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ADMISSIBILITY OF ESTIMATORS OF THE MEAN VECTOR OF A MULTIVARIATE NORMAL DISTRIBUTION WITH QUADRATIC LOSS[1]

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1. Introduction. In this study we shall be concerned with the problem of estimating the mean vector of a multivariate normal distribution. We assume the covariance matrix of the distribution is the identity and the loss function is the sum of squared errors. We will be concerned primarily with the generalized Bayes and/or admissibility properties of estimators. The general problem has been extensively studied, notably by Stein [13], [14], [15], [16], [17].

For the univariate case, one main result offers a necessary and sufficient condition for admissibility of a bounded risk estimator. Whereas Brown [3] also gives an n.a.s.c., the one given here is strictly in terms of the estimator itself. In Brown's case the condition is in terms of the generalized prior with respect to which the estimator is generalized Bayes. This result of Brown's is used to prove our result. The condition is as follows:

Let x be a single observation from a univariate normal population with unknown mean and standard deviation 1. Let $\delta(x)$ be a bounded risk generalized Bayes estimator and define

$$g(x) = \exp\left[\int_0^x (\delta(y) - y)\, dy\right]. \tag{1.1}$$

Then $\delta(x)$ is admissible iff

$$\int^{\infty} (1/g(x))\, dx = \int_{-\infty} (1/g(x))\, dx = \infty. \tag{1.2}$$

Another result that can be used to determine if an estimator can be a candidate for admissibility is a necessary and sufficient condition for an estimator to be generalized Bayes. The condition is the existence of an increasing function $F(\theta)$ such that

$$g(x) = \exp\left[\int_0^x (\delta(y) - y)\, dy\right] = \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta). \tag{1.3}$$

This is essentially equivalent to $g(x/2^{\frac{1}{2}})$ being a Weierstrass transform ([6] page 174). Alternatively $\delta(x)$ is a generalized Bayes estimator iff

$$\exp\left[\int_0^x \delta(y)\, dy\right] \tag{1.4}$$

is the moment generating function for some probability distribution. The above condition will be applied in some examples to demonstrate the inadmissibility of some classes of estimators.

The above results for the univariate case are generalized to the spherically symmetric multivariate case. That is, given one observation on the vector x, we consider only the class of estimators of the form $\delta(x) = h(\|x\|^2)x$. We show that an estimator of this form is generalized Bayes if and only if there exists a spherically symmetric measure $F(\theta)$ (one which assigns equal mass to all orthogonal transformations of any set), such that

$$\begin{aligned} g(\|x\|^2) &= \exp\left[\tfrac{1}{2}\int_0^{\|x\|^2} (h(y) - 1)\, dy\right] \\ &= \int \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] dF(\theta). \end{aligned} \tag{1.5}$$

In addition, if $\delta(x)$ is of bounded risk, it is admissible iff $\int^{\infty}(1/r^{n-1}g(r^2))\, dr = \infty$, where n is the dimension of the multivariate normal distribution.

Next we concern ourselves with questions regarding the admissibility of "shrinkers" and "expanders." Defining a "shrinker" as an estimator $\delta(x) = x - \varepsilon(x)$ where $\varepsilon(x) \geqq 0$ if $x \geqq 0$, and $\varepsilon(x) \leqq 0$ if $x \leqq 0$, we show that all bounded risk generalized Bayes shrinkers are admissible. We generalize the notion of shrinker to "shrinker towards θ_0," i.e. estimators of the form $\theta_0 + \varepsilon(x)(x - \theta_0)$ where $0 \leqq \varepsilon(\cdot) \leqq 1$, and show that the same result holds for "shrinkers towards θ_0." Defining "expander" analogously we show that all bounded risk "expanders" which eventually "expand by at least ε," for any $\varepsilon > 0$, are inadmissible.

For the spherically symmetric multivariate case, the following generalizations of the above results for "shrinkers" and expanders hold: If $\delta(x) = h(||x||^2)x$ is a generalized Bayes estimator of bounded risk such that for some $M > 0, y > M$ implies $h(y) \leqq 1 - (n-2)/y$, then $\delta(x)$ is admissible. If there exists an M such that $y > M$ implies $h(y) \geqq 1 - b/y$ for some $b < n-2$ then $\delta(x)$ is inadmissible.

Finally, we are able to offer some conditions which separate proper Bayes estimators from generalized Bayes estimators resulting from improper prior distributions. For the univariate case the result is that a generalized Bayes estimator $\delta(x)$ is proper Bayes iff $\int_{-\infty}^{\infty} \exp[\int_0^x (\delta(y) - y)\, dy]\, dx < \infty$. From this it follows that if for some ε, $M > 0$, $x > M$ implies $\delta(x) < x - \varepsilon$, then $\delta(x)$ is proper Bayes (provided we know it is generalized Bayes).

For the spherically symmetric multivariate case we have the following result: Given a generalized Bayes estimator $\delta(x) = h(||x||^2)x$, if $h(y) \leqq 1 - \varepsilon/y$ for some $\varepsilon > n$ (the dimension of the space) whenever $x > M$ for some $M > 0$, then $\delta(x)$ is proper Bayes. If on the other hand $h(y) \geqq 1 - n/y$ when $y > M$, then $\delta(x)$ is not proper Bayes.

We now present an ordered summary of the paper. Section 2 contains some mathematical and statistical preliminaries necessary for the development of the succeeding material. In Section 3 we obtain a characterization of generalized Bayes procedures for the univariate case, and obtain a necessary and sufficient condition for admissibility of bounded risk estimators. Section 4 is devoted to analogous results in the multivariate case. Section 5 is concerned with further results for the univariate case, notably the admissibility of shrinkers and expanders. Section 6 treats analogous results for the spherically symmetric multivariate case. Section 5 and Section 6 also contain the results concerned with separating proper Bayes procedures from generalized Bayes procedures resulting from improper priors.

2. Preliminaries.

2.1. *Statistical preliminaries.* This section presents some known results which are used in the main body of the paper. We assume a familiarity with the basic notions of decision theory as found, say, in Ferguson [4].

Given an observation, X, on a multivariate normal distribution with unknown mean, θ, and covariance matrix known to be the identity, I, we consider the problem of estimating θ subject to the squared error loss function, i.e. $\delta(x)$ is a measurable (vector-valued) function; the loss is given by

$$L(\delta(x), \theta) = ||\delta(x) - \theta||^2. \tag{2.1.1}$$

We note that a (proper) Bayes estimator for θ with respect to a (proper) prior distribution has the form

$$(2.1.2)\quad \delta_i(x) = \int_{E^n} \theta_i \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] d\Pi(\theta) \Big/ \int_{E^n} \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] d\Pi(\theta) \quad \text{a.e.,}$$

where $\delta_i(x)$ denotes the ith coordinate of the vector $\delta(x)$. Recall that the essential uniqueness of Bayes estimators imply their admissibility.

We now recall the notion of a generalized Bayes estimator for this case. Given a (possibly non-finite) measure $\Pi(\theta)$ which assigns finite measure to bounded subsets of E^n, a decision function $\delta(x)$ is said to be a generalized Bayes estimator with respect to Π if $\delta(x)$ selects (in perhaps a randomized way) a decision among those values of d which minimize

$$(2.1.3)\quad \int_{E^n} \|d-\theta\|^2 \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] d\Pi(\theta) \Big/ \int_{E^n} \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] d\Pi(\theta),$$

where it is assumed that the expression above is finite for some d. Note that this guarantees the existence of the integral in the denominator.

It is easy to see that the calculation of a generalized Bayes estimator with respect to a prior $\Pi(\theta)$ takes the form (2.1.2). It is not true, however, that a unique generalized Bayes estimator must be admissible. There are many simple examples illustrating this. A somewhat startling counter example was produced by Stein [13], who showed that the sample mean, which is generalized Bayes with respect to the Lebesgue measure on E^n, is inadmissible when $n \geqq 3$. The importance of generalized Bayes estimators lies in the fact that, quite generally, any admissible estimator must be generalized Bayes. We paraphrase a result due to Sacks ([11] page 765) in somewhat less generality than it is valid.

THEOREM 2.1.1. *Let $\Omega = \{\omega : \int e^{x\omega} du(x) < \infty\}$, and $L(d, \omega) = (d-\omega)^2$ where μ is absolutely continuous with respect to Lebesgue measure on E'. Then the class of generalized Bayes estimators is complete.*

Note that the family of univariate normal distributions with variance 1 and unknown mean satisfy the assumptions of the theorem.

We note that the above result holds for the multivariate normal case also ([11] page 767).

We next present two results due to L. Brown [3] which will be extensively used in the sequel.

THEOREM 2.1.2. *For estimating the mean of a univariate normal distribution with variance* 1 *subject to squared error loss, an estimator $\delta(x)$ of bounded risk is admissible* iff *$\delta(x)$ is a generalized Bayes estimator with respect to a prior $F(\theta)$ with the following property;*

$$\text{If } f^*(x) = 1/(2\pi)^{\frac{1}{2}} \int \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta), \text{ then } \int^{\infty} (dx)/(f^*(x)) = \int_{-\infty} (dx)/(f^*(x)) = +\infty.$$

THEOREM 2.1.3. *Suppose $\delta(x)$ is a bounded risk generalized Bayes estimator with respect to a prior $F(\theta)$ for the problem of estimating the mean vector of an n-variate* *normal distribution with covariance matrix the identity and with sum of squared error loss. Then if*

$$f^*(x) = 1/(2\pi)^{\frac{1}{2}n} \int_{E^n} \exp\left[-\tfrac{1}{2}(x-\theta)'(x-\theta)\right] dF(\theta)$$

$$= g(\|x\|), \quad \delta(x) \textit{ is admissible } \text{iff} \int^\infty (dr)/(r^{n-1} g(r)) = \infty.$$

2.2. *Mathematical preliminaries.* In this section we present some necessary mathematical preliminaries. The first result which can be found in Lehmann ([9] page 52) is useful for justifying differentiating under the integral sign of an exponential density.

THEOREM 2.2.1. *Let ϕ be any bounded measurable function on (X, A). Then*

(i) *The integral*

$$\int \phi(x) \exp\left[\textstyle\sum_{j=1}^k \theta_j T_j(x)\right] du(x) \tag{2.2.1}$$

considered as a function of the complex variable $\theta_j = \xi_j + \eta_j$ is an analytic function in each of these variables in the region R of parameter points for which $(\xi_1, \xi_2 \cdots, \xi_k)$ is an interior point of the natural parameter space Ω.

(ii) *The derivatives of all orders with respect to the θ's in* (2.2.1) *can be computed under the integral sign.*

In the above (X, A) is a Euclidean sample space, μ is any σ-finite measure over (X, A), and the natural parameter space is the set of points $(\theta_1, \cdots, \theta_k)$ such that $\int \exp\left[\sum \theta_j T_j(x)\right] du(x) < \infty$.

