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Admissible Majorants for Model Subspaces of H^2, Part II: Fast Winding of the Generating Inner Function

Victor Havin and Javad Mashreghi

Abstract. This paper is a continuation of [6]. We consider the model subspaces $K_\Theta = H^2 \ominus \Theta H^2$ of the Hardy space H^2 generated by an inner function Θ in the upper half plane. Our main object is the class of admissible majorants for K_Θ, denoted by Adm Θ and consisting of all functions ω defined on $\mathbb{R}$ such that there exists an $f \neq 0$, $f \in K_\Theta$ satisfying $|f(x)| \leq \omega(x)$ almost everywhere on $\mathbb{R}$. Firstly, using some simple Hilbert transform techniques, we obtain a general multiplier theorem applicable to any K_Θ generated by a meromorphic inner function. In contrast with [6], we consider the generating functions Θ such that the unit vector $\Theta(x)$ winds up fast as x grows from $-\infty$ to ∞. In particular, we consider $\Theta = B$ where B is a Blaschke product with "horizontal" zeros, *i.e.*, almost uniformly distributed in a strip parallel to and separated from $\mathbb{R}$. It is shown, among other things, that for any such B, any even ω decreasing on $(0, \infty)$ with a finite logarithmic integral is in Adm B (unlike the "vertical" case treated in [6]), thus generalizing (with a new proof) a classical result related to Adm $\exp(i\sigma z)$, $\sigma > 0$. Some oscillating ω's in Adm B are also described. Our theme is related to the Beurling-Malliavin multiplier theorem devoted to Adm $\exp(i\sigma z)$, $\sigma > 0$, and to de Branges' space $\mathcal{H}(E)$.

1 Introduction

1.1 Some Background

This paper is a continuation of [6]. As in [6], we consider the *model subspace* K_Θ of the Hardy space $H^2(\mathbb{C}_+)$ generated by a function Θ which is inner in the upper half plane $\mathbb{C}_+$:

$$K_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+).$$

(Subsection 1.1 of [6] contains a list of papers devoted to model subspaces.)

We call a measurable non-negative function $\omega\colon \mathbb{R} \mapsto [0, \infty)$ an *admissible majorant* for K_Θ, and we write $\omega \in \operatorname{Adm} \Theta$, if there exists a non-zero function $f \in K_\Theta$ satisfying

$$|f(x)| \leq \omega(x) \tag{1.1}$$

almost everywhere on $\mathbb{R}$ ($f(x) = \lim_{\varepsilon \downarrow 0} f(x + i\varepsilon)$; this limit exists almost everywhere on $\mathbb{R}$, and the function $x \mapsto f(x)$ is in $L^2(\mathbb{R})$ [8, p. 114]). The subspace of $L^2(\mathbb{R})$

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formed by all boundary traces of elements of $H^2(\mathbb{C}_+)$ is isometric to $H^2(\mathbb{C}_+)$ and is denoted by $H^2(\mathbb{R})$ [3, pp. 190–191].

In the present paper we consider only *meromorphic* inner functions Θ (*i.e.* Θ is the restriction to $\mathbb{C}_+$ of a meromorphic function with poles in the lower half plane $\mathbb{C}_- = \{\Im z < 0\}$ or, what is the same, $\Theta(z) = e^{i\sigma z}B(z)$ where $\sigma \geq 0$ and B is a Blaschke product in $\mathbb{C}_+$ whose zeros tend to infinity). For any meromorphic inner function Θ there exists a real continuous and increasing function φ on $\mathbb{R}$ such that

$$\Theta(x) = e^{i\varphi(x)}, \quad x \in \mathbb{R}.$$

This function is unique up to an additive constant $2\pi k$, $k \in \mathbb{Z}$. We call it a continuous argument of Θ and denote it by $\arg\Theta$.

In [6] $\operatorname{Adm}\Theta$ was mainly studied for slowly winding unit vectors $\Theta(x)$, or, to be more precise, for Θ's with a slowly growing $\arg\Theta(x)$. The generating inner functions Θ considered in [6] satisfied the estimate

$$(\arg\Theta)'(x) = o(1), \quad |x| \to \infty.$$

Note that in the classical Beurling-Malliavin case of $\Theta(z) = e^{i\sigma z}$, $\sigma > 0$, we have $\arg\Theta(x) = \sigma x$, a *linear* increasing function.

In this paper we turn to Θ's whose argument grows linearly or faster. These assumptions create a completely new situation: the description of $\operatorname{Adm}\Theta$ differs essentially from what we had for slowly winding generating functions Θ. Not only our results, but the technique as well is quite different from [6].

In [6] we were able to find the so called *minimal* elements of $\operatorname{Adm}\Theta$; they had explicit expressions and were *unique* (in a sense); see Theorems 1.3, 1.4 and 1.5 in [6]. A new feature of the results of the present paper (compared with [6]) is the *absence* of minimal elements in $\operatorname{Adm}\Theta$.

E.g., in the Beurling-Malliavin case, $\Theta(z) = e^{i\sigma z}$, $\sigma > 0$, and $K_\Theta = e^{i\sigma z/2}\,\mathrm{PW}_{\sigma/2}$, where $\mathrm{PW}_{\sigma/2}$ is the Paley-Wiener space of entire functions of degree not exceeding $\sigma/2$ and square summable on $\mathbb{R}$, any nonconstant element of K_Θ is an entire function of exponential type at most σ and thus has a zero by the Hadamard factorization theorem [10, p. 16], since otherwise it is a pure exponential (up to a constant factor) in $K_\Theta \subset L^2(dt)$ and thus identically zero. Repeating the argument from the proof of Theorem 3.6 of [6] we conclude that $\omega(x)/(1+|x|)$ is in $\operatorname{Adm} e^{i\sigma z}$ whenever $\omega \in \operatorname{Adm} e^{i\sigma z}$ is positive and continuous.

