for many circumstances $1/(1 + L) < 1$ and x is therefore not admissible. The converse implication is not valid. That is, if γ lies interior to I, then it is not necessarily true that $a_\gamma(x)$ is admissible. A counter-example may be provided as follows: Suppose the measure μ is such that it spreads its entire mass throughout the interval $1 \leqq x \leqq 2$. Then, $\theta(\omega) = E_\omega(x)$ likewise traverses the interval $[1, 2]$ as ω varies over the set $\Omega = (-\infty, \infty)$. No estimate of the form γx ($0 < \gamma < 1$) can be admissible since this entails estimating $\theta(\omega)$ as less than one with positive probability. Whenever the observed $x < (1/\gamma)$, which occurs with positive probability, an immediate improvement of the proposed estimate γx is obtained by estimating $\theta(\omega)$ as 1 in that range. This emphasizes the fact that an estimate $a_{\gamma_0}(x)$, admissible with respect to all estimates $a_\gamma(x)$, need not be universally admissible.

We direct attention to the question of admissibility for $a_\gamma(x)$ where γ is in I. Suppose $g(x)$ is an estimate which satisfies $\rho(\omega, g) \leqq \rho(\omega, a_\gamma)$ for all ω. This inequality may be reduced to the form

$$\int_{-\infty}^{\infty} [g(x) - \gamma x]^2 \beta(\omega) e^{x\omega} \, d\mu(x) \leqq 2 \int_{-\infty}^{\infty} [\gamma x - g(x)][\gamma x \beta(\omega) + \beta'(\omega)] e^{x\omega} \, d\mu(x).$$

Let $dF(\omega) = \beta^{\lambda}(\omega) \, d\omega$ for constant $\lambda \neq -1$, and let $a, b \in \Omega$, $a < b$. Also define $T(\omega) = \int_{-\infty}^{\infty} [g(x) - \gamma x]^2 \beta(\omega) e^{x\omega} \, d\mu(x)$. Then,

$$\begin{aligned} &\int_a^b \beta^{\lambda}(\omega) T(\omega) \, d\omega \\ &\qquad \leqq 2 \int_a^b \beta^{\lambda}(\omega) \left\{ \int_{-\infty}^{\infty} [\gamma x - g(x)][\gamma x \beta(\omega) + \beta'(\omega)] e^{x\omega} \, d\mu(x) \right\} d\omega \\ &\qquad = 2 \int_{-\infty}^{\infty} [\gamma x - g(x)] \left[\frac{\beta^{\lambda+1}(b) e^{xb}}{\lambda + 1} - \frac{\beta^{\lambda+1}(a) e^{xa}}{\lambda + 1} \right] d\mu(x) \\ &\qquad + 2 \int_{-\infty}^{\infty} [\gamma x - g(x)] \left[\gamma x - \frac{1}{\lambda + 1} x \right] \left[\int_a^b \beta^{\lambda+1}(\omega) e^{x\omega} \, d\omega \right] d\mu(x). \end{aligned} \tag{6}$$

Suppose $\gamma = 1/(\lambda + 1)$. Then, the last term in (6) vanishes, and by a proper application of Schwarz's inequality, (6) becomes (for $\gamma = 1/(\lambda + 1)$)

$$\begin{aligned} &\int_a^b \beta^{\lambda}(\omega) T(\omega) \, d\omega \\ &\qquad \leqq \frac{2}{\lambda + 1} \sqrt{\beta^{\lambda}(b)} \sqrt{T(b) \beta^{\lambda}(b)} + \frac{2}{\lambda + 1} \sqrt{\beta^{\lambda}(a)} \sqrt{T(a) \beta^{\lambda}(a)}. \end{aligned} \tag{7}$$

Let c be an interior point of Ω. Suppose $\int_c^b \beta^{-\lambda}(\omega) \, d\omega \to +\infty$ as $b \to \bar{\omega}$ and $\int_a^c \beta^{-\lambda}(\omega) \, d\omega \to +\infty$ as $a \to \underline{\omega}$. Then it follows that (see Cases 1 and 2 below) $T(\omega) = 0$, a.e. But this requires that $g(x) = [1/(\lambda + 1)]x$, a.e.; that is, the estimate $x/(\lambda + 1)$ is an admissible estimate.

CASE 1.

$$\varlimsup_{b \to \bar{\omega}} \beta^{\lambda}(b) \sqrt{T(b)} = \Delta > 0.$$

Fix a and let $H(b) = \int_a^b \beta^{\lambda}(\omega) T(\omega) \, d\omega$. By virtue of (7) we can find an appropriate constant $C > 0$ such that for b sufficiently close to $\bar{\omega}$,

$$H(b) \leqq C \sqrt{\beta^{\lambda}(b)} \sqrt{H'(b)}.$$

This yields by transposition and integration

$$C^2 \left[\frac{1}{H(b_1)} - \frac{1}{H(b_2)} \right] \geqq \int_{b_1}^{b_2} \beta^{-\lambda}(b) \, db,$$

where b_1, b_2 are chosen so that $b_1 < b_2$, $H(b_1) > 0$. As $b_2 \to \bar{\omega}$ the right-hand side tends to $+\infty$ and the left-hand side remains bounded—which is impossible. Thus, Case 1 cannot occur.

CASE 2.

$$\varliminf_{b \to \bar{\omega}} \beta^{\lambda}(b) \sqrt{T(b)} = 0.$$

Let $G(a) = \int_a^{\bar{\omega}} \beta^{\lambda}(\omega)T(\omega)\, d\omega$. By (7) and the assumption of Case 2 it follows that $G(a) \leqq [2/|\lambda + 1|] \sqrt{\beta^{\lambda}(a)} \sqrt{-G'(a)}$. Suppose there exists an a_0 such that $G(a_0) > 0$. Then

$$\left(\frac{2}{\lambda + 1}\right)^2 \left[\frac{1}{G(a_0)} - \frac{1}{G(a_1)}\right] \geqq \int_{a_1}^{a_0} \beta^{-\lambda}(a)\, da, \tag{8}$$

where $a_1 < a_0$. As $a_1 \to \underline{\omega}$ the right-hand side tends to $+\infty$ while the left-hand side remains bounded. This is impossible so $G(a) \equiv 0$, which implies $T(\omega) = 0$, a.e. We summarize the conclusions in the statement of a theorem.

THEOREM 1. *Let $p(x, \omega) = \beta(\omega)e^{x\omega}$ describe the density of the exponential family with respect to a measure μ. If*

$$\int_c^b \beta^{-\lambda}(\omega)\, d\omega \to +\infty \qquad \text{as } b \to \bar{\omega} \tag{i}$$

and

$$\int_a^c \beta^{-\lambda}(\omega)\, d\omega \to +\infty \qquad \text{as } a \to \underline{\omega}, \tag{ii}$$

where c is an interior point of $\Omega = (\underline{\omega}, \bar{\omega})$, then $[1/(\lambda + 1)]x$ is an admissible estimate of $\theta(\omega) = E_{\omega}(x)$.

This theorem subsumes as special consequences all previous known results in this direction (see [4] and [5]). We record several specific applications of this theorem of special interest.

I. If $\Omega = (-\infty, \infty)$ and μ possesses positive measure in each of the intervals $(0, \infty)$ and $(-\infty, 0)$, then $a_{\gamma}(x) = \gamma x$ for each $0 < \gamma \leqq 1$ is admissible. In fact, the assumptions imply that

$$\beta(\omega) = \frac{1}{\int e^{x\omega}\, d\mu(x)}$$

converges to zero as $|\omega| \to \infty$. Consequently (i) and (ii) hold for each $\lambda \geqq 0$.

II. If $\Omega = (-\infty, \infty)$ and there exists positive probability of observing the value zero, then $a_{\gamma}(x) = \gamma x$ for each $0 < \gamma \leqq 1$ is admissible. The proof follows readily from Theorem 1 since $\beta(\omega)$ is bounded above.

III. If $\Omega = (-\infty, \infty)$ with no further conditions specified as to the nature of μ, then at least $a_1(x) = x$ is an admissible estimate of $\theta(\omega)$. This is so since the hypotheses of Theorem 1 are satisfied for $\lambda = 0$.

IV. If $p(x, \omega) = \frac{(-\omega)^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{\omega x}$ for x positive where $\alpha > 0$ is fixed and ω ranges over $\Omega = (-\infty, 0)$, then $\beta(\omega) = (-\omega)^{\alpha}$ and $\theta(\omega) = -\alpha/\omega$. The unique value of λ satisfying (i) and (ii) is equal to $1/\alpha$. Consequently, $\alpha x/(\alpha + 1)$ is the only admissible estimate of $-(\alpha/\omega)$ of the form γx. In the case of n observations $x_1, x_2, \cdots, x_n$ with x_i independently normally distributed, mean 0 and variance σ^2, this result reduces to the well-known fact that

$$\theta(x) = [1/(n+2)] \sum_1^n x_i^2$$

is an admissible estimate of σ^2. The interval I in this case also reduces to a unique point.