Another result we will require comes from Hirschman and Widder [6]. We define a Weierstrass–Stieltjes transform of a function $\alpha(x)$ to be

$$f(x) = (4\pi)^{-\frac{1}{2}} \int \exp\left[-(x-y)^2/4\right] d\alpha(y) \tag{2.2.2}$$

whenever the integral exists. We have

THEOREM 2.2.2. ([6] page 204). *The conditions*

1. $f(x) \in A$
2. $\exp\left[-tD^2\right] f(x) \geqq 0, 0 < t < 1 - \infty < x < \infty$

are necessary and sufficient that (2.2.2) *hold for some non-decreasing $\alpha(x)$ and the integral converges in $a < x < b$.*

In the above $f(x) \in A$ in $a < x < b$ iff it can be extended into the complex plane in such a way that

(i) $f(x+iy)$ is analytic in the strip $a < x < b$
(ii) $f(x+iy) = O(|y| \exp\left[y^2/4\right])$

uniformly in every closed sub-interval of $a < x < b$. And

$$\exp\left[-tD^2\right] f(x) = 1/(4\pi t)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[-y^2/4t\right] f(x+iy)\, dy, \tag{2.2.3}$$

whenever $0 \in (a, b)$. In cases of interest to us $b = -a = \infty$ and hence $0 \in (a, b)$. For the definition of $\exp[-tD^2]f(x)$ when $0 \notin (a, b)$ see ([6] page 179).

3. Admissible and generalized Bayes estimators—the univariate case.

3.1. *A necessary and sufficient condition for an estimator to be generalized Bayes.* We consider the problem of estimating the mean value of a normal distribution based on a single observation. The loss function is taken to be squared error and the variance is assumed to be equal to 1. This section is devoted to a characterization of generalized Bayes estimators. Our principal result is the following:

THEOREM 3.1.1. *A necessary and sufficient condition for an estimator to be generalized Bayes is the existence of a measure $F(\theta)$ such that*

$$\exp[\int_0^x (\delta(y)-y)\,dy = \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-\theta)^2]\,dF(\theta). \tag{3.1.1}$$

PROOF. Suppose $\delta(x)$ is generalized Bayes with respect to the (possibly) generalized prior $F^*(\theta)$. Then as indicated in Subsection 2.1

$$\delta(x) = (\int_{-\infty}^{\infty} \theta \exp[-\tfrac{1}{2}(x-\theta)^2\,dF^*(\theta))/(\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-\theta)^2]\,dF^*(\theta)) \quad \text{a.e.} \tag{3.1.2}$$

By the assumption that $\delta(x)$ is generalized Bayes, it must follow that

$$\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-\theta)^2\,dF^*(\theta) = g(x) < \infty$$

for all real x. Hence by Theorem 2.2.1, $g(x)$, when considered as a function of the complex variable $x = u+vi$, is entire. Also note that for all real x, $g(x) > 0$, and, again by Theorem 2.2.1, may be differentiated under the integral sign.

We have then that

$$\delta(x)-x = (\int_{-\infty}^{\infty} (\theta-x)\exp[-\tfrac{1}{2}(x-\theta)^2]\,dF^*(\theta))/(\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-\theta)^2]\,dF^*(\theta)) = \frac{d}{dx}\log g(x). \tag{3.1.3}$$

We note that for real x, $(d/dx)\log g(x)$ is a continuous function. Hence we have upon integration

$$\int_0^x (\delta(y)-y)\,dy = \log g(x)+\log k \tag{3.1.4}$$

for some $k > 0$, or

$$\begin{aligned} \exp[\int_0^x (\delta(y)-y)\,dy] &= kg(x) \\ &= k\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-\theta)^2]\,dF^*(\theta) \\ &= \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-\theta)^2]\,dF(\theta), \end{aligned} \tag{3.1.5}$$

where in the final step we have absorbed the constant k into the measure $F(\theta)$. This proves the necessity of the condition.

To prove sufficiency, suppose (3.1.1) holds. Noting that the integral on the right-hand side is strictly positive on the whole real line and is an entire function as noted above, we may take logarithms of both sides and obtain a differentiable function. This gives

$$\int_0^x (\delta(x)-x)\,dx = \log \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta), \tag{3.1.6}$$

which upon differentiation yields

$$\delta(x)-x = \left(\int_{-\infty}^{\infty} (\theta-x)\exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta)\right) \Big/ \left(\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta)\right). \tag{3.1.7}$$

The differentiation under the integral sign in the above is again justified by Theorem 2.2.1. Adding x to both sides of (3.2.7), we find that

$$\delta(x) = \left(\int_{-\infty}^{\infty} \theta \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta)\right) \Big/ \left(\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta)\right), \tag{3.1.8}$$

which is equivalent to $\delta(x)$ being generalized Bayes with respect to the (possibly) generalized prior $F(\theta)$. This completes the proof of the Theorem.

The above theorem in conjunction with Theorem 2.2.2 gives immediately a set of conditions for an estimator to be generalized Bayes in terms of the estimator itself.

3.2. *An alternate condition for an estimator to be generalized Bayes.* In the previous section we obtained a condition for an estimator $\delta(x)$ to be generalized Bayes. This section is devoted to the derivation of an alternate condition.

THEOREM 3.2.1. *A necessary and sufficient condition for $\delta(x)$ to be generalized Bayes is that* $\exp\left[\int_0^x \delta(y)\,dy\right]$ $(-\infty < x < \infty)$ *be the moment generating function of some probability distribution.*

PROOF. We have shown in Theorem 3.2.1 that a n.a.s.c. for $\delta(x)$ to be generalized Bayes is that

$$\exp\left[\int_0^x (\delta(y)-y)\,dy\right] = \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(x-\theta)^2\right] dF(\theta) \tag{3.2.1}$$

for some $F(\theta)$. This is equivalent to

$$\exp\left[\int_0^x \delta(y)\,dy - \tfrac{1}{2}x^2\right] = e^{-\frac{1}{2}x^2}\left[\int_{-\infty}^{\infty} e^{x\theta} e^{-\frac{1}{2}\theta^2}\,dF(\theta)\right] \tag{3.2.2}$$

or

$$\begin{aligned} \exp\left[\int_0^x \delta(y)\,dy\right] &= \int_{-\infty}^{\infty} e^{x\theta} e^{-\frac{1}{2}\theta^2}\,dF(\theta) \\ &\doteq \int_{-\infty}^{\infty} e^{x\theta}\,dG(\theta) \end{aligned} \tag{3.2.3}$$

where the measure $G(\theta)$ is defined by

$$G(\theta) = \int_{-\infty}^{\theta} e^{-\frac{1}{2}u^2}\,dF(u). \tag{3.2.4}$$

To complete the proof it remains to show that $G(\theta)$ is a probability measure—i.e. that

$$\int_{-\infty}^{\infty} dG(\theta) = 1. \tag{3.2.5}$$

But this follows from (3.3.3) on setting $x = 0$, thus completing the proof of the Theorem.

We note that $\exp[\int_0^x \delta(y)\,dy]$ is a moment generating function of some measure, and we have identified the measure in terms of the generalized prior distribution. Thus, if $\delta(x)$ is Bayes with respect to a generalized prior $F(\theta)$, then $\exp[\int_0^x \delta(y)\,dy]$ is the moment generating function of the probability measure whose Radon–Nikodym derivative with respect to $F(\theta)$ is

$$e^{-\frac{1}{2}\theta^2}/\int_{-\infty}^{\infty} e^{-\frac{1}{2}\theta^2}\,dF(\theta).$$

3.3. *A necessary and sufficient condition for a bounded risk estimator to be admissible.* In this section we combine our result of Section 3.2 with a result due to Brown [3] to obtain a necessary and sufficient condition for a bounded risk estimator to be admissible. The condition is useful because it is entirely in terms of the estimator itself, and not in terms of generalized prior distributions. The conditions are given in the following Theorem.

THEOREM 3.3.1. *A necessary and sufficient set of conditions for a bounded risk estimator* $\delta(x)$ *to be admissible are* (a) $\delta(x)$ *is generalized Bayes and* (b) $\int^{\infty} dx/g(x) = \int_{-\infty} dx/g(x) = +\infty$ *where* $g(x) = \exp[\int_0^x (\delta(y) - y)\,dy]$.

PROOF. The result follows immediately from Theorem 2.2.1 and (3.1.5).

3.4. *Examples.* We apply the results of Section 3.1 and Section 3.2 to show the inadmissibility of some classes of estimators. It is shown that the classes of estimators in question are not generalized Bayes and hence are not candidates for admissibility by the result of Sacks [1], Theorem 2.1.1.

Consider the class of estimators of the form

$$\delta(x) = x - (ax)/(b + x^2) \qquad a > 0,\ b \geqq 0. \tag{3.4.1}$$

Such estimators have some intuitive appeal because they "shrink" the estimator x towards the origin. We shall see later that all bounded risk generalized Bayes "shrinkers" are admissible. Hence the question of admissibility for this class of estimators reduces to whether the estimators are generalized Bayes.

Note that if $b = 0$ the estimator is discontinuous at the origin and the estimator must be changed on a set of measure greater than zero in order to produce continuity. Hence such an estimator cannot be generalized Bayes by the remarks following (3.1.3). We remark that Karlin and Rubin [8] have shown that a complete class of estimators is given by the class of monotone procedures and the case $b = 0$ is eliminated by this criterion also.

We may therefore assume $b > 0$. Now for such $\delta(x)$,

$$\begin{aligned} g(x) = \exp[\textstyle\int_0^x (\delta(y) - y)\,dy] &= \exp[-a \textstyle\int_0^x (y\,dy)/(b + y^2)] \\ &= \exp[-\tfrac{1}{2}a \log[(x^2 + b)/b]] \\ &= [(x^2 + b)/b]^{-\frac{1}{2}a}. \end{aligned} \tag{3.4.2}$$

This function is not extendable to an entire function because of non-removable singularities at $x = \pm ib^{\frac{1}{2}}$, and hence cannot be generalized Bayes by Theorems 3.1.1 and Condition 1 of Theorem 2.2.2. Therefore such estimators are inadmissible.