Another distinction of the present paper from [6] is the role played by the logarithmic integral

$$\mathcal{L}(\omega) = \int_{-\infty}^{\infty} \frac{\Omega^+(x)}{1+x^2}\,dx, \tag{1.2}$$

where

$$\Omega(x) = -\log\omega(x). \tag{1.3}$$

It is obvious that its convergence is *necessary* for the inclusion $\omega \in \operatorname{Adm}\Theta$.

As shown in [6], this necessary admissibility condition can be quantitatively very far from being sufficient. *E.g.*, if $\Theta = B$, the Blaschke product with zeros in^2, $n = 1, 2, \dots$, then any $\omega(x)$ of the form $e^{-|x|^\alpha}$ with $\alpha \in (1/2, 1)$ is *not* admissible, although $\mathcal{L}(\omega) < \infty$ (see Sections 3.7 and 3.8 of [6] where a complete description of $\omega \in \operatorname{Adm} B$ is given for any Blaschke product with imaginary zeros).

For functions Θ considered in this paper the convergence of $\mathcal{L}(\omega)$ is not so dramatically far from the admissibility of ω, and the non-admissibility is rather of qualitative than quantitative character. It can be only caused by a kind of irregularity of ω. Indeed, we show that if ω is regular, then $\omega \in \operatorname{Adm}\Theta$ provided $\arg\Theta$ grows almost linearly, no matter how fast is the decrease of $\omega(x)$ if only $\mathcal{L}(\omega) < \infty$. In particular, this is applicable to $\Theta = B$ where B is a Blaschke product with zeros almost uniformly distributed in a strip $\{a < \Im z < b\}$, $a > 0$. The regularity means that ω is even and decreases on $[0, \infty)$, so that it does not oscillate at all (see Theorems 1.4, 1.5 and 1.6). The most interesting and difficult admissibility problems arise when $\arg\Theta$ grows almost linearly or faster, and $\omega(x)$ oscillates as $|x|$ tends to infinity. Such situations are considered in Sections 3, 4 and 5.

Our proofs (as in [6]) are constructive; whenever we claim that $\omega \in \operatorname{Adm}\Theta$ we can exhibit explicitly a non-zero $f \in K_\Theta$ whose modulus is majorized by ω on $\mathbb{R}$.

This article is organized as follows. In the short Section 2.1 we study Theorems 1.1 and 1.2 and reduce the former to the latter which is proved in Sections 2.2 and 2 containing some preliminary information on the Hilbert transform of bounded functions. Applications of Theorem 1.1 are collected in the last three sections.

1.2 A Survey of Our Results

In this paper the notion of a *mainly increasing function* plays an essential role. A function $f(x)$ defined on $\mathbb{R}$ is called *mainly increasing* if $f(d_n) = 2\pi n$, $n \in \mathbb{Z}$, for a strictly increasing sequence $\{d_n\}_{n\in\mathbb{Z}}$ such that $lim_{n\to-\infty} d_n = -\infty$, $lim_{n\to\infty} d_n = \infty$, the distances $d_{n+1} - d_n$ are bounded and f does not oscillate too wildly in the intervals (d_n, d_{n+1}) (*e.g.*, if f is Lipschitz or $f' \in C^1(\mathbb{R})$ and f' is uniformly continuous on $\mathbb{R}$; see the details in Section 2.1).

If $\mathcal{L}(\omega) < \infty$, then

$$\tilde{\Omega}(x) = \frac{1}{\pi} \int_{\mathbb{R}} \Big(\frac{1}{x-t} + \frac{t}{1+t^2} \Big) \Omega(t)\, dt,$$

the Hilbert transform of Ω, exists almost everywhere on $\mathbb{R}$. In what follows we assume that $\mathcal{L}(\omega) < \infty$ is fulfilled.

Theorem 1.1 **(Multiplier theorem)** *If* $\arg\Theta(x) + 2\tilde{\Omega}(x)$ *is mainly increasing, then* $\omega \in \operatorname{Adm}\Theta$.

This theorem is a consequence of the following technical result which is interesting in its own right.

Theorem 1.2 *If* $f(x)$ *is mainly increasing, then*

$$f(x) = 2\widetilde{\log \mathfrak{m}}(x) + 2\pi \mathfrak{K}(x)$$

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almost everywhere on $\mathbb{R}$, where $\mathfrak{K}$ is a non-decreasing step function with integer values and $\mathfrak{m} \geq 0$ with $\mathfrak{m} \in L^\infty(dt) \cap L^2(dt)$ and

$$\log \mathfrak{m} \in L^1\Big(\frac{dt}{1+t^2}\Big).$$

An application of Theorem 1.1 is given in Section 4.10 where the Blaschke products B with zeros in a horizontal strip $\{0 < c < \Im z < C\}$ are considered. The zeros are supposed to be almost uniformly distributed in the strip.