V. If $d\mu(x) = \frac{1}{2}e^{-|x|}$, then $\beta(\omega) = 1 - \omega^2$ and the hypotheses of Theorem 1 are satisfied with $\lambda \geqq 1$. It follows that $a_\gamma(x) = \gamma x$ is admissible for $\gamma \leqq \frac{1}{2}$. Also, in this case $I = (0, \frac{1}{2})$ so that no estimate of the form γx may be admissible for $\gamma > \frac{1}{2}$.

Further examples of similar type involving definite biasing can be cited. In numerous examples calculated where Ω has at least one finite boundary we found that $a_1(x) = x$ is not admissible. We propose a stronger assertion which includes this observation. We state in conjecture that the hypotheses of Theorem 1 are also necessary conditions for the admissibility of the corresponding estimate. This would imply in particular that whenever $\beta(\omega)$ approaches infinity exponentially as ω tends to one of its boundaries no estimate of the form γx can be admissible.

2. Extreme value densities. In this section we consider densities of the form

$$p(x, \omega) = \begin{cases} q(\omega)r(x), & 0 \leqq x \leqq \omega, \\ 0, & \text{otherwise}, \end{cases}$$

where $r(x)$ is assumed to be a positive Lebesgue measurable function of x and $q^{-1}(\omega) = \int_0^\omega r(x)\,dx < \infty$ for ω in $\Omega = (0, \infty)$. We further assume that the monotone decreasing function $q(\omega)$ approaches zero as $\omega \to \infty$, or equivalently $\int_0^\infty r(x)\,dx = \infty$.

The problem examined concerns estimating functions of the form $[1/q(\omega)]^\alpha$, $\alpha > 0$. In determining proper estimators attention is directed only to estimates also of the form $\gamma[1/q(x)]^\alpha = a_\gamma(x)$ where γ is a positive constant. It is reasonable and justifiable to consider only a single observation because of the fact that x ordinarily represents a sufficient statistic.

Since $r(x) = -q'(x)/q^2(x)$ almost everywhere, we find

$$\begin{aligned} \rho(\omega, a_\gamma) &= q(\omega) \int_0^\omega \left[\frac{\gamma}{q^\alpha(x)} - \frac{1}{q^\alpha(\omega)}\right]^2 r(x)\,dx \\ &= \left[\frac{\gamma^2}{2\alpha+1} - \frac{2\gamma}{\alpha+1} + 1\right]\frac{1}{q^{2\alpha}(\omega)}. \end{aligned} \tag{9}$$

Hence, the minimum of the quadratic expression is achieved uniformly with respect to ω for the single choice $\gamma = (2\alpha + 1)/(\alpha + 1)$. For comparison purposes we note that within the family of estimates considered the unbiased estimate of $1/q^\alpha(\omega)$ is $(\alpha + 1)/q^\alpha(x)$. The unbiased estimate can therefore be uniformly improved upon in terms of expected risk by applying the bias factor $(2\alpha + 1)/(\alpha + 1)^2 < 1$. We proceed to demonstrate the admissibility of the estimator $[(2\alpha + 1)/(\alpha + 1)1]/q^\alpha(x)$ as an estimate of $1/q^\alpha(\omega)$.

The method of proof follows the same general ideas as used in the preceding section. Suppose $g(x)$ is an estimate which satisfies the property that for all ω

$$\rho(\omega, g) \leqq \rho(\omega, a_\gamma), \qquad \gamma = \frac{2\alpha + 1}{\alpha + 1}.$$

Consequently,

$$(10)\qquad \begin{aligned} a(\omega) &= \int_0^\infty \left(g(x) - \frac{\gamma}{q^\alpha(x)}\right)^2 q(\omega)r(x)\,dx \\ &\leqq 2\int_0^\infty \left[\frac{\gamma}{q^\alpha(x)} - g(x)\right]\left[\frac{\gamma}{q^\alpha(x)} - \frac{1}{q^\alpha(\omega)}\right] q(\omega)r(x)\,dx. \end{aligned}$$

In order to check admissibility for $a_\gamma(x)$ it is enough to show that the only g satisfying this system of inequalities is $g(x) = a_\gamma(x)$, a.e. In view of the formalism indicated in the introduction the aim is to integrate the formula of (10) with respect to an appropriate monotone increasing function in order to cause the right-hand side to vanish. This essentially implies admissibility. Accordingly, we select $dF(\omega) = |\, q'(\omega) \,|\, q^\beta(\omega)\,d\omega$ where $\beta = 2\alpha - 1$. Then,

$$(\beta + 2)/(\beta + 2 - \alpha) = \gamma = (2\alpha + 1)/(\alpha + 1).$$

By direct calculation we obtain

$$(11)\qquad \begin{aligned} &\int_\epsilon^\tau a(\omega)\,|\, q'(\omega)\,|\, q^\beta(\omega)\,d\omega \leqq \\ &\quad 2\int_\epsilon^\tau \left\{\int_0^\infty \left[\frac{\gamma}{q^\alpha(x)} - g(x)\right]\left[\frac{\gamma}{q^\alpha(x)} - \frac{1}{q^\alpha(\omega)}\right] |\, q'(\omega)\,|\, q^{\beta+1}(\omega)r(x)\,dx\right\} d\omega \\ &= \frac{2}{\beta + 2 - \alpha}\left\{\int_0^\tau \left[\frac{\gamma}{q^\alpha(x)} - g(x)\right] r(x)q^{\beta+2-\alpha}(\tau)\left[1 - \frac{q^\alpha(\tau)}{q^\alpha(x)}\right] dx\right\} \\ &\quad - \frac{2}{\beta + 2 - \alpha}\left\{\int_0^\epsilon \left[\frac{\gamma}{q^\alpha(x)} - g(x)\right] r(x)q^{(\beta+2-\alpha)}(\epsilon)\left[1 - \frac{q^\alpha(\epsilon)}{q^\alpha(x)}\right] dx\right\}. \end{aligned}$$

Since $q(x) \geqq q(\tau)$ for $x \leqq \tau$, we deduce with the aid of Schwarz's inequality that

$$(12)\qquad \begin{aligned} &q^\alpha(\tau)\int_0^\tau \left|\frac{\gamma}{q^\alpha(x)} - g(x)\right| \sqrt{r(x)q(\tau)}\,\sqrt{r(x)q(\tau)}\left|1 - \frac{q^\alpha(\tau)}{q^\alpha(x)}\right| dx \\ &\qquad\leqq \sqrt{a(\tau)}\,q^\alpha(\tau) = \sqrt{a(\tau)q^\beta(\tau)}\,\sqrt{q(\tau)}. \end{aligned}$$

In a similar way the second integral of (11) has a bound equal to

$$\sqrt{a(\epsilon)q^\beta(\epsilon)}\,\sqrt{q(\epsilon)}.$$

By combining the relations of (11) and (12) and the last stated bound, we obtain

$$\int_\epsilon^\tau a(\omega) \mid q'(\omega) \mid q^\beta(\omega)\, d\omega \leqq \frac{2}{\alpha+1} \tag{13}$$
$$\times \left[\sqrt{a(\tau)q^\beta(\tau) \mid q'(\tau) \mid} \sqrt{\frac{q(\tau)}{\mid q'(\tau) \mid}} + \sqrt{a(\epsilon)q^\beta(\epsilon) \mid q'(\epsilon) \mid} \sqrt{\frac{q(\epsilon)}{\mid q'(\epsilon) \mid}} \right].$$

The analysis proceeds by examining two possible cases.

CASE 1.

$$\overline{\lim_{\tau \to \infty}} \sqrt{a(\tau) \mid q'(\tau) \mid q^\beta(\tau)} \sqrt{q(\tau)/\mid q'(\tau) \mid} = \Delta > 0.$$

Fix ϵ and set $H(\tau) = \int_\epsilon^\tau a(\omega) \mid q'(\omega) \mid q^\beta(\omega)\, d\omega$. There exists a constant C such that for sufficiently large τ

$$H(\tau) \leqq C\sqrt{H'(\tau)} \sqrt{\frac{q(\tau)}{q'(\tau)}}. \tag{14}$$

We now show that this relation leads to an absurdity. Indeed, squaring the expression of (14) and solving the differential inequality, we deduce that

$$C^2 \left[\frac{1}{H(\beta)} - \frac{1}{H(\alpha)} \right] \leqq \log \frac{q(\beta)}{q(\alpha)}, \tag{15}$$

where $\beta > \alpha$ and α sufficiently large. As $\beta \to \infty$ the left-hand side of (15) remains bounded while the right-hand side tends to $-\infty$ which is impossible. Thus, Case 1 cannot occur.