Another class of "shrinkers toward a point μ_0" has been proposed and studied by Thompson [20]. We demonstrate the inadmissibility of a closely related class. Thompson proposes estimators of the form

$$\delta(x) = [(\bar{x}-\mu_0)^3/[(\bar{x}-\mu_0)^2+s^2/n]]+\mu_0. \tag{3.4.3}$$

Since our work assumes knowledge of σ, and we are working without loss of generality with samples of size $n = 1$ we show inadmissibility of the class of estimators of the form

$$\delta(x) = (x-\mu_0)^3/[(x-\mu_0)^2+1]+\mu_0. \tag{3.4.4}$$

We have that

$$\begin{aligned} \textstyle\int_0^x \delta(y)\,dy &= \mu_0 x+\textstyle\int_0^x ((y-\mu_0)^3/((y-\mu_0)^2+1))\,dy \\ &= \mu_0 x+\textstyle\int_{-\mu_0}^{x-\mu_0} (y^3)/(y^2+1)\,dy \\ &= \mu_0 x+\tfrac{1}{2}[(x-\mu_0)^2-\mu_0{}^2]-\tfrac{1}{2}\log[((x-\mu_0)^2+1)/(\mu_0{}^2+1)]. \end{aligned} \tag{3.4.5}$$

Hence

$$\begin{aligned} \exp[\textstyle\int_0^x (\delta(y)-y)\,dy] &= \exp[\textstyle\int_0^x \delta(y)\,dy]\,e^{-\frac{1}{2}x^2} \\ &= \exp\{\mu_0 x-x^2/2+\tfrac{1}{2}[(x-\mu_0)^2-\mu_0{}^2] \\ &\qquad -\tfrac{1}{2}\log[((x-\mu_0)^2+1)/(\mu_0{}^2+1)]\} \\ &= [\exp\{\mu_0 x-x^2/2+\tfrac{1}{2}[(x-\mu_0)^2-\mu_0{}^2]\}] \\ &\qquad \cdot[(\mu_0{}^2+1)/(x-\mu_0)^2+1]^{\frac{1}{2}} \\ &= [(\mu_0{}^2+1)/(x-\mu_0)^2+1]^{\frac{1}{2}} \end{aligned} \tag{3.4.6}$$

and this has a non-removable singularity at $x = \mu_0 \pm i$. Therefore

$$\exp[\textstyle\int_0^x (\delta(y)-y)\,dy]$$

cannot be extended analytically into the whole complex plane, and $\delta(x)$ cannot be generalized Bayes because condition (i) of Theorem 2.2.2 is violated.

Another class of "shrinkers" which are not inadmissible by virtue of non-analyticity of $\exp[\int_0^x (\delta(y)-y)\,dy]$ are given by,

$$\delta(a) = (1-e^{-ax^2})x, \qquad a > 0. \tag{3.4.7}$$

We show, however, that such estimators are inadmissible since they violate the order condition (ii) of Theorem 2.2.2. That is, note that

$$\begin{aligned} \textstyle\int_0^x (\delta(y)-y)\,dy &= -\textstyle\int_0^x y\,e^{-ay^2}\,dy \\ &= -\tfrac{1}{2}a[1-e^{-ax^2}]. \end{aligned} \tag{3.4.8}$$

Hence

$$g(x) = \exp[\int_0^x (\delta(y)-y)\,dy = e^{-\frac{1}{2}a} e^{\frac{1}{2}a}[e^{-ax^2}]. \tag{3.4.9}$$

Considering $f(x) = g(x/2^{\frac{1}{2}})$ as a function of the complex variable $u+vi$, we have

$$\begin{aligned} f(u+vi) &= (e^{-\frac{1}{2}a})e^{\frac{1}{2}a}[\exp[-a/2(u^2+2uvi-v^2)] \\ &= (e^{-\frac{1}{2}a})(e^{\frac{1}{2}a}[(\exp[-a/2(u^2-v^2)])(\cos auv - i\sin auv)]). \end{aligned} \tag{3.4.10}$$

In order for $\delta(x)$ to be generalized Bayes it must be that uniformly in any finite interval $a \leqq u \leqq b, f(u+iv) = O(|v|\, e^{\frac{1}{2}v^2})$, by Condition 2 of Theorem 2.2.2. But for any fixed values of u, there is a sequence of values of $v \to \pm\infty$ such that $\cos auv - i\sin auv = +1$, and for this sequence $f(u+iv)/(|v|\, e^{\frac{1}{2}v^2}) \to +\infty$, thus violating the condition. Hence the class of estimators of the form (3.4.7) is a class of non-generalized Bayes, and therefore, inadmissible, estimators.

3.5. *A class of generalized Bayes estimators that are not bounded risk admissible.* We now study estimators of the form

$$\delta(x) = x + \alpha(x) \tag{3.5.1}$$

where

$$\exp[\int_0^x \alpha(y)\,dy] \tag{3.5.2}$$

is the moment generating function of some probability distribution function $F(\theta)$. We note first that such estimators are generalized Bayes. To see this, note that

$$\exp[\int_0^x \delta(y)\,dy] = \exp[\int_0^x \alpha(y)\,dy]\, e^{\frac{1}{2}x^2}. \tag{3.5.3}$$

Now, since $e^{\frac{1}{2}x^2}$ is the moment generating function of the standard normal distribution, $\exp[\int_0^x \delta(y)\,dy]$ is the moment generating function of the convolution of the standard normal with $F(\theta)$. Hence by Theorem 3.2.1, $\delta(x)$ is generalized Bayes.

Also note that if $\alpha(x) = 0$, (3.5.2) is the moment generating function of the distribution putting mass 1 at the point 0. This choice of $\alpha(x)$ leads to the estimator x which is bounded risk admissible. We will show that all other choices of $\alpha(x)$ such that (3.5.2) is a moment generating function, lead to estimators which are either of unbounded risk or are bounded risk inadmissible. The classification of such estimators will depend on the upper and lower terminus of the distribution $F(\theta)$ which we now define. The lower terminus of the distribution function $F(\theta)$ is the point $a = \sup_{-\infty \leqq \theta \leqq \infty}\{\theta : F(\theta) = 0\}$. The upper terminus is defined as the point $b = \inf_{-\infty \leqq \theta \leqq \infty}\{\theta : F(\theta) = 1\}$. It may be that $a = -\infty$ and/or $b = +\infty$. We require some preliminary results. We first prove a lemma.

LEMMA 3.5.1. *If $\delta(x)$ is such that $\exp[\int_0^x (\delta(y)-y)\,dy]$ is the moment generating function for some distribution $F(\theta)$ such that $-\infty < a \leqq b < +\infty$, then $\delta(y)$ is of bounded risk.*

PROOF. We will show $|\delta(y)-y|$ is bounded which immediately implies that $\delta(y)$ is of bounded risk. Since a and b are the lower and upper terminus of $F(\theta)$ we have

$$\exp[\int_0^x (\delta(y)-y)\,dy] = \int_a^b e^{x\theta}\, dF(\theta) \tag{3.5.4}$$

by assumption. Taking logarithms and differentiating with respect to x, we obtain

$$(3.5.5)\quad \begin{aligned} |\delta(x)-x| &= |d/dx(\log \textstyle\int_a^b e^{x\theta}\, dF(\theta))| \\ &= |\textstyle\int_a^b \theta\, e^{x\theta}\, dF(\theta)| / \textstyle\int_a^b e^{x\theta}\, dF(\theta) \\ &\leqq \max(|a|, |b|) \textstyle\int_a^b e^{x\theta}\, dF(\theta) / \textstyle\int_a^b e^{x\theta}\, dF(\theta) \\ &= \max(|a|, |b|) < \infty. \end{aligned}$$

We are now able to prove

THEOREM 3.5.1. *If* $\delta(x)$ *is such that* $\exp[\int_0^x (\delta(y)-y)\,dy]$ *is the moment generating function for some probability distribution* $F(\theta)$ *such that* $-\infty < a \leqq b < \infty$ *and either* a *or* b *is non-zero, then* $\delta(x)$ *is a bounded risk, generalized Bayes inadmissible estimator.*

PROOF. We have already remarked that $\delta(x)$ is generalized Bayes, and the Lemma showed that $\delta(x)$ has bounded risk. It remains to show inadmissibility. We have by assumption that

$$(3.5.6)\qquad g(x) = \exp[\textstyle\int_0^x (\delta(y)-y)\,dy] = \textstyle\int_a^b e^{x\theta}\, dF(\theta),$$

where both $|a|$, $|b| < \infty$ and one of a or b does not equal zero.

Assume $-\infty < a < 0$. Then for $x < 0$, and $a < c < 0$ we have

$$(3.5.7)\qquad \begin{aligned} g(x) = \textstyle\int_a^b e^{x\theta}\, dF(\theta) &\geqq \textstyle\int_a^c e^{x\theta}\, dF(\theta) \\ &\geqq e^{cx} \textstyle\int_a^c dF(\theta) = \varepsilon\, e^{cx} \end{aligned}$$

where $\varepsilon > 0$ by definition of a. Hence

$$(3.5.8)\qquad \textstyle\int_{-\infty}^0 dx/g(x) \leqq \varepsilon^{-1} \textstyle\int_{-\infty}^0 e^{-cx}\, dx < \infty.$$

Thus by Theorem 3.3.1, $\delta(x)$ is inadmissible. A similar argument proves inadmissibility of $0 < b < \infty$ which completes the proof of the theorem.

We note that the conditions $-\infty < a$ in the above theorem is not necessary for the convergence of the integrals in (3.5.6), and similarly $b < \infty$ is not needed for convergence of the upper integral. Hence we could have stated the theorem in the seemingly stronger form, that if $\delta(x)$ is such that $\exp[\int_0^x (\delta(y)-y)\,dy]$ is the moment generating function of a probability distribution function $F(\theta)$ which does not assign mass 1 to the point $\theta = 0$, and if $\delta(x)$ is of bounded risk, then $\delta(x)$ is inadmissible. We show next however that the admissible estimators included by allowing infinite values for a and b are all of unbounded risk and hence no new admissible estimators of bounded risks are included. We require some preliminary results which we give as Lemmas. The first is a result due to Brown [3].

LEMMA 3.5.2. *If* $\delta(x)$ *is bounded risk admissible then* $|\delta(x)-x| < M - \infty < x < \infty$ *for some* $M > 0$.

We also require the following Lemma which follows from Lemma 3.5.2.

LEMMA 3.5.3. *If* $\delta(x)$ *is bounded risk admissible, then* $\exp[\int_0^x (\delta(y)-y)\,dy] < e^{M|x|}$ *for some* $M > 0$.

Finally we require the following lemma.

LEMMA 3.5.4. *If* $F(\theta)$ *is a probability distribution admitting an entire moment generating function, and if either terminus of the distribution is at (plus or minus) infinity, then there is no* M *such that* $\int_{-\infty}^{\infty} e^{x\theta}\,dF(\theta) < e^{M|x|}$.