Definition 1.3 A sequence $\{z_k\}_{k\in\mathbb{Z}}$ in the strip $\{0 < c < \Im z < C\}$ has an almost uniform distribution if there exist numbers $L, K > 0$ such that for any $a \in \mathbb{R}$ the rectangle $[a, a+L] \times [c, C]$ contains at least one and no more than K zeros.

Theorem 1.4 *Let B be a Blaschke product with almost uniformly distributed zeros in a horizontal strip. If $\mathcal{L}(\omega) < \infty$, $\tilde{\Omega}$ is Lipschitz and $\|\tilde{\Omega}'\|_\infty < \delta$, ($\delta$ depends on c, C, L and K), then $\omega \in \operatorname{Adm} B$.*

The geometric conditions imposed on the distribution of the zeros of B entail the almost linear growth of $\arg B$, *i.e.*,

$$0 < c_1 < (\arg B)'(x) < C_1$$

for all $x \in \mathbb{R}$. A more general version of Theorem 1.4 is stated in Section 3.4.

The classical Beurling-Malliavin case (*i.e.* $\Theta(z) = e^{i\sigma z}$, $\sigma > 0$) is discussed in many books and papers. Theorem 1.1 applied to this case yields the following result.

Theorem 1.5 *If $\mathcal{L}(\omega) < \infty$, $\tilde{\Omega}$ is Lipschitz and $\|\tilde{\Omega}'\|_\infty < \sigma/2$, then $\omega \in \operatorname{Adm} e^{i\sigma z}$.*

Beurling and Malliavin proved that $\omega \in \bigcap_{\sigma>0} \operatorname{Adm} e^{i\sigma z}$ provided $\mathcal{L}(\omega) < \infty$ and Ω (not $\tilde{\Omega}$) is Lipschitz, no matter how big $\|\Omega'\|_\infty$ is. Another application of Theorem 1.1 is the following result on the *nice* ω's.

Theorem 1.6 *Let B be a Blaschke product with almost uniformly distributed zeros in a horizontal strip. If ω is even and decreasing on $[0,\infty)$ with $\mathcal{L}(\omega) < \infty$, then $\omega \in \operatorname{Adm} B$.*

In this case the convergence of $\mathcal{L}(\omega)$ is sufficient for ω to be admissible for K_B. Our method yields a new proof for the admissibility of nice ω's for $K_{e^{i\sigma z}}$ as well. Three other proofs of this result (for the particular case of $\Theta = e^{i\sigma z}$) are available in [5, p. 276], [9, p. 97] and [10, p. 159]. See also further remarks on [5, p. 393].

Two further applications of Theorem 1.5 are the following results which refer to some irregular (oscillating) majorants ω.

Theorem 1.7 *Suppose $\mathcal{L}(\omega) < \infty$. Denote by h_Ω the modulus of continuity of $\Omega = \log 1/\omega$ and suppose*

$$\int_0^1 \frac{h_\Omega(s)}{s}\,ds + \int_1^\infty \frac{h_\Omega(s)}{s^2}\,ds < \infty. \tag{1.4}$$

Then $\omega \in \operatorname{Adm} e^{i\sigma z}$ for any $\sigma > 0$.

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Theorem 1.8 *If ω satisfies the conditions of Theorem 1.7, then $\omega \in \operatorname{Adm} B$ for any Blaschke product B with zeros almost uniformly distributed in a horizontal strip.*

The condition (1.4) is called the *Zygmund condition* and is well known in the theories of approximation and singular integrals [14, p. 121], [2]; (1.4) if fulfilled if Ω satisfies the Lipschitz condition of order $\alpha \in (0, 1)$, *i.e.*

$$h_\Omega(t) \le Ct^\alpha, \quad t > 0. \tag{1.5}$$

Thus we get the following corollary.

Corollary 1.9 *If $\mathcal{L}(\omega) < \infty$ and Ω satisfies (1.5), then the admissibility conditions of Theorems 1.7 and 1.8 hold.*

In fact $\omega \in \bigcap_{\sigma>0} \operatorname{Adm} e^{i\sigma z}$ whenever $\mathcal{L}(\omega) < \infty$ and $\Omega \in \operatorname{Lip} 1$ (*i.e.* $h_\Omega(t) \le Ct$, $t > 0$). This is a famous result of Beurling and Malliavin implied by their multiplier theorem [1]. Several proofs are known now. For the present state of this topic see books [5], [9], [10] and [7]. The Lip 1 case is much more difficult than the Lip α case, $0 < \alpha < 1$. The authors believe, however, that this case can be also deduced from our Theorem 1.5.

In Section 5.2 we consider the Blaschke products B_α, $1/2 < \alpha < 1$, with horizontal zeros $\{|k|^\alpha \operatorname{sgn} k + i\}_{k\in\mathbb{Z}}$ (The convergence condition for Blaschke products forces $\alpha > 1/2$; $\alpha = 1$ is a particular case of Theorem 1.4, and $\alpha > 1$ is studied in [6]; $\arg B$ grows slowly in that case and a unique minimal majorant for K_B can be constructed explicitly). For $1/2 < \alpha < 1$, $\arg B_\alpha$ grows much faster than the almost linearly growing arguments due to case $\alpha \ge 1$, and that's why the admissibility conditions for K_{B_α} are milder.