CASE 2.

$$\overline{\lim_{\tau \to \infty}} \sqrt{a(\tau) \mid q'(\tau) \mid q^\beta(\tau)} \sqrt{q(\tau)/\mid q'(\tau) \mid} = 0.$$

Let τ tend to $+\infty$ along a sequence $\{\tau_n\}$ for which

$$\lim_{n \to \infty} \sqrt{a(\tau_n) \mid q'(\tau_n) \mid q^\beta(\tau_n)} \sqrt{q(\tau_n)/\mid q'(\tau_n) \mid} = 0.$$

Then, by (13)

$$G(\epsilon) = \int_\epsilon^\infty a(\omega) \mid q'(\omega) \mid q^\beta(\omega)\, d\omega \leqq \frac{2}{\alpha+1} \sqrt{a(\epsilon) \mid q'(\epsilon) \mid q^\beta(\epsilon)} \sqrt{q(\epsilon)/\mid q'(\epsilon)\mid}. \tag{16}$$

Suppose $G(\epsilon_0) > 0$. Then $G(\epsilon) \geqq G(\epsilon_0) > 0$ for $\epsilon \leqq \epsilon_0$. (16) can be written as $G(\epsilon) \leqq [2/(\alpha+1)]\sqrt{-G'(\epsilon)}\sqrt{q(\epsilon)/\mid q'(\epsilon) \mid}$. Transposition of terms in this expression and integration over (ϵ_1, ϵ_0) yields

$$\left(\frac{2}{\alpha+1}\right)^2 \left[\frac{1}{G(\epsilon_1)} - \frac{1}{G(\epsilon_0)} \right] \leqq \log \frac{q(\epsilon_0)}{q(\epsilon_1)}. \tag{17}$$

As $\epsilon_1 \to 0$ the left-hand side remains bounded but the right-hand side tends to $-\infty$ which is an absurdity. Thus the supposition that $G(\epsilon_0) > 0$ for some $\epsilon_0 > 0$

is erroneous, and therefore $G(\epsilon) \equiv 0$. Consequently, $a(\omega) = 0$, a.e., which implies $g(x) = \gamma/q^\alpha(x)$, a.e.

We have thus established the truth of

THEOREM 2. *There exists a single admissible estimate of $1/q^\alpha(\omega)$ of the form $\gamma/q^\alpha(x)$, and this is given by $\gamma = (2\alpha + 1)/(\alpha + 1)$.*

The following specific application might be of some interest. Let $r(x) = nx^{n-1}$. Then $[(2 + n)/(1 + n)]x$ is an admissible estimate of ω. Furthermore, this is the only admissible estimate which is a multiple of x.

This states that if $x_1, x_2, \cdots, x_n$ represents n independent observations from a rectangular density spread on the interval $(0, \omega)$, then

$$[(n + 2)/(n + 1)] \max_i x_i$$

is an admissible estimate of ω with respect to squared error.

To pinpoint the reasons for the validity of the preceding methodology it seems worth emphasizing that although for any γ it is possible to construct a measure $q^\beta(\omega)| q'(\omega)|$ which formally gives

$$\frac{\gamma}{q^\alpha(x)} = \frac{\int_x^\infty q^{\beta+1-\alpha}(\omega)q'(\omega)\,d\omega}{\int_x^\infty q^{\beta+1}(\omega)q'(\omega)\,d\omega},$$

nevertheless, the reader will find that it is only possible to justify the formalism for the special choices $\beta = 2\alpha - 1$ and $\gamma = (2\alpha + 1)/(\alpha + 1)$ as we have done. The estimate $[(2\alpha + 1)/(\alpha + 1)]/q^\alpha(x)$ of $1/q^\alpha(\omega)$, although uniquely admissible with respect to square error, is still not altogether acceptable. It is disturbing to note that the estimate $[(2\alpha + 1)/(\alpha + 1)]/q^\alpha(x)$ is very closely tied to the measure of error described by quadratic loss. If the risk function is given by

$$\rho(\omega, a_\gamma(x)) = q(\omega) \int_0^\omega \left[\frac{\gamma}{q^\alpha(x)} - \frac{1}{q^\alpha(\omega)}\right]^{2k} r(x)\,dx$$
$$= \left\{\sum_{r=0}^{2k} (-1)^r \frac{\gamma^r}{r\alpha + 1} \binom{2k}{r}\right\} \frac{1}{q^{2k\alpha}(\omega)},$$

it can be shown that the minimum is achieved uniformly in ω at a value γ_k which strictly varies with k. This implies that an admissible estimate with respect to square error need not be admissible when considered for the error function involving 4th powers. It is found that when $\alpha = 1$ in the case of square error the best estimate of the type $\gamma/q(x)$ is $\frac{3}{2}\, 1/q(x)$ while for the loss function of fourth powers the best estimate is $\gamma^*/q(x)$ where $\gamma^* > 0$ satisfies

$$\tfrac{4}{5}\gamma^3 - 3\gamma^2 + 4\gamma - 2 = 0,$$

which is slightly larger than $\frac{3}{2}$.

We close this section with a brief discussion of the problem of admissibility for the density

$$p(x, \omega) = \begin{cases} q(\omega)r(x), & x \geqq \omega, \\ 0, & \omega_0 \leqq x < \omega, \end{cases}$$

where $r(x)$ is a positive measurable function of x and $q^{-1}(\omega) = \int_\omega^\infty r(x)\, dx < \infty$ for ω in $\Omega = (\omega_0, \infty)$.

One important such example is furnished by taking $r(x) = e^{-x}$, $q(\omega) = e^{\omega}$, and $\Omega = (-\infty, \infty)$. Another example is obtained by setting $r(x) = 1/x^{\delta}$, $\delta > 1$ and $\omega_0 = 0$. As before our problem is to estimate the quantity $1/q^{\alpha}(\omega)$ by using estimates of the form $\gamma/q^{\alpha}(x)$. We assume in what follows that $q(\omega_0) = 0$ or equivalently $\int_{\omega_0}^\infty r(x)\, dx = \infty$.

THEOREM 3. *If*

$$p(x, \omega) = \begin{cases} q(\omega)r(x), & x \geqq \omega, \\ 0, & \omega_0 \leqq x < \omega, \end{cases}$$

where $q^{-1}(\omega) = \int_\omega^\infty r(x)\, dx$ *and* $q(\omega_0) = 0$, *then* $[(2\alpha + 1)/(\alpha + 1)]/q^{\alpha}(x)$ *is an admissible estimate of* $1/q^{\alpha}(\omega)$ *with respect to quadratic loss.*

The proof of Theorem 3 parallels that of Theorem 2 subject to simple obvious modifications and will therefore be omitted.

3. Translation parameter problem: single observation. The random variable X is distributed according to the probability density $p(x, \omega) = p(x + \omega)$ where $\omega \in \Omega$ is the unknown state of nature and $p(\xi)$ is a known, fixed density function which satisfies $\int_{-\infty}^\infty \xi p(\xi)\, d\xi = 0$. The analogous problem where X is an integer-valued random variable and the parameter likewise ranges over the set of discrete integers will be discussed later. The problem is to estimate the parameter $-\omega$. If x is the single observed value, then the usual (unbiased, invariant) estimate of $-\omega$ is $\delta(x) = x$. The property of unbiasedness is easily verified and for its relationship to invariance the reader is referred to [1]. The principal goal of this section is to establish the admissibility of this estimate, $\delta(x) = x$, subject to appropriate smoothness conditions.

This formulation of the translation parameter problem differs notationally from the customary version. If ω is substituted for $-\omega$, then the familiar form of the problem will emerge. The difference in the formulation of the problem is not significant in any way and on the other hand is helpful in that it leads to a more convenient form for applying theorems on Fourier transforms.

To establish admissibility it is sufficient to show that the inequality

$$\rho(\omega, g) \leqq \rho(\omega, \delta),$$

or equivalently

$$\int_{-\infty}^{\infty} [x - g(x)]^2 p(x + \omega)\, dx \leqq 2 \int_{-\infty}^{\infty} [x - g(x)][x + \omega] p(x + \omega)\, dx, \tag{18}$$

implies $g(x) = x$, a.e.

To accomplish this it is necessary to impose the following assumption.

ASSUMPTION I.

$$\int_{-\infty}^{\infty} \xi^2 p(\xi)\, d\xi < \infty, \qquad \int_{-\infty}^{\infty} \xi^2 p^2(\xi)\, d\xi < \infty, \quad \text{and} \quad \int_{-\infty}^{\infty} \xi p(\xi)\, d\xi = 0.$$

The meaning and relevance of the last condition was discussed above. The first integrability requirement is indispensible in order that (18) define a meaningful relationship. The second finiteness condition represents a slight further restriction beyond that of the first integral. For instance, the second integrability condition would be an immediate consequence of the first integrability condition and boundedness of the density $p(\xi)$.

We further assume initially that we deal only with alternative estimates $g(x)$ satisfying $| g(x) - x | \leqq M < \infty$. The nature of this restriction is considerably milder than might appear at first glance. It will later be shown that this constraint may be completely eliminated or, equivalently, we will show the only estimates for which (18) is possible must satisfy this restraint.

Unless stated to the contrary we suppose hereafter that Assumption I and the boundedness requirement on competing estimates are satisfied.

LEMMA 1. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta)$ *for all* ω, *then* $\int_{-\infty}^{\infty} (g(x) - x)^2\, dx < \infty$.