PROOF. Suppose $b = +\infty$, then given any $M > 0$, $\int_M^\infty dF(\theta) = \varepsilon_m > 0$ by the definition of the upper terminus, and for $x \geqq 0$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{x\theta}\,dF(\theta) &\geqq \int_M^\infty e^{x\theta}\,dF(\theta) \\ &\geqq e^{MX}\int_M^\infty dF(\theta) \\ &= \varepsilon_M e^{MX}. \end{aligned} \tag{3.5.9}$$

But suppose the Lemma is true. Then for some M^*, $E(e^{x\theta}) \leqq e^{M^*X}$. For any $M > M^*$, however, we may choose X so large that $\exp[(M^*-M)X] < \varepsilon_M$. Hence for such an x we have

$$\begin{aligned} E(e^{x\theta}) &\geqq \varepsilon_M e^{MX} \\ &= \varepsilon_M \exp[(M-M^*)X]\,e^{M^*X} \\ &> \varepsilon_M(\varepsilon_M{}^{-1})\,e^{M^*X} \\ &\geqq e^{M^*X}. \end{aligned} \tag{3.5.10}$$

This contradiction establishes the lemma in the case that the upper terminus is $+\infty$. The proof is similar when the lower terminus is $-\infty$. This completes the proof of the Lemma. We note that this result is related to a similar result for entire characteristic functions and could have been derived from it. See for instance Lukacs ([10] page 141).

We are now able to prove the following result.

THEOREM 3.5.2. *If* $\delta(x)$ *is such that* $\exp[\int_0^x (\delta(y)-y)\,dy]$ *is the moment generating function of a probability distribution* $F(\theta)$ *such that either terminus is at (plus or minus) infinity, then if* $\delta(x)$ *is admissible it must be of unbounded risk.*

PROOF. If $\delta(x)$ is to be bounded risk admissible it must be that

$$\exp[\int_0^x (\delta(y)-y)\,dy] \leqq e^{M|x|} \tag{3.5.11}$$

for some $M > 0$, by Lemma 3.5.3. However if $\exp[\int_0^x (\delta(y)-y)\,dy]$ is the moment generating function of $F(\theta)$ where either the upper terminus is at plus infinity or the lower terminus is at minus infinity, we have by Lemma 3.5.4. that it is impossible for $\exp[\int_0^x (\delta(y)-y)\,dy]$ to satisfy (3.5.11), a contradiction. Hence if $\delta(x)$ is to be admissible, it must be of unbounded risk. This completes the proof of the Theorem.

We note that the class of estimators covered by Theorem 3.5.1 contains all estimators of the form $X+a$ for $a \neq 0$. For this particular class of estimators it is easy to see they are bounded risk, and a simple calculation shows that X is a better estimator.

4. Admissible and generalized Bayes estimators—the spherically symmetric multivariate case.

4.1. *Relations between symmetry of prior distribution, symmetry of convolutions with the normal density and symmetry of generalized Bayes estimators.* We investigate the relationship between spherically symmetric generalized Bayes estimators, spherically symmetric generalized prior distributions, and convolutions of such priors with the normal density. We shall show that these three notions are related by the fact that spherical symmetry of any one implies spherical symmetry of the other two.

We require a fact about the continuity of partial derivatives of

$$\int_{E^n} \exp[-\tfrac{1}{2}||x-\theta||^2]\, dF(\theta)$$

which is not strictly contained in Theorem 2.2.1. The proof is very similar however and we omit the details.

LEMMA 4.1.1. *The partial derivatives of* $g(x) = \int_{E^n} \exp[-\tfrac{1}{2}||x-\theta||^2]\, dF(\theta)$ *are jointly continuous for all values of x interior to the region of convergence of the integral.*

We now proceed to the main result of this section.

THEOREM 4.1.1. *If* $\delta(x)$ *is generalized Bayes with respect to the generalized prior* $F(\theta)$, *the following are equivalent*:

(i) $\delta(x)$ *is spherically symmetric* a.e. *That is* $\delta(x) = h(||x||^2)x$, *or* $\delta(Px) = P\,\delta(x)$ *for any orthogonal transformation P, for almost all x.*

(ii) $F(\theta)$ *is a spherically symmetric measure. That is, for any measurable set* $A \in E^n$, $F(A) = F(PA)$ *where P is any orthogonal transformation.*

(iii) *The convolution of* $F(\theta)$ *with the standard normal density is a function only of* $||x||$, *i.e.,*

$$g(x) = \int \exp[-\tfrac{1}{2}||x-\theta||^2]\, dF(\theta) = g(Px)$$

for any orthogonal transformation P.

PROOF. Let $dG(\theta) = \int \exp[-\tfrac{1}{2}||\theta||^2]\, dF(\theta)$ and note that $\int dG(\theta) < \infty$ since $g(0) < \infty$. Setting $\varphi_G(x) = \int e^{x'\theta}\, dG(\theta)$, (iii) is equivalent to

$$\varphi_G(Px) = \varphi_G(x) \tag{4.1.1}$$

for all x and P orthogonal. From the unicity of the Laplace transform, (4.1.1) is equivalent to $G(A) = G(PA) \equiv G_P(A)$ which is clearly equivalent to (ii). Since

$$\begin{aligned} \delta(Px) &= \int \theta\, e^{x'P'\theta}\, dG(\theta) / \int e^{x'P'\theta}\, dG(\theta) \\ &= \int P\eta\, e^{x'\eta}\, dG_P(\eta) / \int e^{x'\eta}\, dG_P(\eta) \\ &= P\left(\int \eta\, e^{x'\eta}\, dG_P(\eta) / \int e^{x'\eta}\, dG_P(\eta)\right) \end{aligned} \tag{4.1.2}$$

we have that (iii) implies (i).

For (i) implies (iii), $\delta(x) = P'\,\delta(PX)$ so that

$$(4.1.3)\qquad \begin{aligned} P'\delta(Px) &= \int P'\theta\, e^{x'P'\theta}\, dG(\theta)/\int e^{x'P'\theta}\, dG(\theta) \\ &= \int \theta\, e^{x'\theta}\, dG_P(\theta)/\int e^{x'\theta}\, dG_P(\theta) \\ &= \delta(x) = \int \theta\, e^{x'\theta}\, dG(\theta)/\int e^{x'\theta}\, dG(\theta). \end{aligned}$$

Hence $\nabla \log \varphi_G(x) = \nabla \log \varphi_{G_P}(x)$ where ∇ is the gradient operator. Since $\varphi_G(0) = \varphi_{G_P}(0) > 0$ and φ_G and φ_{G_P} are analytic, $\varphi_G(x) = \varphi_{G_P}(x)$ so that $G = G_P$. This completes the proof.

We remark that Bochner and Chandrasekheran [2] prove theorems which are closely related to our results on the equivalence of spherically symmetric convolutions and spherically symmetric measures, for the absolutely continuous case.

4.2. *A necessary and sufficient condition for a spherically symmetric estimator to be generalized Bayes.* In this section we develop a necessary and sufficient condition for a spherically symmetric estimator to be generalized Bayes. The main result is the following theorem.

THEOREM 4.2.1. *An estimator of the form $\delta(x) = h(\|x\|^2)x$ is generalized Bayes if and only if there exists a measure $F(\theta)$ such that*

$$(4.2.1)\qquad \exp\left[\tfrac{1}{2}\int_0^{\|x\|^2} (h(y)-1)\, dy\right] = \int \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] dF(\theta).$$

PROOF. If $\delta(x) = h(\|x\|^2)x$ is generalized Bayes with respect to the prior $F'(\theta)$, then,

$$(4.2.2)\qquad \delta(x) = h(\|x\|^2)x = \left[(2g'(\|x\|^2)/g(\|x\|^2)) + 1\right] x,$$

where $g(\|x\|^2) = \int \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] dF'(\theta)$. From (4.2.2) we have

$$(4.2.3)\qquad \tfrac{1}{2}\left[h(\|x\|^2) - 1\right] = g'(\|x\|^2)/g(\|x\|^2)$$

or

$$(4.2.4)\qquad \begin{aligned} \tfrac{1}{2}\left[h(y) - 1\right] &= g'(y)/g(y) \\ &= d/dy \log g(y). \end{aligned}$$

Noting again that the continuity of the partial derivative of $g(\|x\|^2)$ implies the continuity of $d/dy\, g(y)$, we integrate (4.2.4) to obtain

$$(4.2.5)\qquad \begin{aligned} g(\|x\|)^2 &= k \exp\left[\tfrac{1}{2}\int_0^{\|x\|^2} (h(y)-1)\, dy\right] \\ &= \int_{E^n} \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] dF'(\theta). \end{aligned}$$

Absorbing the constant $1/k$ into the measure $F'(\theta)$ we obtain

$$(4.2.6)\qquad \exp\left[\tfrac{1}{2}\int_0^{\|x\|^2} (h(y)-1)\, dy\right] = \int_{E^n} \exp\left[-\tfrac{1}{2}\|x-\theta\|^2\right] dF(\theta).$$

This proves the necessity portion of the theorem.

To prove sufficiency, assume that

$$\exp\left[\tfrac{1}{2}\int_0^{||x||^2}(h(y)-1)\,dy\right] = \int_{E^n}\exp\left[-\tfrac{1}{2}||x-\theta||^2\right]dF(\theta), \tag{4.2.7}$$

for some measure $F(\theta)$. Then, taking logarithms, we find

$$\tfrac{1}{2}\int_0^{||x||^2}(h(y)-1)\,dy = \log\int_{E^n}\exp\left[-\tfrac{1}{2}||x-\theta||^2\right]dF(\theta). \tag{4.2.8}$$

By Theorem 2.2.1 we may take partial derivatives under the integral sign and we have for almost all x,

$$h(||x||^2)x_i = \left(\int_{E^n}\theta_i\exp\left[-\tfrac{1}{2}||x-\theta||^2\right]dF(\theta)\right)\Big/\left(\int_{E^n}\exp\left[-\tfrac{1}{2}||x-\theta||^2\right]dF(\theta)\right). \tag{4.2.9}$$

Since $\delta(x) = h(||x||^2)x$, (4.2.9) is equivalent to the fact that $\delta(x)$ is a generalized Bayes estimator with respect to $F(\theta)$. This completes the proof of the theorem.

4.3. *An alternate necessary condition for an estimator to be generalized Bayes.* We use the results of the previous section to prove an alternate necessary and sufficient condition for an estimator to be generalized Bayes. We then obtain an alternate necessary condition which is one-dimensional in nature and perhaps, therefore, somewhat easier to apply. We show that the one-dimensional condition is not sufficient.

Our first result follows directly from Theorem 4.2.1.

THEOREM 4.3.1. *A necessary and sufficient condition for a spherically symmetric estimator of the form* $\delta(x) = h(||x||^2)x$ *to be generalized Bayes is that* $\exp\left[\tfrac{1}{2}\int_0^{||x||^2} h(y)\,dy\right]$ *be the moment generating function of some probability distribution* $G(\theta)$.

We next use the above to find a one-dimensional necessary condition for a spherically symmetric estimator to be generalized Bayes.