Theorem 1.10 *Suppose $\int_{\mathbb{R}} |\Omega(x)|/(1+x^2)\,dx < \infty$, $\tilde{\Omega} \in C^1(\mathbb{R})$ and that*

$$-c_\alpha < \liminf_{|x|\to\infty} \frac{\tilde{\Omega}'(x)}{|x|^{\frac{1}{\alpha}-1}} \le \limsup_{|x|\to\infty} \frac{\tilde{\Omega}'(x)}{|x|^{\frac{1}{\alpha}-1}} < C_\alpha,$$

where $c_\alpha = \pi/\alpha$ and C_α is a positive constant. Moreover, suppose that

$$\lambda_t(t^{1-\frac{1}{\alpha}}) \le K,$$

where K is a positive constant and λ_t is the modulus of continuity of $\tilde{\Omega}'$ on $\mathbb{R} \setminus (-t, t)$. Then $\omega \in \operatorname{Adm} B_\alpha$.

We believe conditions imposed on $\tilde{\Omega}$ in Theorem 1.10 can be expressed in terms of Ω (maybe with some losses). Another open question is the sharpness of the conditions. The authors hope to return to these problems.

2 The Relation Between $\arg\Theta$ and the Hilbert Transform

2.1 Mainly Increasing Functions

We need this notion to state our main result, Theorem 1.1. A function $f\colon \mathbb{R} \mapsto \mathbb{R}$ is called *mainly increasing* if

1. f is absolutely continuous.

2. There is a strictly increasing sequence $\{d_n\}_{n\in\mathbb{Z}}$ with $\lim_{n\to-\infty} d_n = -\infty$, $\lim_{n\to\infty} d_n = \infty$ such that, for each n, $f(d_n) = 2\pi n$. Moreover, there is a constant L such that

$$\ell_n = \frac{d_n - d_{n-1}}{2} \le L$$

for each $n \in \mathbb{Z}$.

3. There is a constant C such that

$$\frac{1}{2\ell_n} \int_{d_{n-1}}^{d_n} |f'(x) - f'(t)|\, dt \le C$$

for all $x \in (d_{n-1}, d_n)$ and for all $n \in \mathbb{Z}$.

Note that for $f \in C^1(\mathbb{R})$, property (3) is implied by

$$\operatorname{osc}_{I_n} f' \le \text{Const}, \quad n \in \mathbb{Z}. \tag{2.1}$$

Here $\operatorname{osc}_E \varphi$ denotes the oscillation of the function φ on a set E, *i.e.*,

$$\sup\{\varphi(x_2) - \varphi(x_1) : x_1, x_2 \in E\} = \sup_E \varphi - \inf_E \varphi.$$

Property (2.1) is, in its turn, implied by the following simpler property:

$$\operatorname{osc}_I f' \le \text{Const} \tag{2.2}$$

for any interval of length at most one. Finally (2.2) is implied by the uniform continuity of f'. Any Lipschitz function f satisfies (3) with Const $= 2\|f'\|_\infty$.

Any f satisfying (2) increases by 2π as its argument jumps from d_{n-1} to d_n whereas f does not vary too intensely between d_{n-1} and d_n due to (3). This phenomenon justifies our choice of the term *mainly increasing*.

2.2 Hilbert Transform of a Bounded Function

The definition and some simple theorems on the Hilbert transform in a form adjusted to our purposes is in [6]. The facts collected in this paragraph are stated in a slightly more general form than we really need.

Lemma 2.1 *Let $u \in L^1_{\text{loc}}(dt)$, and suppose that $U(x) = \int_0^x u(t)\, dt$ belongs to $L^1\big(dt/(1+t^2)\big)$ and $U(t) = o(t)$ as $|t| \to \infty$. Then*

$$\lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b \frac{t}{1+t^2} u(t)\, dt$$

exists, and is finite.

Proof By integration by parts. For each pair a, b, $-\infty < a < b < \infty$,

$$\int_a^b \frac{t}{1+t^2} u(t)\,dt = \frac{t}{1+t^2} U(t)\Big|_a^b - \int_a^b \frac{1-t^2}{(1+t^2)^2} U(t)\,dt$$
$$= \frac{b}{1+b^2} U(b) - \frac{a}{1+a^2} U(a) - \int_a^b \frac{1-t^2}{(1+t^2)^2} U(t)\,dt.$$

Let $a \to -\infty$ and $b \to \infty$. Since $U(t) = o(t)$ as $|t| \to \infty$, we have

$$\lim_{|t|\to\infty} \frac{t}{1+t^2} U(t) = 0.$$

On the other hand

$$\left| \frac{1-t^2}{(1+t^2)^2} U(t) \right| \leq \frac{|U(t)|}{1+t^2}.$$

Thus the right hand integral is absolutely convergent. Hence

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b \frac{t}{1+t^2} u(t)\,dt = \int_{-\infty}^{\infty} \frac{t^2-1}{(1+t^2)^2} U(t)\,dt,$$

which is finite. ■

Theorem 2.2 *Suppose $u \in L^1\big(dt/(1+t^2)\big)$ satisfies the conditions of Lemma 2.1. Then for almost all $x \in \mathbb{R}$*

$$\tilde{u}(x) = \lim_{\substack{a\to-\infty \\ b\to\infty \\ \varepsilon\to 0}} \int_{I_{ab\varepsilon}} \frac{u(t)}{x-t}\,dt,$$

where $I_{ab\varepsilon} = (a, x-\varepsilon) \cup (x+\varepsilon, b)$. The order in which the passages to the limit are taken is immaterial.