PROOF. Define $\Phi(u) = \int_{-\infty}^{u} \xi p(\xi)\, d\xi$. Then

$$\begin{aligned} &\int_{-n}^{n} \left[\int_{-\infty}^{\infty} [x - g(x)]^2 p(x + \omega)\, dx \right] d\omega \\ (19) \qquad &\leqq 2 \int_{-\infty}^{\infty} | x - g(x) |\, dx \left| \int_{-n}^{n} (x + \omega) p(x + \omega)\, d\omega \right| \\ &\leqq 2M \int_{-\infty}^{\infty} | \Phi(x + n) - \Phi(x - n) |\, dx \leqq 4M \int_{-\infty}^{\infty} (-\Phi(u))\, du \end{aligned}$$

as $-\Phi(u)$ is positive. But,

$$\left| u \int_{-\infty}^{u} \xi p(\xi)\, d\xi \right| \leqq \int_{-\infty}^{u} \xi^2 p(\xi)\, d\xi \qquad \text{as } u \to -\infty$$

and

$$\left| u \int_{-\infty}^{u} \xi p(\xi)\, d\xi \right| = \left| u \int_{u}^{\infty} \xi p(\xi)\, d\xi \right| \leqq \int_{u}^{\infty} \xi^2 p(\xi) d\xi \quad \text{as } u \to +\infty$$

so

$$| u\Phi(u)| \to 0 \qquad \text{as} \qquad | u | \to \infty.$$

Hence, integration by parts yields

$$\int_{-\infty}^{\infty} (-\Phi(u))\, du = \int_{-\infty}^{\infty} \xi^2 p(\xi)\, d\xi < \infty.$$

Allowing n to tend to infinity in (19) after interchanging the order of integration produces the desired result.

THEOREM 4. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta)$ *for all* ω, *then* $g(x) = \delta(x)$, *a.e.*

PROOF. By Lemma 1 $\theta_1(\xi) = \xi - g(\xi) \,\varepsilon\, L^2$ so by Plancherel's theorem its Fourier transform $\varphi_1(u)$ is defined and belongs to L^2.

$$\varphi_1(u) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iu\xi}\theta_1(\xi)\, d\xi.$$

According to Assumption I $\theta_2(\xi) = \xi p(\xi) \,\varepsilon\, L^2$ so also its Fourier transform $\varphi_2(u)$ exists and

$$\varphi_2(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iu\xi}\theta_2(\xi)\, d\xi$$

(since the integral exists). The function $\theta(\omega) = \int_{-\infty}^{\infty} \theta_1(x)\theta_2(x + \omega)\, dx$, which is essentially a convolution of θ_1 and θ_2, also belongs to L^2. It can be readily verified that its Fourier transform is $\varphi(u) = \varphi_1(-u)\varphi_2(u)(\varepsilon L^2)$.

Since φ_1 and φ_2 both belong to L^2, by Schwarz's inequality

$$\varphi = \varphi_1(-u)\varphi_2(u) \,\varepsilon\, L^1.$$

By the inversion theorem on Fourier transforms

$$\int_{-\infty}^{\infty} (x - g(x))(x + \omega)p(x + \omega)\, dx = \int_{-\infty}^{\infty} e^{-i\omega u}\varphi_1(-u)\varphi_2(u)\, du \tag{20}$$

for real ω as both sides represent continuous functions, the first by virtue of the fact that $\xi p(\xi)$ is in L^1 and the second since $\varphi_1 \cdot \varphi_2$ is in L^1 as established. If both sides of (20) are integrated from $-n$ to n and the order of integration is reversed on the right-hand side (which is permissible for reasons indicated below)

$$\begin{aligned} &\int_{-n}^{n} \left\{ \int_{-\infty}^{\infty} (x - g(x))(x + \omega)p(x + \omega)\, dx \right\} d\omega \\ &\qquad = \int_{-\infty}^{\infty} \frac{\varphi_1(-u)\,\varphi_2(u)}{iu} [e^{-inu} - e^{inu}]\, du. \end{aligned} \tag{21}$$

To justify the interchange on the right-hand side we observe first that $\varphi_2(0) = 0$ while $\varphi_2'(u) = [i/\sqrt{2\pi}]\int_{-\infty}^{\infty} e^{iu\xi}\xi^2 p(\xi)\, d\xi$ is bounded independently of u. By the mean value theorem $\varphi_2(u)/u = \varphi_2'(\tilde{u})$ where $0 \leqq \tilde{u} \leqq u$. Thus,

$$\lim_{u \to 0} \varphi_2(u)/u < \infty$$

and $\int_{-\epsilon}^{\epsilon} \varphi_2(u)/u\, du < \infty$ for $\epsilon > 0$. This implies $[\varphi_1(-u)\varphi_2(u)]/u \,\varepsilon\, L^1$. But the Riemann-Lebesgue theorem asserts that for any $q \,\varepsilon\, L^1, \int_{-\infty}^{\infty} e^{iu\omega} q(u)\, du \to 0$ as $\omega \to \pm\infty$. Therefore,

$$\lim_{n \to \infty} \int_{-n}^{n} \left\{ \int_{-\infty}^{\infty} (x - g(x))(x + \omega)p(x + \omega)\, dx \right\} d\omega = 0,$$

and on account of (18) we may infer that

$$\int_{-\infty}^{\infty} (g(x) - x)^2\, dx = \int_{-\infty}^{\infty} (g(x) - x)^2 \left\{ \int_{-\infty}^{\infty} p(x + \omega)\, d\omega \right\} dx \leqq 0,$$

which implies $g(x) = x$, a.e.

As mentioned previously, there are two general cases in which the boundedness assumption is satisfied by all g which need be considered.

CASE 1: $$p(\xi)\begin{cases} = 0, & \xi < a, \xi > b, -\infty < a < b < \infty, \\ \geqq 0, & \text{otherwise.}\end{cases}$$

This type of density is fairly general and will occur, for instance, when any distribution is truncated at finite endpoints.

Suppose x is the observed value. Then, because of the form of $p(\xi)$

$$x - b \leqq -\omega \leqq x - a.$$

Any estimate which assumes values outside the interval $[x - b, x - a]$ can be improved upon by an estimate $h(x)$ which satisfies $x - b \leqq h(x) \leqq x - a$ for all x. Intuitively this is clear; a rigorous proof may be readily supplied by the reader. Thus if $\rho(\omega, g) \leqq \rho(\omega, \delta)$ for all ω and g does not satisfy the boundedness assumption, there exists another estimate $h(x)$ such that

$$| h(x) - x | \leqq | a | + | b |$$

and $\rho(\omega, h) \leqq \rho(\omega, g) \leqq \rho(\omega, \delta)$ for all ω. Since this implies $h(x) = x$, a.e., and hence $\rho(\omega, h) \equiv \rho(\omega, \delta)$, $\rho(\omega, g) \equiv \rho(\omega, \delta)$ which implies that δ is admissible.

In addition, note that Assumption I is automatically satisfied in this case.

CASE 2: $p(x, \omega) = p(x + \omega)$ has a monotone likelihood ratio.

LEMMA 2: *If $\rho(\omega, g) \leqq \rho(\omega, \delta)$ for all ω, then there exists a constant C such that*

$$\int_{-\infty}^{\infty} (g(x) - x)^2 p(x + \omega)\, dx \leqq C$$

for all ω (under Assumption I).

PROOF. By Schwarz's inequality

$$\frac{1}{2}\int_{-\infty}^{\infty} (g(x) - x)^2 p(x + \omega)\, dx < \int_{-\infty}^{\infty} (x - g(x))(x + \omega)p(x + \omega)\, dx$$
$$\leqq \left[\int_{-\infty}^{\infty} (g(x) - x)^2 p(x + \omega)\, dx\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} (x + \omega)^2 p(x + \omega)\, dx\right]^{\frac{1}{2}}.$$

It follows easily that

$$\int_{-\infty}^{\infty} (g(x) - x)^2 p(x + \omega)\, dx \leqq 4 \int_{-\infty}^{\infty} \xi^2 p(\xi)\, d\xi = C < \infty.$$

Without loss of generality it can be assumed that g is a monotone estimate (i.e., $x_1 < x_2$ implies $g(x_1) \leqq g(x_2)$). Since the monotone estimates constitute a complete class (cf. [3]), any estimate which improves upon δ and is not monotone is in turn improved upon by a monotone estimate.

We add for the purposes of convenience the following assumption, which is so exceptionally weak as not to constitute any real restriction.

ASSUMPTION II. There exist constants a_1, a_2, b such that $a_1 < a_2$, $a_2 - a_1 < 1$, $b > 0$, and $p(\xi) \geqq b$ for $a_1 \leqq \xi \leqq a_2$.