THEOREM 4.3.2. *A necessary condition for a spherically symmetric estimator* $\delta(x) = h(||x||^2)x$ *to be generalized Bayes, is that* $\exp\left[\tfrac{1}{2}\int_0^{t^2} h(y)\,dy\right]$ *is the moment generating function of a one-dimensional symmetric probability distribution.*

PROOF. We know by Theorem 4.4.1 that if $\delta(x) = h(||x||^2)x$ is generalized Bayes that $\exp\left[\tfrac{1}{2}\int_0^{||x||^2} h(y)\,dy\right]$ is the moment generating function of some probability distribution $G(\theta)$. Hence there is a random vector $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$ such that $E_G(e^{\theta' x}) = \exp\left[\tfrac{1}{2}\int_0^{||x||^2} h(y)\,dy\right]$. But this function is spherically symmetric and hence $E_G(e^{\theta' x}) = E_G(e^{\theta' P x})$ for any orthogonal transformation P. Choosing for every x, the orthogonal transformation P_x which carries x into $(||x||, 0, 0, \cdots, 0)$, we have $\exp\left[\tfrac{1}{2}\int_0^{||x||^2} h(y)\,dy\right] = E_G(e^{\theta_1 ||x||})$ which is just the moment generating function of the one-dimensional random variable θ_1. This completes the proof of the theorem.

We note that the condition of Theorem 4.3.2 is not sufficient since there exist many one-dimensional symmetric distributions which are not the marginal distribution of any spherically symmetric multivariate distribution.

4.4. *A necessary and sufficient condition for a bounded risk spherically symmetric estimator to be admissible.* We find in this section a necessary and sufficient condition for a bounded risk spherically symmetric estimator to be admissible. Our proof depends on Theorem 2.1.3. The result is the following:

THEOREM 4.4.1. *Necessary and sufficient conditions for a bounded risk spherically symmetric estimator* $\delta(x) = h(||x||^2)x$ *to be admissible are*

(i) $g(||x||^2) = \exp[\frac{1}{2}\int_0^{||x||^2}(h(y)-1)\,dy] \equiv \int_{E^n}\exp[-\frac{1}{2}||x-\theta||^2]\,dF(\theta)$ *for some measure* $F(\theta)$, *and*

(ii) $\int^{\infty}1/r^{n-1}g(r^2)\,dr = \infty$.

PROOF. Condition (i) implies that $\delta(x)$ is generalized Bayes with respect to a prior $F(\theta)$ by Theorem 4.2.1. Theorem 4.1.1. implies $\int_{E^n}\exp[-\frac{1}{2}||x-\theta||^2]\,dF(\theta)$ is a function of $||x||^2$ only.

Recall, by Theorem 2.1.3, that for a bounded risk, generalized Bayes estimator such that $\int_{E^n}\exp[-\frac{1}{2}||x-\theta||^2]\,dF(\theta) = g(||x||^2)$ a necessary and sufficient condition for admissibility is that $\int^{\infty}1/r^{n-1}g(r^2)\,dr = \infty$. Condition (i) is also necessary for admissibility since if it fails $\delta(x)$ cannot be generalized Bayes, and hence not admissible. This completes the proof of the theorem.

For completeness we restate the theorem in a form that will be useful for later applications.

4.5. *Example.* In this section we show inadmissibility of a class of estimators by showing that the estimators are not generalized Bayes. We accomplish this by showing that the necessary condition of Theorem 4.3.2. is violated.

Consider estimators of the form

$$\delta(x) = [1-a/(b+x'x)]x \qquad a>0, b \geqq 0 \tag{4.5.1}$$

which are of the form $h(||x||^2)x$ with $h(y) = 1-a/(b+y)$. Assume that $b > 0$. In order for such an estimator to be generalized Bayes it is necessary by Theorem 4.3.2, that $\exp[\frac{1}{2}\int_0^{t^2}h(y)\,dy]$ be a moment generating function of some one-dimensional symmetric distribution. It follows since moment generating functions are analytic in their domain of definition, that $\exp[\frac{1}{2}\int_0^{t^2}h(y)\,dy]$ must be extendable to an entire function. We show that this is not the case. Note that

$$\begin{aligned}\exp[\tfrac{1}{2}\textstyle\int_0^{t^2}h(y)\,dy] &= \exp[\tfrac{1}{2}\textstyle\int_0^{t^2}[1-a/(b+y)]\,dy]\\ &= \exp\{\tfrac{1}{2}t^2 - a/2\log[(b+t^2)/b]\}\\ &= (e^{\frac{1}{2}t^2})[b/(b+t^2)]^{\frac{1}{2}a},\end{aligned} \tag{4.5.2}$$

which has a non-removable singularity at $\pm ib^{\frac{1}{2}}$. Hence $\exp[\frac{1}{2}\int_0^{t^2}h(y)\,dy]$ cannot be extended analytically into the entire complex plane which implies the estimator is not generalized Bayes, and therefore is inadmissible.

If $b = 0$, $\delta(x) = [1-a/x'x]x$. At the origin the estimator is badly discontinuous and the discontinuity cannot be removed by any redefinition of the estimator on a set of measure zero. But we demonstrated in the course of proving Theorem 4.2.1 that such a situation is impossible. Hence these estimators also are not generalized Bayes and consequently inadmissible.

We note that the class of estimators considered above have been considered by Stein [13]. James and Stein [7] consider the case $b = 0$ and show that the estimator $\delta(x) = (1-(n-2)/(x'x))x$ has uniformly lowest risk among all estimators of the form $\delta(x) = (1-a/x'x)x$. This result immediately implies inadmissibility of the usual estimator $\delta(x) = x$ in 3 or higher dimensions. Sclove and Baranchik [12] consider a generalization of the class (4.6.1) where $b = 0$, but a is assumed to be a monotone function of $||x||$.

5. Further results on generalized Bayes estimators—the univariate case.

5.1. *All bounded risk generalized Bayes "shrinkers" are admissible.* We define a "shrinker" to be an estimator $\delta(x) = x-\varepsilon(x)$ where $\varepsilon(x) \geqq 0$ of $x \geqq 0$ and $\varepsilon(x) \leqq 0$ if $x \leqq 0$. We have the following result.

THEOREM 5.1.1. *Any bounded risk generalized Bayes shrinker is admissible.*

PROOF. Let $\delta(x)$ be a bounded risk generalized Bayes shrinker. Then

(5.1.1) $$\delta(x)-x \leqq 0 \quad \text{if} \quad x \geqq 0$$
$$\delta(x)-x \geqq 0 \quad \text{if} \quad x \leqq 0.$$

Hence for any x, $-\infty \leqq x \leqq \infty$,

(5.1.2) $$\int_0^x (\delta(y)-y)\,dy \leqq 0,$$ and

(5.1.3) $$\exp\left[\int_0^x (\delta(y)-y)\,dy\right] \leqq 1.$$

Hence for any $M > 0$

(5.1.4) $$\int_M^\infty 1/\exp\left[\int_0^x (\delta(y)-y)\,dy\right]dx \geqq \int_M^\infty 1/1\,dx = \infty,$$

and

(5.1.5) $$\int_{-\infty}^{-M} 1/\exp\left[\int_0^x (\delta(y)-y)\,dy\right]dx \geqq \int_{-\infty}^{-M} 1/1\,dx = \infty.$$

Hence $\delta(x)$ is admissible by Theorem 3.3.1. This completes the proof of the theorem.

As a generalization of the above we define the concept of a "shrinker towards θ_0" as an estimator of the form $\delta(x) = \theta_0+\varepsilon(x)(x-\theta_0)$, where $0 \leqq \varepsilon(x) \leqq 1$. In terms of this definition, our original definition of a "shrinker" would be described as a "shrinker towards 0." We next prove that all generalized Bayes shrinkers towards θ_0, which are of bounded risk, are admissible. We need first a lemma.

LEMMA 5.1.1. *If $\delta(x)$ is generalized Bayes and/or admissible, then so is $\delta(x+a)-a$ for any $-\infty < a < \infty$.*

PROOF. Suppose $\delta(x)$ is admissible but that $\delta(x+a)-a$ is not. Assume $\delta(x+a)-a$ is beaten by $\delta'(x)$. We evaluate the risk of $\delta(x+a)-a$ at $\theta-a$ and show it equal to the risk of $\delta(x)$ at θ. We then evaluate the risk of $\delta'(x)$ at $\theta-a$ and show it is equal to the risk of $\delta'(x-a)+a$ at θ. This would imply the inadmissibility of $\delta(x)$ since $\delta'(x-a)+a$ would beat it. This contradiction shows that $\delta(x+a)-a$ must be admissible. We now show the indicated equalities, by a simple change of variable. The risk of $(\delta(x+a)-a)$ is

$$(5.1.6)\qquad (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[\delta(x+a)-a-(\theta-a)]^2\exp[-\tfrac{1}{2}[x-(\theta-a)]^2]\,dx$$
$$= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[\delta(x)-\theta]^2\exp[-\tfrac{1}{2}[x-\theta]^2]\,dx.$$

The risk of $\delta'(x)$ at $(\theta-a)$ is

$$(5.1.7)\qquad (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[\delta'(x)-(\theta-a)]^2\exp[-\tfrac{1}{2}[x-(\theta-a)]^2]\,dx$$
$$= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[\delta'(x-a)+a-\theta]^2\exp[-\tfrac{1}{2}[x-\theta]^2]\,dx.$$

This completes the proof that admissibility of $\delta(x)$ implies admissibility of $\delta(x+a)-a$.

Now assume $\delta(x)$ is generalized Bayes with respect to the prior $F(\theta)$. Then we show $\delta(x+a)-a$ is generalized Bayes with respect to the prior $F(\theta+a)$. Note that

$$\Big(\int_{-\infty}^{\infty}\theta\exp[-\tfrac{1}{2}(x-\theta)^2]\,dF(\theta+a)\Big)\Big/\Big(\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}(x-\theta)^2]\,dF(\theta+a)\Big)$$
$$(5.1.8)\qquad = \Big(\int_{-\infty}^{\infty}(u-a)\exp[-\tfrac{1}{2}[x-(u-a)]^2]\,dF(u)\Big)\Big/$$
$$\Big(\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}[x-(u-a)]^2]\,dF(u)\Big)$$
$$= \delta(x+a)-a.$$

This completes the proof of the lemma.

We can now state the theorem.