Proof We have, for almost all $x \in \mathbb{R}$,

$$\tilde{u}(x) = \lim_{\varepsilon\to 0} \int_{|x-t|>\varepsilon} \Big(\frac{1}{x-t} + \frac{t}{1+t^2} \Big) u(t)\,dt$$
$$= \lim_{\substack{a\to-\infty \\ b\to\infty \\ \varepsilon\to 0}} \int_{I_{ab\varepsilon}} \Big(\frac{1}{x-t} + \frac{t}{1+t^2} \Big) u(t)\,dt.$$

Thus, by Lemma 2.1,

$$\tilde{u}(x) = \lim_{\substack{a\to-\infty \\ b\to\infty \\ \varepsilon\to 0}} \int_{I_{ab\varepsilon}} \frac{u(t)}{x-t}\,dt + c,$$

where c is a constant. Absorb c into $\tilde{u}$. ■

We need the following version of Theorem 2.2.

Theorem 2.3 *Suppose $u \in L^\infty(dt)$, $U(x) = \int_0^x u(t)\,dt \in L^1\big(dt/(1+t^2)\big)$, and*

$$U(a_n) = U(b_n) = 0, \quad n = 1, 2, \dots,$$

where $\lim_{n\to\infty} a_n = -\infty$ and $\lim_{n\to\infty} b_n = \infty$. Then

$$\tilde{u}(x) = \lim_{\substack{\varepsilon\to 0\\ n\to\infty}} \frac{1}{\pi} \int_{\{|t-x|\ge\varepsilon\}\cap(a_n,b_n)} \frac{u(t)}{x-t}\,dt.$$

The proof is an obvious modification of the proofs of Lemma 2.1 and Theorem 2.2.

2.3 Systems of Short Intervals

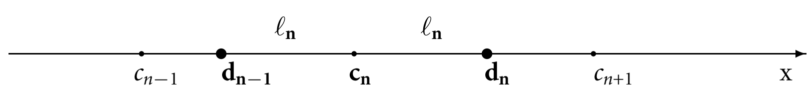
Suppose that $\{d_n\}_{n\in\mathbb{Z}}$ is a strictly increasing sequence of real numbers with

$$\lim_{n\to-\infty} d_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} d_n = \infty.$$

Put $\ell_n = \frac{d_n - d_{n-1}}{2}$ and $c_n = \frac{d_n + d_{n-1}}{2}$. Suppose, furthermore, there is a number $L > 0$ such that

$$\ell_n \le L \tag{2.3}$$

for all $n \in \mathbb{Z}$.



In the following, we study functions which are uniformly Lipschitz on each interval (d_{n-1}, d_n); they have finite left-hand and right-hand limits at every d_n.

We need the following assumption on d_n.

$$\sum_{d_{n-1}d_n>0} \frac{\ell_n^2}{d_{n-1}d_n} < \infty \tag{2.4}$$

It clearly holds whenever

$$\sum_{d_{n-1}>0} \frac{\ell_n^2}{d_{n-1}^2} + \sum_{d_n<0} \frac{\ell_n^2}{d_n^2} < \infty \tag{2.5}$$

The following fact will be used later.

Lemma 2.4 *Suppose $\{\ell_n\}_{n\in\mathbb{Z}}$ is a bounded sequence of positive numbers, i.e. $\ell_n \le L$ for all $n \in \mathbb{Z}$. Then (2.4) and (2.5) are fulfilled.*

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Proof We estimate the first sum in (2.5). The estimate of the second is quite similar. First, note that if $d_{n-1} > 0$, then $d_n = d_{n-1} + 2\ell_n \le d_{n-1} + 2L \le 2d_{n-1}$, if n is sufficiently large, say $n > N$, since $\lim_{n\to\infty} d_{n-1} = \infty$. Thus, for $n > N$,

$$\frac{\ell_n^2}{d_{n-1}^2} \le \frac{4L\ell_n}{d_n^2}.$$

Now

$$\frac{2\ell_n}{d_n^2} \le \int_{d_{n-1}}^{d_n} \frac{dx}{x^2},$$

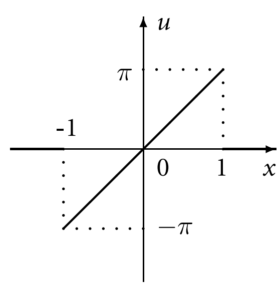
whence

$$\sum_{n>N} \frac{\ell_n^2}{d_{n-1}^2} \le 2L \int_{d_{N-1}}^{\infty} \frac{dx}{x^2} \le \frac{2L}{d_{N-1}} < \infty.$$ ■

In fact we only need (2.3), although (2.4) and (2.5) are satisfied by some (but not all) *unbounded* sequences $\{\ell_n\}_{n\in\mathbb{Z}}$. We mention (2.4) and (2.5) for two reasons. First, (2.5) means the sequence $\{I_n\}_{n\in\mathbb{Z}}$ is a so-called *system of short intervals*, a notion occurring in some important theorems of Fourier analysis [5, p. 399]. Second, (2.4) and (2.5) may be useful for possible generalizations of Theorem 1.10.

2.4 Sawtooth Functions

Let $u(x) = \pi x \kappa_{(-1,1)}(x)$, where κ is the characteristic function of $[-1, 1]$.