Suppose there exists a sequence $\{x_i\}$ for which $g(x_i) - x_i \to \infty$ as $i \to \infty$. Then, there must exist for any n an index i_n such that $g(x_{i_n}) - x_{i_n} \geqq n$. Since g is monotone, $g(\xi) - \xi \geqq n - 1$ for $x_{i_n} \leqq \xi \leqq x_{i_n} + 1$. Let

$$\tilde{\omega} = (a_1 + a_2)/2 - (x_{i_n} + \tfrac{1}{2}).$$

Then,

$$b(a_2 - a_1)(n - 1)^2 \leqq \int_{-\infty}^{\infty} (g(x) - x)^2\, p(x + \tilde{\omega})\, dx. \tag{22}$$

But by Lemma 2 the integral is bounded by $C < \infty$. Since n is arbitrary, this leads to a contradiction.

A similar argument applies if there exists a subsequence $\{x_i\}$ such that

$$g(x_i) - x_i \to -\infty$$

as $i \to \infty$. Thus, $g(x) - x$ must remain bounded and the admissibility of

$$\delta(x) = x$$

then follows according to Theorem 4 in the case where $p(x + \omega)$ has a monotone likelihood ratio.

The preceding argument also shows that in order for $| g(x) - x |$ to be unbounded and consistent with the result of Lemma 2 it must peak up very sparsely for durations of increasingly shorter lengths. Such pathologies are not excluded readily by means of our methods except for the two cases discussed. It seems unreasonable to admit such estimates for consideration.

A third case for which Theorem 4 is valid without the assumption of boundedness being necessary corresponds to the situation where $p(\xi)$ tends to zero exceptionally fast. More precisely, we assume that

$$\frac{\left| \int_{-\infty}^{u} \xi p(\xi)\, d\xi \right|}{p(u)} \leqq C. \tag{23}$$

For example this is satisfied by the standard normal distribution. The boundedness assumption was only used to prove Lemma 1 which on closer inspection is also valid whenever we can show that the expressions

$$\int_{-\infty}^{\infty} | g(x) - x | \, [-' \Phi (x + n)]\, dx = A(n)$$

and

$$\int_{-\infty}^{\infty} | g(x) - x | \, [- \Phi (x - n)]\, dx = B(n)$$

are uniformly bounded. We study only the case of $A(n)$, the argument being similar for $B(n)$. Invoking Schwarz's inequality, we obtain

$$A(n) \leqq \sqrt{\int [g(x) - x]^2 (-\Phi(x+n))\, dx} \sqrt{\int (-\Phi(u))\, du}$$

$$\leqq C\left\{\int [g(x) - x]^2\, p(x+n)\, dx\right\}^{\frac{1}{2}} \leqq C',$$

where the last inequality is valid because of Lemma 2.

What we have shown for the general problem is that $\delta(x) = x$ is admissible within the class of all estimates g satisfying $|g(x) - x| \leqq M$ for all x where M is any finite constant. This means that x is admissible with respect to all estimates which do not differ too wildly from it. An appropriate formulation of the conclusions may be made in terms of a concept of local admissibility.

We close this section with a brief discussion of the case where the observation is integer-valued and the parameter ω also traverses the set of all integers. The analysis is considerably simpler.

In this case we can deduce immediately from the analog of Lemma 2 and equation (22) that if $\rho(\omega, g) \leqq \rho(\omega, \delta)$ for all integral ω then $|g(x) - x|$ is uniformly bounded.

Assumption I may be slightly weakened and now takes the form:

$$\sum_x x^2\, p(x) < \infty \quad \text{and} \quad \sum x\, p(x) = 0. \tag{24}$$

The role of Fourier transforms in the analysis is now replaced by Fourier series and the general line of the arguments carries over to the discrete case *mutatis mutandis*. Summing up we get:

THEOREM 5. *Suppose x and ω are discrete and integer-valued, and the conditions* (24) *are satisfied. If $\rho(\omega, g) \leqq \rho(\omega, \delta)$ where $\delta(x) = x$, then $g(x) \equiv x$, a.e.*

4. Translation parameter problem for the loss function $L(a, \omega) = (a + \omega)^{2N}$ with one observation. In the preceding section $\delta(x) = x$ was seen to be an admissible estimate of $-\omega$ in the translation parameter problem when the loss function is $L(a, \omega) = (a + \omega)^2$. Under suitable assumptions which are analogous to Assumption I and the boundedness restriction it will now be shown that $\delta(x) = x$ is also admissible for the loss function $L(a, \omega) = (a + \omega)^{2N}$. Note that consideration is still restricted to the case of a single observation.

The assumptions imposed are the following:

ASSUMPTION III. $p(\xi)$ is symmetric, i.e., $p(\xi) = p(-\xi)$.

Because of this assumption the odd moments of $p(\xi)$ vanish. This property is used in a crucial way.

If $h(x)$ is an estimate which presumably improves on $\delta(x)$, then we shall assume

ASSUMPTION IV. There exists a constant $M > 1$ such that $|h(x) - x| \leqq M$ for all x.

Several remarks will be appended pertaining to this assumption after the completion of the theorem. For suitable general classes of densities p we will find as before that Assumption IV is unnecessary.

ASSUMPTION V. $\int_{-\infty}^{\infty} \xi^{2N} p(\xi)\, d\xi < \infty$, $\int_{-\infty}^{\infty} \xi^{2N} p^2(\xi)\, d\xi < \infty$.

It is readily verified that the analogue of equation (18) is the following:

$$\sum_{k=1}^{N} \binom{2N}{2k} \int_{-\infty}^{\infty} (x - h(x))^{2k} (x + \omega)^{2N-2k} p(x + \omega)\, dx \leqq \sum_{k=0}^{N-1} \binom{2N}{2k+1} \int_{-\infty}^{\infty} (x - h(x))^{2k+1} (x + \omega)^{2N-2k-1} p(x + \omega)\, dx. \tag{25}$$

The proofs of this section are completely analogous to those of the preceding section. Consequently, the detailed proofs will be shortened appropriately.

LEMMA 3. *If* $\rho(\omega, h) \leqq \rho(\omega, \delta)$ *for all* ω, *then* $\int_{-\infty}^{\infty} | x - h(x)|^{\alpha}\, dx < \infty$ *for* $\alpha \geqq 2$, (*under Assumptions* III, IV, *and* V).

PROOF. By (25), Fubini's theorem, and Assumption IV

$$\sum_{k=1}^{N} \binom{2N}{2k} \int_{-\infty}^{\infty} (x - h(x))^{2k} \left\{\int_{-n}^{n} (x + \omega)^{2N-2k} p(x + \omega)\, d\omega\right\} dx \leqq M^{2N-1} \sum_{k=0}^{N-1} \binom{2N}{2k+1} \int_{-\infty}^{\infty} \left| \Phi_k (x + n) - \Phi_k (x - n) \right| dx = K(n),$$

where $\Phi_k(u) = \int_{-\infty}^{u} \xi^{2N-2k-1} p(\xi)\, d\xi$. By Assumptions III and V it follows that $\Phi_k(u) \in L^1$. Thus

$$K(n) \leqq M^{2N-1} \sum_{k=0}^{N-1} \binom{2N}{2k+1} \left[\int_{-\infty}^{\infty} | \Phi_k (x + n) |\, dx + \int_{-\infty}^{\infty} | \Phi_k (x - n) |\, dx\right] < C < \infty,$$

where C is a constant independent n. Therefore,

$$0 \leqq \sum_{k=1}^{N} \binom{2N}{2k} \left[\int_{-\infty}^{\infty} \xi^{2N-2k} p(\xi)\, d\xi\right] \left[\int_{-\infty}^{\infty} (x - h(x))^{2k}\, dx\right] < C.$$

THEOREM 6. *Let Assumptions* III, IV, *and* V *be satisfied.* $\rho(\omega, h) \leqq \rho(\omega, \delta)$ *for all* ω *implies that* $h(x) = \delta(x)$, *a.e.*

PROOF. The proof is obtained by adapting appropriately the methods employed in the discussion of Theorem 4. The details are omitted.

A few remarks promised earlier concerning Assumption IV will now be given. As in Section 3 for the case

$$p(\xi) = \begin{cases} 0, & \xi < a, \xi > b, -\infty < a < b < \infty, \\ \geqq 0, & \text{otherwise,} \end{cases}$$

the only type of estimate which need be considered is an estimate $h(x)$ satisfying Assumption IV. The proof is the same as before. The argument for the second case in which $p(x, \omega) = p(x + \omega)$ has a monotone likelihood ratio is almost the same. It depends on Lemma 4 which may be derived with the aid of the Hölder inequality.

LEMMA 4. *If* $\rho(\omega, h) \leqq \rho(\omega, \delta)$ *for all* ω, *then there exists a constant* C *such that*

$$\int_{-\infty}^{\infty} (x - h(x))^{2N} p(x + \omega)\, dx \leqq C$$

for all ω *(under Assumption* V).

5. Translation parameter problem: n observations. The problem studied in this section is the multi-observation analogue of the problem treated in Section 3. Let $x_1, \cdots, x_n$ be n independent observations on the random variable X where X is distributed according to the density function $p(x, \omega) = p(x + \omega)$, $\omega \,\varepsilon\, (-\infty, \infty)$, with $p(\xi)$ a known prescribed density. Alternatively, X is allowed to be an integer-valued random variable with ω likewise assuming only integer values. $P\{X = i \mid \omega\} = p(i + \omega)$ where the probabilities $p(j)$, $j = 0, \pm 1, \pm 2, \cdots$, are assumed known. As previously the location parameter $-\omega$ is to be estimated.