THEOREM 5.1.2. *All generalized Bayes bounded risk "shrinkers towards θ_0" are admissible.*

PROOF. Let $\delta(x)$ be a bounded risk, generalized Bayes "shrinker towards θ_0." Then $\delta(x+\theta_0)-\theta_0$ is easily seen to be bounded risk, and is generalized Bayes by the lemma. We now show $\delta(x+\theta_0)-\theta_0$ is a "shrinker" by our original definition—i.e. that $x-\delta(x+\theta_0)-\theta_0 \geqq 0$ for $x \geqq 0$ and $x-\delta(x+\theta_0)-\theta_0 \leqq 0$ for $x \leqq 0$. Note that $\delta(x) = \varepsilon(x)(x-\theta_0)+\theta_0$ where $0 \leqq \varepsilon(x) \leqq 1$. Hence $\delta(x+\theta_0)-\theta_0 = \varepsilon(x+\theta_0)x$, where $0 \leqq \varepsilon(x+\theta_0) \leqq 1$. Therefore $x-\delta(x+\theta_0)-\theta_0 = x-\varepsilon(x+\theta_0)x = [1-\varepsilon(x+\theta_0)]x$. Since $1-\varepsilon(x+\theta_0) \geqq 0$ the result follows. We have then that $\delta(x+\theta_0)-\theta_0$ is a bounded risk, generalized Bayes "shrinker" and it is therefore admissible by Theorem 5.2.1. But this implies that $\delta(x)$ is admissible by Lemma 5.1.1. This completes the proof of the theorem.

Shrinkers toward a point θ_0 have been discussed by Tukey [21] and Thompson [20] among others.

5.2. *Bounded risk "expanders."* We define an "expander" to be an estimator of the form $\delta(x) = x+\varepsilon(x)$ where $\varepsilon(x) \geqq 0$, $x \geqq 0$ and $\varepsilon(x) \leqq 0$ for $x \leqq 0$. It is well known that one class of estimators, namely estimators of the form $\delta(x) = ax$, $a > 1$ are all inadmissible and in fact are beaten by the estimator $\delta(x) = x$. It is not true however that all "expanders" are beaten by x. We present an example of a pair of estimators which are bounded risk admissible, one of which is an expander and the other of which is a shrinker. The estimators are the generalized Bayes estimators with respect to the generalized prior density $dF(\theta)/d\theta = 1 \pm e^{\frac{1}{2}\theta^2}$. The estimators are given by

$$\delta(x) = \frac{(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\theta[1 \pm e^{-\frac{1}{2}\theta^2}]\exp[-1[x-\theta]^2/2]\,d\theta}{(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[1 \pm e^{-\frac{1}{2}\theta^2}]\exp[-1[x-\theta]^2/2]\,d\theta}$$

$$= \frac{x \pm (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\theta\exp[-\frac{1}{2}[2\theta^2-2x\theta+x^2]]\,d\theta}{1 \pm (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp[-\frac{1}{2}[2\theta^2-2x\theta+x^2]]\,d\theta}$$

$$(5.2.1)\qquad = \frac{x \pm e^{-\frac{1}{4}x^2}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\theta\exp[-2/2[\theta-x/2]^2]\,d\theta}{1 \pm e^{-\frac{1}{4}x^2}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp[-2/2[\theta-x/2]^2]\,d\theta}$$

$$= \frac{x \pm e^{-\frac{1}{4}x^2}(x/2)}{1 \pm e^{-\frac{1}{4}x^2}(1/2^{\frac{1}{2}})}$$

$$= x\left[\frac{1 \pm \frac{1}{2}e^{-\frac{1}{4}x^2}}{1 \pm 1/2^{\frac{1}{2}}e^{-\frac{1}{4}x^2}}\right].$$

This estimator is an "expander" when the plus sign is taken, and a "shrinker" when the minus sign is taken.

The estimators are of bounded risk since

$$|\delta(x)-x| = \left|x - x\left[\frac{1 \pm \frac{1}{2}e^{-\frac{1}{4}x^2}}{1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}}\right]\right|$$

$$(5.2.2)\qquad = |x|\left|\frac{(1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}) - (1 \pm \frac{1}{2}e^{-\frac{1}{4}x^2})}{1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}}\right|$$

$$= |x|\left|\frac{e^{-\frac{1}{4}x^2}(\pm 2^{-\frac{1}{2}} \mp \frac{1}{2})}{1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}}\right|$$

$$\leq |x|(2^{-\frac{1}{2}}+\tfrac{1}{2})\frac{e^{-\frac{1}{4}x^2}}{(1-2^{-\frac{1}{2}})}.$$

Clearly the last expression in (5.2.2) is bounded and as noted in the proof of Lemma 3.5.1 this implies $\delta(x)$ is of bounded risk. To show that the estimators are admissible we use Brown's result, Theorem 2.1.3, since it is more convenient to apply in this case than our own Theorem 3.3.2. We note however that Theorem 5.1.1 already proves the admissibility of the estimator when the minus sign is used. We compute the convolution

$$(5.2.3)\qquad (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}[1 \pm e^{-\frac{1}{2}\theta^2}]e^{-[x-\theta]^2}\,d\theta = 1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2},$$

and for admissibility we need

$$(5.2.4)\qquad \int^{\infty}(1/(1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}))\,dx = \int_{-\infty}(1/(1 \pm 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2}))\,dx + \infty.$$

But this is clearly so since the integrand approaches 1 as $x \to \pm\infty$. Hence we have an example of a pair of bounded risk admissible estimators, one of which is an "expander."

We next show that most expanders of bounded risk are inadmissible. Specifically we have the following theorem.

THEOREM 5.2.1. *If $\delta(x)$ is a bounded risk "expander" of the form $\delta(x) = x+\varepsilon(x)$ where $|x| > M$ implies $|\varepsilon(x)| \geqq \varepsilon > 0$, then $\delta(x)$ is inadmissible.*

PROOF. We assume that $\delta(x)$ is generalized Bayes, otherwise it is clearly inadmissible. We show that $\int^{\infty} 1/\exp[\int_0^x(\delta(y)-y)\,dy]\,dx < \infty$, which by Theorem 3.3.1 suffices to show inadmissibility. Now $\delta(y)-y = \varepsilon(y)$, which is greater than ε for $y > M$ and less than $-\varepsilon$ for $x < -M$. Therefore

$$\begin{aligned}
\int^{\infty} 1/\exp[\textstyle\int_0^x(\delta(y)-y)\,dy]\,dx &= K+\textstyle\int_M^{\infty} 1/\exp[\int_0^M \varepsilon(y)\,dy+\int_M^x \varepsilon(y)\,dy]\,dx \\
&= K+\textstyle\int_M^{\infty} 1/K' \exp[\int_M^x \varepsilon(y)\,dy]\,dx \qquad (5.2.5)\\
&= K+1/K' \textstyle\int_M^{\infty} \exp[-\int_M^x \varepsilon(y)\,dy]\,dx \\
&\leqq K+1/K' \textstyle\int_M^{\infty} e^{-\varepsilon(x-M)}\,dx < \infty.
\end{aligned}$$

This completes the proof of the theorem.

5.3. *A characterization of proper Bayes estimators.* In this section we investigate the difference between generalized Bayes estimators where the prior distribution gives finite mass to the real line, so called proper Bayes estimators, and estimators where the prior gives infinite mass to the line, or improper Bayes procedures. That is, given that $\delta(x)$ is generalized Bayes with respect to a distribution $F(\theta)$, we find necessary and sufficient conditions in terms of the estimator $\delta(x)$, so that $\int_{-\infty}^{\infty} dF(\theta) < \infty$—i.e. so that $\delta(x)$ is proper Bayes. Our first result is the following:

LEMMA 5.3.1. *If $\delta(x)$ is generalized Bayes with respect to a prior $F(\theta)$, then $\delta(x)$ is proper Bayes if and only if $\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\exp[-\frac{1}{2}(x-\theta)^2]\,dF(\theta)]\,dx < \infty$.*

PROOF. We wish to show $\int_{-\infty}^{\infty} dF(\theta) < \infty$ iff.

$$\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}(x-\theta)^2]\,dF(\theta)]\,dx < \infty,$$

but this follows immediately from Tonelli's theorem (Dunford and Schwarz [4] page 194).

An immediate consequence of this lemma and Theorem 3.1.1 is the following:

THEOREM 5.3.1. *If $\delta(x)$ is a generalized Bayes estimator, it is proper Bayes if and only if $\int_{-\infty}^{\infty}\exp[\int_0^x(\delta(y)-y)\,dy]\,dx < \infty$.*

5.4. *Generalized Bayes "shrinkers" of unbounded risk.* In Subsection 5.1 we proved that all bounded risk generalized Bayes shrinkers are admissible. In this section we extend this result to a wider class of generalized Bayes shrinkers, many of which are of unbounded risk. Specifically we will show that a large class of shrinkers which "shrink enough" are actually proper Bayes, and hence are admissible.

THEOREM 5.4.1. *Let $\delta(x)$ be a generalized Bayes estimator. If there is an M such that $|x| > M$ implies $|\delta(x)| \leqq |x|-(1+\varepsilon)/|x|$ for some $\varepsilon > 0$, then $\delta(x)$ is proper Bayes, and hence admissible.*

PROOF. We use Theorem 5.3.1 after showing that $\int_{-\infty}^{\infty} \exp[\int_0^x (\delta(y)-y)\,dy]\,dx < \infty$. Now

$$\begin{aligned}
\int_M^\infty \exp[\int_0^x (\delta(y)-y)\,dy]\,dx &= \int_M^\infty \exp[\int_0^M(\delta(y)-y)\,dy]\exp[\int_M^x(\delta(y)-y)\,dy]\,dx \\
&= K\int_M^\infty \exp[\int_M^x(\delta(y)-y)\,dy]\,dx \\
&\leqq K\int_M^\infty \exp[-\int_M^x(1+\varepsilon)/y\,dy]\,dx \qquad (5.4.1)\\
&= K\int_M^\infty \exp[-(1+\varepsilon)\log x/M]\,dx \\
&= K\int_M^\infty (M/x)^{1+\varepsilon}\,dx < \infty,
\end{aligned}$$

since $\int_M^\infty 1/x^{1+\varepsilon}\,dx < \infty$. Similarly

$$\int_{-\infty}^{-M} \exp[\int_0^x (\delta(y)-y)\,dy]\,dx < \infty. \qquad (5.4.2)$$

Since $\delta(y)-y$ is a continuous function it is bounded on $[-M, M]$. Hence and hence

$$\int_{-M}^{M} \exp[\int_0^x (\delta(y)-y)\,dy]\,dx < \infty. \qquad (5.4.3)$$

Therefore adding (5.4.1), (5.4.2) and (5.4.3) we have

$$\int_{-\infty}^{\infty} \exp[\int_0^x (\delta(y)-y)\,dy]\,dx < \infty, \qquad (5.4.4)$$

which completes the proof of the theorem. We also note, if for $|x| > M$, $|\delta(x)| \geqq |x|-(1-\varepsilon)/|x|$ for some $\varepsilon \geqq 0$ that the above inequalities may be reversed and (5.4.1) would become

$$\begin{aligned}
\int_M^\infty \exp[\int_0^x (\delta(y)-y)\,dy] &\geqq K'\int_M^\infty \exp[(1-\varepsilon)\log(x'M)]\,dx \qquad (5.4.5)\\
&= K'\int_M^\infty (M/x)^{1-\varepsilon}\,dx = \infty.
\end{aligned}$$

Hence such estimators cannot be proper Bayes. Therefore, $\delta(x) = X-1/x$ is the dividing line between proper and improper Bayes estimators in the sense that generalized estimators which eventually "expand" $\delta(x)$ are improper Bayes, and those which "shrink" it are proper Bayes.