Then by a direct calculation, we have

$$\tilde{u}(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(t)}{x-t}\, dt = -2 + x \log\left|\frac{x+1}{x-1}\right|. \tag{2.6}$$

The function $x \mapsto x \log|\frac{x+1}{x-1}|$ is even; it is positive in $(0, 1)$ and hence in $(-1, 1)$. Thus, for $|x| < 1$,

$$\tilde{u}(x) \ge -2. \tag{2.7}$$

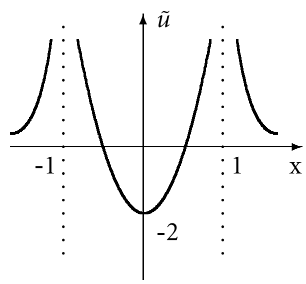
Now, for $|x| > 1$,

$$\tilde{u}(x) > 0, \tag{2.8}$$

since $\tilde{u}(x) = \frac{2}{3x^2} + \frac{2}{5x^4} + \frac{2}{7x^6} + \cdots$ there. This expansion also yields

$$\tilde{u}(x) \le \frac{2}{x^2 - 1} \tag{2.9}$$

for $|x| > 1$.

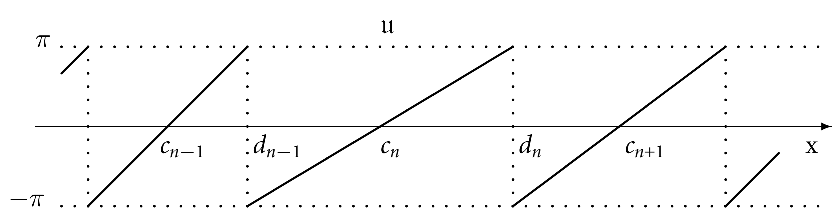


Let $U(x) = 0$ for $x \le -1$, $U(x) = \int_{-1}^{x} u(t)\,dt$ for $x \ge -1$, so that $U(x) = \frac{\pi(x^2-1)}{2}\kappa_{(-1,1)}(x)$ for all $x \in \mathbb{R}$; U is a primitive of u with

$$-\frac{\pi}{2} \le U(x) \le 0 \tag{2.10}$$

for all $x \in \mathbb{R}$. Put

$$\mathfrak{u}(x) = \sum_{n=-\infty}^{\infty} u\Big(\frac{x - c_n}{\ell_n}\Big).$$

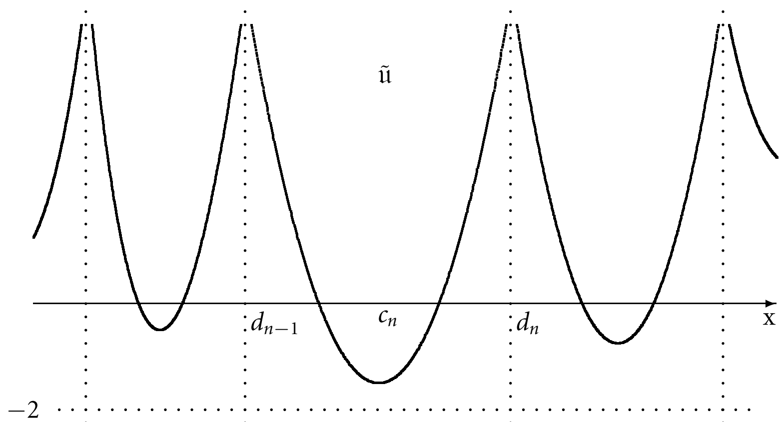


For each x different from the d_k at most one of the terms $u(\frac{x-c_n}{\ell_n})$ is non-zero. The sum $\mathfrak{u}$ is thus a function with a graph shaped like a saw blade which grows linearly from $-\pi$ to π on each (d_{n-1}, d_n) and jumps downward by 2π at each d_n. The slope of each line is at least π/L if $\ell_n \le L$ for all $n \in \mathbb{Z}$. The sawtooth function u plays a crucial role in the proof of Theorem 1.2. We show that any mainly increasing function f can be sufficiently well approximated modulo 2π by a sawtooth function u. On the other hand $\tilde{u}$ behaves sufficiently well to enable us to obtain the desired multiplier m.

Lemma 2.5 *Suppose $\{\ell_n\}_{n\in\mathbb{Z}}$ satisfies (2.4). Then for each $x \in \mathbb{R} \setminus \{d_n\}$, $\tilde{\mathfrak{u}}(x)$ exists and*

$$\tilde{\mathfrak{u}}(x) = \sum_{n=-\infty}^{\infty} \tilde{u}\Big(\frac{x - c_n}{\ell_n}\Big).$$

Furthermore, $\tilde{\mathfrak{u}}$ *is bounded from below.*



Proof For each $x \in \mathbb{R}$ at most one term in the series $\sum_{n=-\infty}^{\infty} \ell_n U(\frac{x-c_n}{\ell_n})$ is nonzero. The series thus represents a continuous function $\mathfrak{U}(x)$ which is zero at each d_n. Furthermore, for every $x \in (d_{k-1}, d_k)$,

$$\frac{d\mathfrak{U}(x)}{dx} = \frac{d}{dx}\sum_{n=-\infty}^{\infty} \ell_n U\Big(\frac{x-c_n}{\ell_n}\Big) = U'\Big(\frac{x-c_k}{\ell_k}\Big) = u\Big(\frac{x-c_k}{\ell_k}\Big) = \mathfrak{u}(x),$$