Define $y_i = x_i - x_1$, $i = 2, \cdots, n$. An appealing estimate for the parameter $-\omega$ which was proposed by Pitman and has the property of being invariant with respect to translations of the observations x_i is

$$\delta^*(x_1, x_2, \cdots, x_n) = x_1 - T(y_2, \cdots, y_n), \tag{26}$$

where

$$T(y_2, \cdots, y_n) = \frac{\int_{-\infty}^{\infty} \xi p(\xi) \prod_{i=2}^{n} p(y_i + \xi)\, d\xi}{\int_{-\infty}^{\infty} p(\xi) \prod_{i=2}^{n} p(y_i + \xi)\, d\xi}. \tag{27}$$

Invariance of δ^* means that

$$\delta^*(x_1 + c, x_2 + c, \cdots, x_n + c) = c + \delta^*(x_1, x_2, \cdots, x_n)$$

for each constant c, an obviously desirable property when dealing with an unknown location parameter. It is well-known that δ^* is an invariant minimax estimator of $-\omega$ (cf. [5]).

Girshick and Savage [5] in discussing estimating procedures associated with quadratic loss conjectured that the estimator (26) is unique minimax. Since the risk of the estimate δ^* is identically constant it follows that in order to substantiate this conjecture it is enough to show that δ^* is admissible. This has been verified by Blackwell for the special case where both X and ω are essentially integer-valued and where $p(i)$ vanished except for at most a finite number of i [7]. He also constructed an example in which X traversed a discrete set and the range of ω was also discrete with values incommensurate with the possible X values, and he showed that δ^* need not be admissible in this case. This is not at all surprising in view of the fact that the usual demands corresponding to invariance in essence necessitate that the possible values of X and the ω values should comprise the same group structure. This characteristic was violated in the example of Blackwell.

We shall establish the admissibility of δ^* as an estimate of $-\omega$ in three separate cases which include most of the common distributions. In two of the cases we deal with densities of a continuous real variable for which ω traverses the real line. The third case examined is the general discrete problem where X and ω range over the integers. Blackwell's result for discrete densities with bounded domain emerges as a special case.

The convolution character of the location parameter problem suggests a representation of the problem in terms of Fourier integrals. It is therefore natural for our arguments to appeal to the powerful developed techniques of Fourier analysis which we, in fact, use abundantly. Our methods consequently apply to a considerably wider class of distributions which includes most of the common situations. The sequence of lemmas established follows principally the line of reasoning of the analogous single observation case and may be considered an extension thereof.

The three cases require separate analysis because of the different regularity assumptions needed for each. To establish the admissibility of $\delta^*(x, y_2, \cdots, y_n)$ we must show that if the inequality

$$
\begin{aligned}
\rho(\omega, g) &= \int \cdots \int [g(x, y_2, \cdots, y_n) + \omega]^2 \\
&\qquad \times p(x + \omega)p(x + \omega + y_2) \cdots p(x + \omega + y_n)\, dx dy_2 \cdots dy_n \\
&\leqq \int \cdots \int [\delta^*(x, y_2, \cdots, y_n) + \omega]^2 \qquad (28)\\
&\qquad \times p(x + \omega)p(x + \omega + y_2) \cdots p(x + \omega + y_n)\, dx dy_2 \cdots dy_n \\
&= \rho(\omega, \delta^*) = c
\end{aligned}
$$

is valid for each ω then $g = \delta^*$, a.e. For the discrete case (i.e., X and ω are in teger-valued) the integral is to be replaced by the appropriate summation. The inequality (29) below is equivalent to (28).

$$
\begin{aligned}
&\int \cdots \int [g(x, y_2, \cdots, y_n) - \delta^*(x, y_2, \cdots, y_n)]^2 \\
&\qquad \times p(x + \omega) \prod_{i=2}^{n} p(x + \omega + y_i)\, dx dy_2 \cdots dy_n \qquad (29)\\
&\leqq 2 \int \cdots \int [\delta^* - g][\delta^* + \omega] p(x + \omega) \prod_{i=2}^{n} p(x + \omega + y_i)\, dx dy_2 \cdots dy_n .
\end{aligned}
$$

CASE 1.

$$
p(\xi) \begin{cases} \geqq 0, & -\infty < a \leqq \xi \leqq b < \infty, \\ = 0, & \xi < a,\ b < \xi. \end{cases}
$$

Only estimators $g(x_1, y_2, \cdots, y_n)$ of $-\omega$ which satisfy

$$
| \delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)| \leqq M
$$

for some constant $M < \infty$ need be considered. The argument is analogous to that given for the one-dimensional case in Section 3. The underlying reason is that the boundedness of the spectrum determines for each set of observations $x_1, x_2, \cdots, x_n$, an interval within which the true value of $-\omega$ must lie and any estimator which produces a value outside this interval can be improved upon. Thus any estimate which differs from δ^* and which improves in terms of risk on δ^* must only differ by a fixed constant from δ^*, regardless of the observed values of x. The single regularity assumption required in this case is that for all ξ, $0 \leqq p(\xi) \leqq C < \infty$.

LEMMA 5. *If* $\rho(\omega, g) \leqq \rho(\omega,\delta^*)$ *for all* ω, *then*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots y_n)]^2 \times \left[\int_{-\infty}^{\infty} p(\xi)p(y_2 + \xi) \cdots p(y_n + \xi)d\xi\right] dx_1dy_2 \cdots dy_n < \infty. \tag{30}$$

PROOF. By the fundamental inequality (29)

$$\begin{aligned}
&\int_{-n}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]^2 \\
&\quad \times p(x_1 + \omega) \cdots p(y_n + x_1 + \omega)\, dx_1 \cdots dy_n\, d\omega \\
&\quad \leqq 2 \int_{-n}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)] \\
&\quad \times [\delta^*(x_1, y_2, \cdots, y_n) + \omega]p(x_1 + \omega) \cdots p(y_n + x_1 + \omega)\, dx_1dy_2 \cdots dy_n\, d\omega \\
&\quad \leqq 2M \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} | \Phi(x_1 + n, y_2, \cdots, y_n) \\
&\qquad - \Phi(x_1 - n, y_2, \cdots, y_n) |\, dx_1dy_2 \cdots dy_n,
\end{aligned} \tag{31}$$

where

$$\Phi(u, y_2, \cdots, y_n) = \int_{-\infty}^{u} [\xi - T(y_2, \cdots, y_n)]p(\xi)p(y_2 + \xi) \cdots p(y_n + \xi)\, d\xi.$$

By direct calculation we observe that

$$\Phi(\infty, y_2, \cdots, y_n) = \Phi(-\infty, y_2, \cdots, y_n) = 0. \tag{32}$$

If we can show that $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} | \Phi(x_1, y_2, \cdots, y_n)|\, dx_1dy_2 \cdots dy_n < \infty$, then it follows that the expression of (31) is uniformly bounded with respect to n which clearly implies the sought-for conclusion. The remainder of the proof consists in verifying the finiteness of this integral.

Note that $\Phi(x_1, y_2, \cdots, y_n) \leqq 0$ for all $x_1, y_2, \cdots, y_n$, and for fixed $y_2, \cdots, y_n$ there exists a constant N such that $| x_1 | \geqq N$ implies

$$\Phi(x_1, y_2, \cdots, y_n) = 0.$$

Integration by parts with respect to x_1 yields

$$0 \leqq -\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(x_1, y_2, \cdots, y_n)\, dx_1 dy_2 \cdots dy_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1[x_1 - T(y_2, \cdots, y_n)]p(x_1) \cdots p(y_n + x_1)\, dx_1 \cdots dy_n .$$

But the last integral converges absolutely since $p(\xi)$ vanishes outside a finite interval and $| T(y_2, \cdots, y_n)| \leqq | a | + | b |$.