A Corollary which follows by an easy calculation is

COROLLARY 5.4.1. *Let $\delta(x)$ be a generalized Bayes estimator. Let $\delta'(x)$ denote the derivative of $\delta(x)$. If $0 \leqq \limsup_{|x|\to\infty} \delta'(x) < 1$, then $\delta(x)$ is proper Bayes and admissible.*

5.5. *Generalized Bayes "expanders" of unbounded risk.* In this section we extend the class of inadmissible expanders studied in Subsection 5.3, to include a certain class of unbounded risk estimators. Specifically we will show that if $\delta(x) = a(x)x$ where $a(x) \geqq a > 1$ for $|x| > M$, then $\delta(x)$ is inadmissible. We require first a lemma.

LEMMA 5.5.1. *If $\delta(x) = a(x)x$ where $|x| > M$ implies $a(x) \geqq a > 1$, then*

$$\lim_{\theta\to\infty} \int_\theta^\infty (\delta(x)-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx \to \infty.$$

PROOF. If $x > \theta > M$ then

$$[\delta(x)-\theta] \geqq (ax-\theta) > 0, \tag{5.5.1}$$

and hence

$$[\delta(x)-\theta]^2 > [ax-\theta]^2. \tag{5.5.2}$$

We therefore have

$$\begin{aligned}\int_\theta^\infty [\delta(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx &\geqq \int_\theta^\infty (ax-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &= \int_0^\infty [au+(a-1)\theta]^2 e^{-\frac{1}{2}u^2}\,du\\ &= a^2\int_0^\infty u^2 e^{-\frac{1}{2}u^2}\,du + 2a(a-1)\theta\int_0^\infty u\,e^{-\frac{1}{2}u^2} + (a-1)^2\theta^2\int_0^\infty e^{-\frac{1}{2}u^2}\,du\\ &= (2\pi)^{\frac{1}{2}}\left[\frac{a^2}{2}+\frac{2(a-1)a\theta}{(2\pi)^{\frac{1}{2}}}+\frac{(a-1)^2\theta^2}{2}\right],\end{aligned} \tag{5.5.3}$$

which approaches $+\infty$ as $\theta \to \infty$. This completes the proof of the lemma. Now we may state and prove the basic result of the section.

THEOREM 5.5.1. *If $\delta(x)$ is an estimator such that $\delta(x) = a(x)x$ where $|x| > M$ implies $a(x) \geqq a > 1$, then $\delta(x)$ is inadmissible.*

PROOF. By the lemma we can choose $\theta_0 > M$ so large that $\theta > \theta_0$ implies

$$\int_\theta^\infty [\delta(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx > 2(2\pi)^{\frac{1}{2}} \quad \text{if} \quad \theta > \theta_0. \tag{5.5.4}$$

Define an estimator

$$\begin{aligned}\delta^*(x) &= \delta(x) \qquad \text{if} \quad x < \theta_0;\\ &= x \qquad\quad\ \text{if} \quad x \geqq \theta_0.\end{aligned} \tag{5.5.5}$$

We show that $\delta^*(x)$ is better than $\delta(x)$.

CASE a. $\theta < \theta_0$. If $x < \theta_0$ then $\delta(x) = \delta^*(x)$ and hence $(\delta(x)-\theta)^2 = (\delta^*(x)-\theta)^2$. If $x \geqq \theta_0$ and $\theta < \theta_0$ then $(\delta(x)-\theta) \geqq (ax-\theta) > (x-\theta) = (\delta^*(x)-\theta)$ and $(\delta(x)-\theta)^2 > (\delta^*(x)-\theta)^2$. Therefore

$$\begin{aligned}&(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty (\delta(x)-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &\qquad > (2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty (\delta^*(x)-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx.\end{aligned} \tag{5.5.6}$$

CASE b. $\theta \geqq \theta_0$. If $x < \theta_0$, $\delta(x) = \delta^*(x)$ and therefore $(\delta(x)-\theta)^2 = (\delta^*(x)-\theta)^2$. We have then that

$$\begin{aligned}&(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty [\delta(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &\qquad\qquad -(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty [\delta^*(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &= (2\pi)^{-\frac{1}{2}}\Big[\int_{\theta_0}^\infty [\delta(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &\qquad\qquad -\int_{\theta_0}^\infty [\delta^*(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\Big]\\ &\geqq (2\pi)^{-\frac{1}{2}}\Big[\int_\theta^\infty [\delta(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &\qquad\qquad -\int_{\theta_0}^\infty [\delta^*(x)-\theta]^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\Big]\\ &\geqq 2-(2\pi)^{-\frac{1}{2}}\int_{\theta_0}^\infty (\delta^*(x)-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &= 2-(2\pi)^{-\frac{1}{2}}\int_{\theta_0}^\infty (x-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &\geqq 2-(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty (x-\theta)^2 \exp[-\tfrac{1}{2}(x-\theta)^2]\,dx\\ &= 1 > 0.\end{aligned} \tag{5.5.7}$$

Hence the differences in the risks for $\theta \geqq \theta_0$ is always strictly positive, and $\delta^*(x)$ is better than $\delta(x)$. This completes the proof of the theorem.

A Corollary is the following

COROLLARY 5.5.1. *If* $\delta(x)$ *is such that* $\liminf_{|x|\to\infty} \delta'(x) > 1$ *then* $\delta(x)$ *is inadmissible.*

In Section 5 we have shown that all bounded risk generalized Bayes "shrinkers" and all "shrinkers" that eventually "shrink by at least ε" are admissible. We have also shown that all bounded risk expanders that eventually expand by at least ε are inadmissible, as are all expanders of the form $a(x)x$ where $a(x)$ is eventually bounded below by $a > 1$. We have also shown that bounded risk generalized Bayes shrinkers towards a point θ_0 are admissible. We might remark that Lemma 5.1.1 also allows us to extend our other results on admissibility and inadmissibility to "shrinkers towards θ_0" and an analogously defined "expanders about θ_0." We shall not do so formally as the basic idea involved has been presented in the proof of Theorem 5.1.2 and carries over to these other cases.

Finally we remark that Lemma 5.5.1 and Theorem 5.5.1 hold not only for the normal case, but for the case of a translation parameter family of distribution functions $F(x-\theta)$, where the parameter space is $(-\infty, \infty)$, $\int_0^\infty dF(x) > 0$, and $\int_{-\infty}^{\infty} x^2\, dF(x) < \infty$.

6. Further results on generalized Bayes estimators—the spherically symmetric multivariate case.

6.1. *A dividing line between admissible and inadmissible bounded risk estimators.* We showed by Theorem 4.4.1 that a necessary and sufficient condition for a bounded risk generalized Bayes estimator of the form $\delta(x) = h(\|x\|^2)x$ to be admissible is that $\int^\infty 1/(r^{n-1}g(r^2))\, dr = \infty$ where n is the dimension of the mean vector of the multivariate normal distribution and $g(r^2) = \exp[\frac{1}{2}\int_0^{r^2}[h(y)-1]\, dy]$. We now use this result to obtain a dividing line between bounded risk admissible, and bounded risk inadmissible estimators. We have the following result.

THEOREM 6.1.1. *Let* $\delta(x)$ *be a bounded risk generalized Bayes estimator of the form* $\delta(x) = h(\|x\|^2)x$. *The following results hold*:

(a) *If there exists an* M *such that* $y > M$ *implies* $h(y) \leqq 1-(n-2)/y$, *then* $\delta(x)$ *is admissible, and*

(b) *If there exists an* M *such that* $y > M$ *implies* $h(y) \geqq 1-b/y$ *for some* $b < n-2$, *then* $\delta(x)$ *is inadmissible.*

PROOF. By Theorem 4.1.1 inadmissibility is equivalent to the divergence of

$$\begin{aligned} I &= \int_M^\infty 1/r^{n-1} \exp\left[\tfrac{1}{2}\left[\int_0^{M^2}(h(y)-1)\, dy + \int_{M^2}^{r^2}(h(y)-1)\, dy\right]\right] dr \\ &= \int_M^\infty 1/Ar^{n-1} \exp\left[\tfrac{1}{2}\int_{M^2}^{r^2}[h(y)-1]\, dy\right] dr \\ &= 1/A \int_M^\infty \left[\exp\left[\tfrac{1}{2}\int_{M^2}^{r^2}(1-h(y))\, dy\right]/r^{n-1}\right] dr. \end{aligned} \tag{6.1.1}$$

We show that the integral diverges when the assumptions of (a) holds and converges when (b) holds, thus proving admissibility and inadmissibility respectively.

CASE a. $h(y) \leqq 1-(n-2)/y$ if $y > M$. In this case

$$1-h(y) \geqq (n-2)/y \tag{6.1.2}$$ and

$$\tfrac{1}{2}\int_{M^2}^{r^2}(1-h(y))\,dy \geqq \tfrac{1}{2}\int_{M^2}^{r^2}(n-2)/y\,dy = (n-2)/2\log[r^2/M^2]. \tag{6.1.3}$$

Hence

$$\begin{aligned} I &\geqq 1/A\int_M^\infty(\exp\{(n-2)/2\log[r^2/M^2]\})/(r^{n-1}) \\ &= 1/A\int_M^\infty(r/M)^{n-2}\,1/r^{n-1}\,dr \\ &= 1/AM^{n-2}\int_M^\infty dr/r = \infty. \end{aligned} \tag{6.1.4}$$

CASE b. $h(y) \geqq 1-b/y$ where $b < n-2$ for $y > M$. In this case

$$(1-h(y)) \leqq b/y \tag{6.1.5}$$ and

$$\tfrac{1}{2}\int_{M^2}^{r^2}(1-h(y))\,dy \leqq b/2\int_{M^2}^{r^2}dy/y = b/2\log r^2/M^2. \tag{6.1.6}$$

Hence

$$\begin{aligned} I &\leqq 1/A\int_M^\infty(\exp[b/2\log r^2/M^2])/(r^{n-1})\,dr \\ &= 1/A\int_M^\infty[r/M]^b\,1/r^{n-1}\,dr \\ &= 1/AM^b\int_M^\infty 1/r^{n-1-b}\,dr < \infty, \end{aligned} \tag{6.1.7}$$

since $n-1-b > (n-1)-(n-2) = 1$ and such integrals converge. This completes the proof of the theorem.