so that $\mathfrak{U}$ is a primitive of $\mathfrak{u}$. We also have

$$\int_{d_{n-1}}^{d_n} \frac{|\mathfrak{U}(t)|}{t^2}\,dt = \ell_n \int_{d_{n-1}}^{d_n} \frac{\big|U\big((t-c_n)/2\big)\big|}{t^2}\,dt = \frac{\pi\ell_n^2}{2}\int_{-1}^{1}\frac{d\tau}{(c_n+\ell_n\tau)^2} = \frac{\pi\ell_n^2}{d_{n-1}d_n},$$

if $0 \notin [d_{n-1}, d_n]$, whence, by (2.4), $\mathfrak{U} \in L^1(dt/(1+t^2))$. Therefore, by Theorem 2.2, for almost all $x \in \mathbb{R}$,

$$\tilde{\mathfrak{u}}(x) = \lim_{\varepsilon\to 0, N\to\infty} \frac{1}{\pi}\int_{I_{N\varepsilon}} \frac{\mathfrak{u}(t)}{x-t}\,dt,$$

where $I_{N\varepsilon} = (d_{-N}, x-\varepsilon) \cup (x+\varepsilon, d_N)$, and the passages to the limit can be taken in any order. Hence,

$$\begin{aligned}\tilde{\mathfrak{u}}(x) &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \frac{1}{\pi}\int_{I_{N\varepsilon}} \frac{\mathfrak{u}(t)}{x-t}\,dt \\ &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \frac{1}{\pi}\int_{I_{N\varepsilon}} \frac{\sum_{n=-N}^{N} u\big((t-c_n)/\ell_n\big)}{x-t}\,dt.\end{aligned}$$

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If $d_{m-1} < x < d_m$, the integral on the right is equal, for large N and $\varepsilon > 0$ small enough, to

$$\tilde{u}(x) = \frac{1}{\pi} \sum_{|n| \leq N, n \neq m} \int_{d_{n-1}}^{d_n} \frac{u\big((t-c_n)/\ell_n\big)}{x-t}\, dt + \frac{1}{\pi} \int_{\substack{d_{m-1}<t<d_m \\ |t-x|>\varepsilon}} \frac{u\big((t-c_m)/\ell_m\big)}{x-t}\, dt.$$

For $n \neq m$, the substitution $\tau = \frac{t-c_n}{\ell_n}$ converts the corresponding term of the summation to

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u(\tau)}{(\frac{x-c_n}{\ell_n}) - \tau}\, d\tau = \tilde{u}\Big(\frac{x-c_n}{\ell_n}\Big),$$

and the remaining integral is similarly seen to equal

$$\frac{1}{\pi} \int_{\substack{|\tau|<1 \\ |\tau - \frac{x-c_m}{\ell_m}| > \frac{\varepsilon}{\ell_m}}} \frac{u(\tau)}{(\frac{x-c_m}{\ell_m}) - \tau}\, d\tau.$$

Since $u(\tau) = 0$ for $|\tau| > 1$, this tends to $\tilde{u}(\frac{x-c_m}{\ell_m})$ when $\varepsilon \to 0$. We therefore have

$$\lim_{\varepsilon \to 0} \int_{I_{N\varepsilon}} \frac{\mathrm{u}(t)}{x-t}\, dt = \sum_{n=-N}^{N} \tilde{u}\Big(\frac{x-c_n}{\ell_n}\Big),$$

and finally

$$\tilde{\mathrm{u}}(x) = \sum_{n=-\infty}^{\infty} \tilde{u}\Big(\frac{x-c_n}{\ell_n}\Big).$$

On $(c_m - \ell_m, c_m + \ell_m)$, $\tilde{u}(\frac{x-c_m}{\ell_m}) \geq -2$. All other terms in the summation are positive. Thus $\tilde{\mathrm{u}}$ is also bounded below. ∎

Fix an integer m and let us look at the local behavior of $\tilde{\mathrm{u}}(x)$ near d_m. We have $d_m = c_m + \ell_m = c_{m+1} - \ell_{m+1}$. Here, for $x \in (c_m, c_{m+1})$, we have, by (2.6),

$$\begin{aligned} \tilde{u}\Big(\frac{x-c_m}{\ell_m}\Big) &= -2 + \Big(\frac{x-d_m+\ell_m}{\ell_m}\Big) \log\Big|\frac{x-d_m+2\ell_n}{x-d_m}\Big| \\ &= -\log|x-d_m| + \varphi_m(x), \end{aligned}$$

and again

$$\begin{aligned} \tilde{u}\Big(\frac{x-c_{m+1}}{\ell_{m+1}}\Big) &= -2 + \Big(\frac{x-d_m-\ell_{m+1}}{\ell_{m+1}}\Big) \log\Big|\frac{x-d_m}{x-d_m-2\ell_{m+1}}\Big| \\ &= -\log|x-d_m| + \psi_m(x), \end{aligned}$$

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where φ_m and ψ_m are analytic at d_m. Therefore, for each $x \in (c_m, c_{m+1})$,

$$\tilde{\mathfrak{u}}(x) = -2\log|x - d_n| + \varphi_m(x) + \psi_m(x) + \rho_m(x),$$

where

$$\rho_m(x) = \Big(\sum_{n=-\infty}^{m-1} + \sum_{n=m+2}^{\infty}\Big)\,\tilde{u}\Big(\frac{x - c_{m+1}}{\ell_{m+1}}\Big) = \tilde{\mathfrak{v}}(x). \tag{2.11}$$

Note that $\mathfrak{v}(x) = \mathfrak{u}(x)$ off (d_{m-1}, d_{m+1}) and $\mathfrak{v}(x) = 0$ in (d_{m-1}, d_{m+1}), and thus, ρ_m is analytic at d_m. Hence $\tilde{\mathfrak{u}}$ is a continuous function on $\mathbb{R} \setminus \{d_n\}_{n\in\mathbb{Z}}$ and, in a neighborhood of each d_n, it is equal to $-2\log|x - d_n| + \varphi_n(x)$, where φ_n is a continuous function in that neighborhood.