THEOREM 7. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta^*)$ *for all* ω, *then* $g = \delta^*$, *a.e.*

PROOF. Let

$$G(x_1, y_2, \cdots, y_n) = [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)] \times \left[\int_{-\infty}^{\infty} p(\xi) \cdots p(y_n + \xi)\, d\xi\right]^{1/2},$$

$$H(u, y_2, \cdots, y_n) = [u - T(y_2, \cdots, y_n)] \left[\int_{-\infty}^{\infty} p(\xi) \cdots p(y_n + \xi)\, d\xi\right]^{-1/2} \times [p(u)p(y_2 + u) \cdots p(y_n + u)].$$

By Lemma 5, $G(x_1, y_2, \cdots, y_n) \in L^2$, and by direct calculation we see that $H(u, y_2, \cdots, y_n) \in L^2$. Therefore, the Fourier transforms $\tilde{G}(t_1, \cdots, t_n)$ and $\tilde{H}(t_1, \cdots, t_n)$ of G and H, respectively, are well-defined and belong to L^2. Consider the expression

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)] \\ &\qquad \times [x_1 + \omega_1 - T(y_2 + \omega_2, \cdots, y_n + \omega_n)] \\ &\qquad \times p(x_1 + \omega_1) \cdots p(y_n + \omega_n + x_1 + \omega_1)\, dx_1 dy_2 \cdots dy_n \end{aligned} \tag{33}$$

where for our purposes we shall need to evaluate this expression only for the values $\omega_2 = \cdots = \omega_n = 0$. (33) is essentially a convolution of G and H so its Fourier transform exists and is equal to

$$\tilde{G}(-t_1, -t_2, \cdots, -t_n)\tilde{H}(t_1, t_2, \cdots, t_n) \in L^1.$$

By the inversion property of Fourier transforms we obtain

$$\begin{aligned} &\int_{-n}^{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)] \\ &\qquad \times [x_1 + \omega_1 - T(y_2, \cdots, y_n)] \\ &\qquad \times p(x_1 + \omega_1) \cdots p(y_n + x_1 + \omega_1)\, dx_1 \cdots dy_n\, d\omega_1 \end{aligned} \tag{34}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\tilde{G}(-t_1, \cdots, -t_n)\tilde{H}(t_1, \cdots, t_n)}{it_1} [e^{-it_1 n} - e^{it_1 n}]\, dt_1 \cdots dt_n$$

which is defined everywhere since $\tilde{G}\cdot\tilde{H}$ belongs to L^1. The last integral is an absolutely convergent integral. To substantiate this assertion we note that $\tilde{H}(0, t_2, \cdots, t_n) = 0$, and

$$\frac{\partial}{\partial t_1}\tilde{H}(t_1, \cdots, t_n)$$

$$= i\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\,[i(t_1x_1 + \cdots + t_nx_n)]x_1[x_1 - T(y_2, \cdots, y_n)]$$

$$\times\left[\int_{-\infty}^{\infty}p(\xi)\cdots p(y_n+\xi)\,d\xi\right]^{-1/2} p(x_1)p(y_2+x_1)\cdots p(y_n+x_1)\,dx_1\cdots dy_n$$

is bounded independently of $t_1, \cdots, t_n$. Hence, by the mean value theorem $\tilde{G}(-t_1, \cdots, -t_n)\tilde{H}(t_1, \cdots, t_n)/t_1$ as a function of t_1 is integrable in a neighborhood about the origin for all $t_2, \cdots, t_n$. Therefore,

$$\tilde{G}(-t_1, \cdots, -t_n)\tilde{H}(t_1, \cdots, t_n)/t_1 \,\varepsilon\, L^1.$$

By virtue of the Riemann-Lebesgue lemma and Lebesgue's theorem of dominated convergence we see that the expression in (34) tends to zero as $n \to \infty$. Hence, on comparison of (29) and (34), we deduce

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots y_n)]^2$$

$$\times\left[\int_{-\infty}^{\infty}p(x_1+\omega)\cdots p(y_n+x_1+\omega)\,d\omega\right]dx_1\cdots dy_n \leqq 0,$$

which establishes the theorem.

CASE 2. Discrete case.

$$X = \{0, \pm1, \pm2, \cdots\}, \qquad \Omega = \{0, \pm1, \pm2, \cdots\},$$

and

$$P\{x = i \mid \omega\} = p(i+\omega),$$

where $p(j) \geqq 0$, $\sum_{j=-\infty}^{\infty} p(j) = 1$.

We impose the following regularity assumption.

ASSUMPTION VI. $\sum_{j=-\infty}^{\infty} j^2\sqrt{p(j)} < \infty$.

Unfortunately, we do not know whether this assumption may be relaxed to the obviously weaker and more natural condition $\sum_{-\infty}^{\infty} j^2p(j) < \infty$. The weaker requirement was indeed sufficient for the case of a single observation whenever $\sum_{-\infty}^{\infty} jp(j) = 0$. [See Section 3.]

LEMMA 6. *If $\rho(\omega, g) \leqq \rho(\omega, \delta^*)$ for all ω, then there exists a constant C such that*

$$\sum_{y_n}\cdots\sum_{x_1}[\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]^2$$

$$\times p(x_1+\omega)\cdots p(y_n+x_1+\omega) \leqq C$$

for all ω (under Assumption VI).

The proof is analogous to that of Lemma 2 of Section 3 so it is omitted.

LEMMA 7. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta^*)$ *for all* ω, *then*

$$\sum_{y_n} \cdots \sum_{x_1} [\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]^2 \times [\sum_{\omega} p(x_1 + \omega) \cdots p(y_n + x_1 + \omega)] < \infty.$$

PROOF. As a consequence of Lemma 6,

$$| x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n) | \leqq \frac{C^{1/2}}{\sqrt{p(x_1 + \omega) \cdots p(y_n + x_1 + \omega)}}$$

for all ω. Since ω is arbitrary,

$$\max_{x_1} | x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n) | \leqq \frac{C^{1/2}}{\max_j \sqrt{p(j)p(y_1 + j) \cdots p(y_n + j)}}.$$

Define for integers u,

$$\Phi(u, y_2, \cdots, y_n) = \sum_{j=-\infty}^{u} [j - T(y_2, \cdots, y_n)]p(j)p(y_2 + j) \cdots p(y_n + j).$$

By the fundamental inequality (29)

$$\begin{aligned} \sum_{\omega=-n}^{n} \sum_{y_n} \cdots \sum_{x_1} & [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]^2 \\ & \times p(x_1 + \omega) \cdots p(y_n + x_1 + \omega) \\ & \leqq 2 \sum_{y_n} \cdots \sum_{x_1} \frac{C^{1/2}}{\max_j \sqrt{p(j)p(y_2 + j) \cdots p(y_n + j)}} \\ & \times | \Phi(x_1 + n, y_2, \cdots, y_n) - \Phi(x_1 - n, y_2, \cdots, y_n) |. \end{aligned} \tag{35}$$

It is easily checked that $\Phi(u, y_2, \cdots, y_n) \leqq 0$ for all $u, y_2, \cdots, y_n$, and as $| u | \to \infty$ $| u\Phi(u, y_2, \cdots, y_n)| \to 0$ by Assumption VI with the aid of the fact that $\Phi(\infty, y_2, \cdots, y_n) = 0$. Summation by parts with respect to x_1 yields

$$\begin{aligned} -\sum_{y_n} \cdots \sum_{x_1} & \frac{\Phi(x_1, y_2, \cdots, y_n)}{\max_j \sqrt{p(j)p(y_2 + j) \cdots p(y_n + j)}} \\ & = \sum_{y_n} \cdots \sum_{x_1} \frac{x_1[x_1 - T(y_2, \cdots, y_n)]p(x_1)p(y_2 + x_1) \cdots p(y_n + x_1)}{\max_j \sqrt{p(j)p(y_2 + j) \cdots p(y_n + j)}} \\ & \leqq 2 \sum_{y_n} \cdots \sum_{x_1} \frac{x_1^2 p(x_1)p(y_2 + x_1) \cdots p(y_n + x_1)}{\max_j \sqrt{p(j)p(y_2 + j) \cdots p(y_n + j)}} \\ & \leqq 2 \sum_{y_n}{}' \cdots \sum_{x_1}{}' \frac{x_1^2 p(x_1) \cdots p(y_n + x_1)}{\sqrt{p(x_1) \cdots p(y_n + x_1)}}, \end{aligned} \tag{36}$$

where $\sum'_{y_n} \cdots \sum'_{x_1}$ denotes summation over all $x_1, \cdots, y_n$ for which

$$p(x_1)p(y_2 + x_1) \cdots p(y_n + x_1) > 0.$$

But by Assumption VI

$$\sum_{y_n} \cdots \sum_{x_1} x_1^2 \sqrt{p(x_1) \cdots p(y_n + x_1)} < \infty. \tag{37}$$

Hence, (35), (36), and (37) in conjunction yield the desired result.

THEOREM 8. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta^*)$ *for all* ω, *then*

$$g(x_1, y_2, \cdots, y_n) = \delta^*(x_1, y_2, \cdots, y_n)$$

for all $y_2, \cdots, y_n$ *such that* $\sum_j p(j)p(y_2 + j) \cdots p(y_n + j) > 0$.