In the above, then, we have shown that $[1-(n-2)/(x'x)]x$ is in a sense the dividing line between admissible and inadmissible bounded risk generalized Bayes estimators. If an estimator shrinks this estimator sufficiently and it is bounded risk generalized Bayes, then it is admissible. If it expands it, and is of bounded risk, it is inadmissible. The estimator $\delta(x) = [1-(n-2)/(x'x)]x$ is precisely the estimator that James and Stein [7] showed was the "best" in the class of all estimators of the form $\delta(x) = [1-a/x'x]x$. We have noted before that for $a > 0$ all such estimators are not generalized Bayes, and hence not admissible themselves.

6.2. *A characterization of proper Bayes estimators.* We now give a characterization of those spherically symmetric generalized Bayes estimators which are proper Bayes. We require first a multivariate generalization of Lemma 5.3.1.

LEMMA 6.2.1. *An estimator $\delta(x)$ which is generalized Bayes with respect to the prior $F(\theta)$, is proper Bayes if and only if* $\int_{E^n}[\int_{E^n}\exp[-\frac{1}{2}||x-\theta||^2]\,dF(\theta)]\,dx_1\,dx_2\cdots dx_n < \infty$.

PROOF. The proof is essentially the same as given for Lemma 5.3.1. We now obtain the desired characterization.

THEOREM 6.2.1. *A generalized Bayes estimator $\delta(x)$ of the form $\delta(x) = h(||x||^2)x$ is proper Bayes if and only if* $\int_{E^n}\exp[\frac{1}{2}\int_0^{||x||^2}(h(y)-1)\,dy]\,dx_1\cdots dx_n < \infty$.

PROOF. This is an immediate consequence of Theorem 4.2.1 and Lemma 6.2.1.

We now apply the result to obtain a dividing line between proper and improper Bayes estimators. We first require a lemma to help in computations. Its proof follows by a change to polar coordinates.

LEMMA 6.2.2. *Let $L(r)$ be a continuous function on E'. Then*

$$\int_{\{\sum_1^n x_i^2 < R^2\}} L(\textstyle\sum_1^n x_i^2)^{\frac{1}{2}}\, dx_1 \cdots dx_n = nK \int_0^R L(r) r^{n-1}\, dr,$$

where K is the volume of the sphere $\sum_1^n x_i^2 \leqq 1$.

THEOREM 6.2.2. *Let $\delta(x) = h(\|x\|)^2 x$ be a generalized Bayes estimator. Then the following are true:*

(a) *If $h(y) \leqq 1-\varepsilon/y$ for $y > M$ where $\varepsilon > n$, the dimension of the mean vector being estimated, then $\delta(x)$ is proper Bayes and hence admissible.*
(b) *If $h(y) \geqq 1-n/y$ for $y > M$ then $\delta(x)$ is not proper Bayes.*

PROOF. If $\delta(x)$ is to be proper Bayes it must follow from Theorem 6.2.1 that

$$\int_{E^n} \exp\left[\tfrac{1}{2}\int_0^{\|x\|^2} (h(y)-1)\, dy\right] dx_1 \cdots dx_n < \infty$$

or using the lemma that

$$\int_0^\infty r^{n-1} \exp\left[\tfrac{1}{2}\int_0^{r^2} (h(y)-1)\, dy\right] dr < \infty.$$

The calculations required to show convergence in case (a) and divergence in case (b) are similar to those of Theorem 6.1.1 and are omitted.

We therefore have that the estimator $\delta(x) = [1-n/x'x]x$ is the dividing line between proper Bayes and improper Bayes estimators. Generalized Bayes estimators which "shrink" this estimator sufficiently for large $\|x\|$ are proper Bayes, and hence admissible, and those which "expand" this estimator sufficiently are not proper Bayes. As a corollary we have

COROLLARY 6.2.1. *Let $\delta(x) = h(\|x\|^2)x$ be a generalized Bayes estimator. Then*

(a) *If* $0 \leqq \limsup_{\|x\|\to\infty} h(\|x\|^2) < 1$, *then $\delta(x)$ is proper Bayes and hence admissible.*

(b) *If* $\liminf_{\|x\|\to\infty} h(\|x\|^2) > 1$ *then $\delta(x)$ is inadmissible if $\delta(x)$ is of bounded risk.*

We note that actually the hypothesis of bounded risk is not necessary in claiming inadmissibility of an estimator that does not satisfy the hypothesis of Brown's Theorem 2.1.3.

6.3. *A generalization of a result due to L. Brown.* We extend the result of L. Brown, Theorem 2.1.3, to estimators of the form $\delta(x) = \theta_0 + h(\|x-\theta_0\|^2)(x-\theta_0)$. We first require a lemma.

LEMMA 6.3.1. *An estimator of the mean vector of the multivariate normal distribution $\delta(x)$ being admissible and/or generalized Bayes with respect to the prior $F(\theta)$ implies that the estimator $\delta_a^*(x) = a+\delta(x-a)$, $\infty < a < \infty$, is also admissible and/or is generalized Bayes with respect to the prior $F(\theta+a)$.*

PROOF. The proof is essentially the same as for Lemma 5.1.1. Now we prove

THEOREM 6.3.1. *Let $\delta(x)$ be a bounded risk generalized Bayes estimator with respect to a prior $F(\theta)$ such that $\int_{E^n} \exp[-\frac{1}{2}||x-\theta||^2]\,dF(\theta) = g(||x-\theta_0||^2)$. Then $\delta(x)$ is admissible if and only if $\int^\infty 1/r^{n-1} g(r^2)\,dr = \infty$.*

PROOF. Consider the estimator $\delta^*(x) = \delta(x+\theta_0)-\theta_0$. By the lemma $\delta^*(x)$ is generalized Bayes with respect to the prior $F(\theta-\theta_0)$, and the two estimators are both either admissible or inadmissible, and it is also easy to see that both are of bounded risk. Hence it is sufficient to show $\delta^*(x)$ is admissible. But $\delta^*(x)$ is generalized Bayes with respect to $F(\theta-\theta_0)$. We have that

$$
\begin{aligned}
\int_{E^n} \exp[-\tfrac{1}{2}||x-\theta||^2]\,dF(\theta+\theta_0) &= \int_{E^n} \exp[-\tfrac{1}{2}||x-(\theta-\theta_0)||^2]\,dF(\theta) \\
&= \int_{E^n} \exp[-\tfrac{1}{2}||(x-\theta_0)-\theta||^2]\,dF(\theta) \qquad (6.3.1)\\
&= g(||(x-\theta_0)-\theta_0||^2) \\
&= g(||x||^2).
\end{aligned}
$$

But Theorem 2.1.3 implies that such an estimator $\delta^*(x)$ of bounded risk is admissible if and only if $\int^\infty 1/r^{n-1} g(r^2)\,dr = \infty$. This completes the proof of the theorem.

We have also the related result,

THEOREM 6.3.1. *A necessary and sufficient condition for a bounded risk, generalized Bayes estimator $\delta(x) = h(||x-\theta_0||^2)(x-\theta_0)+\theta_0$ to be admissible is that*

$$\int^\infty 1/r^{n-1} \exp[\tfrac{1}{2}[\textstyle\int_0^{r^2} (h(y)-1)\,dy]]\,dr = \infty.$$

PROOF. If $\delta(x)$ is generalized Bayes bounded risk then so is $\delta^*(x) = h(||x||^2)x = \delta(x+\theta_0)-\theta_0$. And $\delta^*(x)$ is admissible if and only if $\delta(x)$ is. But by Theorem 4.4.1 a necessary and sufficient condition for admissibility of $\delta^*(x)$ is that $\int^\infty 1/r^{n-1} \exp[\frac{1}{2}[\int_0^{r^2}(h(y)-1)\,dy]]\,dr = \infty$. This completes the proof of the theorem.

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REFERENCES

[1] APOSTOL, T. M. (1957). *Mathematical Analysis*. Addison-Wesley, Reading.

[2] BOCHNER, S. and CHANDRASEKHARAN, K. (1949). *Fourier Transforms*. Princeton Univ. Press.

[3] BROWN, L. (1967). Admissible procedures in simple estimation and testing problems. Invited paper IMS meeting, Washington, D.C.

[4] DUNFORD, H. and SCHWARZ, J. T. (1958). *Linear Operators, Part* 1: *General Theory*. Interscience, New York.

[5] FERGUSON, T. S. (1967). *Mathematical Statistics*. Academic Press, New York.

[6] HIRSCHMANN, I. I. and WIDDER, D. V. (1955). *The Convolution Transform*. Princeton Univ. Press.

[7] JAMES, W. and STEIN, CHARLES (1961). Estimation with quadratic loss. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 361–379.

[8] KARLIN, S. and RUBIN, H. (1956). The theory of decision procedures for distributions with monotone likelihood ratio. *Ann. Math. Statist.* **27** 272–299.

[9] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.
[10] LUKACS, E. (1962). *Characteristic Functions.* Charles Griffin and Company, Ltd., London.
[11] SACKS, J. (1963). Generalized Bayes solutions in estimation problems. *Ann. Math. Statist.* **34** 751–768.
[12] SCLOVE, STANLEY L. and BARANCHIK, A. J. (1966). A family of minimax estimators of the mean of a multivariate normal distribution, with an application to regression on principal components. Unpublished.
[13] STEIN, CHARLES (1955). Inadmissibility of the usual estimator for the mean vector of a multivariate normal distribution. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **1** 197–206.
[14] STEIN, CHARLES (1959). The admissibility of Pitman's estimator for a single location parameter. *Ann. Math. Statist.* **30** 970–979.
[15] STEIN, CHARLES (1962). Confidence sets for the mean of a multivariate normal distribution. *J. Roy. Statist. Soc.* Ser. B **24** 265–296.
[16] STEIN, CHARLES (1965). Approximation of improper prior measures by probability measures, *Bayes, Bernoulli, Laplace.* Springer-Verlag, Berlin. 217–240.
[17] STEIN, CHARLES (1966). An approach to recovery of inter-block information in balanced incomplete block designs, *Research Papers in Statistics.* Wiley, New York. 351–366.
[18] STRAWDERMAN, WILLIAM E. (1969a). Generalized Bayes estimators and admissibility of estimators of the mean vector of a multivariate normal distribution with quadratic loss. Ph.D. dissertation, Rutgers, the State University.
[19] STRAWDERMAN, WILLIAM E. (1969b). On linear combinations of generalized Bayes estimators of the mean of a normal distribution. Submitted *J. Amer. Statist. Assoc.*
[20] THOMPSON, JAMES R. (1968). Some shrinkage techniques for estimating the mean. *J. Amer. Statist. Assoc.* **63** 113–122.
[21] TUKEY, JOHN W. (1960). Where do we go from here? *J. Amer. Statist. Assoc.* **55** 80–93.
[22] WIDDER, D. V. (1941). *The Laplace Transform.* Princeton Univ. Press.