PROOF. Let

$$G(x_1, y_2, \cdots, y_n) = [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)] \times [\sum_j p(j) \cdots p(y_n + j)]^{1/2},$$

$$H(u, y_2, \cdots, y_n) = [u - T(y_2, \cdots, y_n)] [\sum_j p(j)p(y_2 + j) \cdots p(y_n + j)]^{-1/2} \times [p(u)p(y_2 + u) \cdots p(y_n + u)].$$

Since $\sum_{y_n} \cdots \sum_{x_1} G^2(x_1, y_2, \cdots, y_n) < \infty$ by Lemma 7,

$$\sum_{y_n} \cdots \sum_{x_1} \exp [i(t_1 x_1 + \cdots + t_n y_n)]G(x_1, y_2, \cdots, y_n)$$

converges in quadratic mean to a function $\tilde{G}(t_1, \cdots, t_n) \varepsilon L^2(-\pi, \pi)$. Also, by Assumption VI $\sum_{y_n} \cdots \sum_{x_1} | H(x_1, y_2, \cdots, y_n)| < \infty$. Indeed, inspection of the series shows that its convergence would be a consequence of the convergence of the related series

$$\sum_{u, y_2, \cdots, y_n} \frac{| u | p(u)p(u + y_2) \cdots p(u + y_n)}{\sqrt{\sum_\xi p(\xi)p(\xi + y_2) \cdots p(\xi + y_n)}} = J.$$

This follows in view of the inequality of Schwarz, the uniform boundedness of

$$\frac{p(u)p(u + y_2) \cdots p(u + y_n)}{\sum_\xi p(\xi)p(\xi + y_2) \cdots p(\xi + y_n)},$$

and Assumption VI. Hence,

$$\sum_{y_n} \cdots \sum_{x_1} \exp [i(t_1 x_1 + \cdots + t_n y_n)]H(x_1, y_2, \cdots, y_n)$$

converges uniformly and absolutely to a function $\tilde{H}(t_1, \cdots, t_n) \varepsilon L^2(-\pi, \pi)$.

The expression

$$I(\omega_1, \cdots, \omega_n) = \sum_{y_n} \cdots \sum_{x_1} [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)] \times [x_1 + \omega_1 - T(y_2 + \omega_2, \cdots, y_n + \omega_n)] \times p(x_1 + \omega_1) \cdots p(y_n + \omega_n + x_1 + \omega_1),$$

where in actuality $\omega_2 = \cdots = \omega_n = 0$, is essentially a convolution of G and H so its Fourier series converges absolutely to a function $\tilde{I}(t_1, \cdots, t_n)$, and $\tilde{I}(t_1, \cdots, t_n) = \tilde{G}(-t_1, \cdots, -t_n)\tilde{H}(t_1, \cdots, t_n)$, a.e. Since

$$I(\omega_1, 0, \cdots, 0) = 1/(2\pi)^n \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp(-it_1\omega_1)\tilde{I}(t_1, \cdots, t_n)\, dt_1 \cdots dt_n$$

and $\sum_{-n}^{n} e^{-itk} = [e^{itn} - e^{-it(n+1)}]/(1 - e^{-it})$, it follows that

$$\sum_{\omega_1=-n}^{n} I(\omega_1, 0, \cdots, 0) = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \tilde{I}(t_1, \cdots, t_n) \times \left[\frac{e^{it_1n} - e^{-it_1(n+1)}}{1 - e^{-it_1}}\right] dt_1 \cdots dt_n. \tag{38}$$

The interchange of summation and integration signs on the right-hand side of (38) is valid since by virtue of Assumption VI

$$\lim_{t_1 \to 0} \tilde{I}(t_1, \cdots, t_n)/(1 - e^{-it_1}) < \infty$$

and $\tilde{I}(t_1, \cdots, t_n) \varepsilon L^1(-\pi, \pi)$. But by the Riemann-Lebesgue lemma the right-hand side of (38) converges to zero as $n \to \infty$. By the fundamental inequality

$$\sum_{y_n} \cdots \sum_{x_1} [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]^2 [\sum_j p(j) \cdots p(y_n + j)] \leqq 0$$

from which the desired result follows.

CASE 3. General density functions.

This case will include all density functions which satisfy the following regularity conditions:

ASSUMPTION VII:

$$0 \leqq p(\xi) \leqq C, \int_{-\infty}^{\infty} \xi^2 \sqrt{p(\xi)}\, d\xi < \infty.$$

ASSUMPTION VIII:

$$\frac{p(\xi)p(y_2 + \xi) \cdots p(y_n + \xi)}{\int_{-\infty}^{\infty} p(\theta)p(y_2 + \theta) \cdots p(y_n + \theta)\, d\theta} \leqq C' \quad \text{for all } \xi, y_2, \cdots, y_n.$$

Assumption VIII asserts that the conditional density of x_1 given $y_2, \cdots, y_n$ must remain bounded for all $x_1, y_2, \cdots, y_n$. This assumption is a bit stronger than necessary; it could be replaced by an assumption of finiteness of a number of definite integrals involving the conditional density. However, there seems to be no gain involved in such a generalization. The class of densities which satisfy Assumptions VII and VIII includes as two of its important members the normal and negative exponential distributions as well as any density which asymptotically dies off like a power.

It will be shown by Theorem 9 below that δ^* is admissible with respect to the class of all estimators $g(x_1, y_2, \cdots, y_n)$ which satisfy the following additional requirement:

ASSUMPTION IX. There exists a constant $M < \infty$ such that for all $x_1, y_2, \cdots, y_n$ $|\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)| \leqq M$.

This will establish a suitably broad form of the concept of "local" admissibility for the estimator δ^*. This concept of "local" admissibility was introduced earlier in Section 3. As yet suitable supplementary conditions on the form of $p(\xi)$ for the relaxation of this assumption have not been obtained.

LEMMA 8. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta^*)$ *for all* ω, *and* g *satisfies Assumption* IX, *then*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\delta^*(x_1, y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]^2 \times \left[\int_{-\infty}^{\infty} p(\xi)p(y_2 + \xi) \cdots p(y_n + \xi)\, d\xi\right] dx_1 dy_2 \cdots dy_n < \infty.$$

PROOF. The proof is analogous to that of Lemma 5. It is sufficient to prove that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Phi(x_1, y_2, \cdots, y_n)|\, dx_1 dy_2 \cdots dy_n < \infty,$$

where Φ is defined as in Lemma 5. $\Phi(x_1, y_2, \cdots, y_n) \leqq 0$, and as $|u| \to \infty$, $|u\Phi(u, y_2, \cdots, y_n)| \to 0$. As $u \to -\infty$

$$\begin{aligned} 0 &\leqq u\,\Phi(u, y_2, \cdots, y_n) \\ &\leqq \int_{-\infty}^{u} \xi^2 p(\xi)p(y_2 + \xi) \cdots p(y_n + \xi)\, d\xi \\ &\qquad + |T(y_2, \cdots, y_n)| \int_{-\infty}^{u} \xi p(\xi)p(y_2 + \xi) \cdots p(y_n + \xi)\, d\xi. \end{aligned} \tag{39}$$

Both integrals in (39) vanish as $u \to -\infty$ by Assumption VII. A similar analysis is valid as $u \to \infty$ since

$$\int_{-\infty}^{u} [\xi - T(y_2, \cdots, y_n)]p(\xi) \cdots p(y_n + \xi)\, d\xi = -\int_{u}^{\infty} [\xi - T(y_2, \cdots, y_n)]\, p(\xi) \cdots p(y_n + \xi)\, d\xi.$$

Integration by parts with respect to x_1 and an application of Schwarz's inequality yields

$$0 \leqq -\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi(x_1, y_2, \cdots, y_n)\, dx_1\, dy_2 \cdots dy_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1 [x_1 - T(y_2, \cdots, y_n)]\, p(x_1) \cdots p(y_n + x_1)\, dx_1 \cdots dy_n$$

$$\leqq 2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^2\, p(x_1) \cdots p(y_n + x_1)\, dx_1 \cdots dy_n .$$

The final expression is therefore finite by virtue of Assumption VII.

THEOREM 9. *If* $\rho(\omega, g) \leqq \rho(\omega, \delta^*)$ *for all* ω, *and* g *satisfies Assumption* IX, *then* $g = \delta^*$, *a.e.*

PROOF. Define $G(x_1, y_2, \cdots, y_n)$ and $H(x_1, y_2, \cdots, y_n)$ as in Theorem 7. By Lemma 8, $G(x_1, y_2, \cdots, y_n) \in L^2$, and by Assumptions VII and VIII, $H(x_1, y_2, \cdots, y_n) \in L^2$. Therefore, the Fourier transforms $\tilde{G}(t_1, \cdots, t_n)$ and $\tilde{H}(t_1, \cdots, t_n)$ of G and H, respectively, are well-defined and belong to L^2. The Fourier transform $\tilde{I}(t_1, \cdots, t_n)$ of

$$I(\omega_1, \cdots, \omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [x_1 - T(y_2, \cdots, y_n) - g(x_1, y_2, \cdots, y_n)]$$
$$\times [x_1 + \omega_1 - T(y_2 + \omega_2, \cdots, y_n + \omega_n)]\, p(x_1 + \omega_1) \cdots p(y_n + \omega_n + x_1 + \omega_1)$$
$$\times\, dx_1 dy_2 \cdots dy_n$$

is well-defined and equals $\tilde{G}(-t_1, \cdots, -t_n)\tilde{H}(t_1, \cdots, t_n)$. By an argument analogous to that of Theorem 7 it follows that

$$\lim_{n \to \infty} \int_{-n}^{n} I(\omega_1, 0, \cdots, 0)\, d\omega_1 = 0,$$

which proves the result.

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