From Lemma 2.5 and the preceding computation, we get, finally

Lemma 2.6 *Let $\mathfrak{u}$ be as in Lemma 2.5. Then for each $m \geq 0$, the ratio*

$$\frac{e^{-\tilde{\mathfrak{u}}(x)}}{\prod_{k=1}^{m}(x - d_{j_k})^2}$$

formed using arbitrary distinct d_{j_k} from among the d_n, is a bounded continuous function.

2.5 Distorted Sawtooth Functions

Let $\mathfrak{g}$ be a real function such that for each d_n,

$$\lim_{x\to d_n^+} \mathfrak{g}(x) = -\pi$$

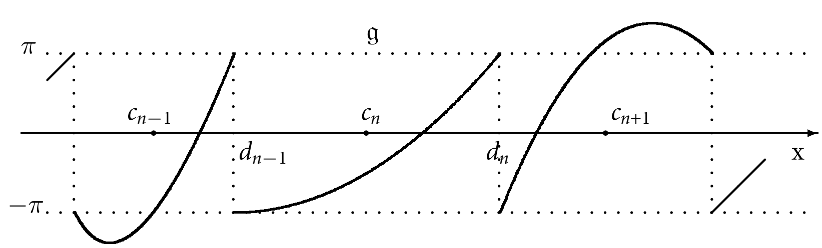
and

$$\lim_{x\to d_n^-} \mathfrak{g}(x) = \pi.$$

Moreover, suppose $\mathfrak{g}$ is absolutely continuous on every $I_n = (d_{n-1}, d_n)$, and for a $C > 0$

$$\frac{1}{d_n - d_{n-1}} \int_{d_{n-1}}^{d_n} |\mathfrak{g}'(t) - \mathfrak{g}'(x)|\, dt \leq C \tag{2.12}$$

for all $x \in I_n$ and for all $n \in \mathbb{Z}$.



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If $g|_{I_n}$ is in $C^1(I_n)$, then (2.12) is implied by

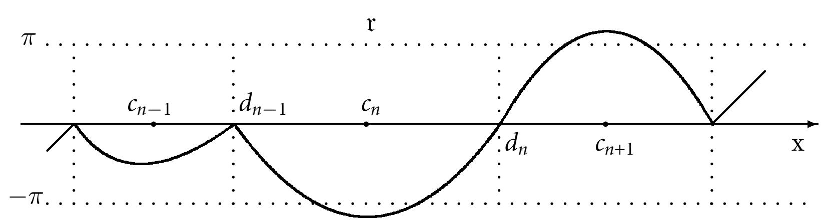
$$\sup_{n\in\mathbb{Z}} \operatorname{osc}_{I_n} \mathfrak{g}' \le C.$$

The estimate (2.12) also holds if $\mathfrak{g}$ satisfies a uniform Lipschitz condition on any I_n, *i.e.*, for each $x_1, x_2 \in I_n$, we have

$$|\mathfrak{g}(x_2) - \mathfrak{g}(x_1)| \le \operatorname{Lip}_{\mathfrak{g}} |x_2 - x_1|, \tag{2.13}$$

where the constant $\operatorname{Lip}_{\mathfrak{g}}$ does not depend on n (then $C = \operatorname{Lip}_{\mathfrak{g}}$).

Let $\mathfrak{u}$ be the linear sawtooth function constructed in Section 2.4 with discontinuities at the d_n. Put $\mathfrak{r} = \mathfrak{g} - \mathfrak{u}$ on $\mathbb{R} \setminus \{d_n : n \in \mathbb{Z}\}$ and $\mathfrak{r}(d_n) = 0$, $n \in \mathbb{Z}$. Then $\mathfrak{r}$ is continuous on $\mathbb{R}$.



Lemma 2.7 *Suppose the sequence $\{\ell_n\}_{n\in\mathbb{Z}}$ is bounded, and $\mathfrak{g}$ satisfies (2.13). Then $\mathfrak{r}$ is a bounded Lipschitz function on $\mathbb{R}$.*

Proof Clearly $\mathfrak{r}$ is absolutely continuous. We only need to prove that $\mathfrak{r}' \in L^\infty(dt)$ since it entails $\mathfrak{r} \in L^\infty(dt)$:

$$|\mathfrak{r}(x)| = \Big| \int_{d_{n-1}}^{x} \mathfrak{r}'(t)\,dt \Big| \le 2\ell_n \|\mathfrak{r}'\|_\infty \le 2L\|\mathfrak{r}'\|_\infty, \tag{2.14}$$

if $x \in I_n$. Now, for $x \in I_n$, we have

$$\mathfrak{r}(x) = \mathfrak{g}(x) - \mathfrak{g}(d_{n-1}^+) - \frac{\mathfrak{g}(d_n^-) - \mathfrak{g}(d_{n-1}^+)}{d_n - d_{n-1}}(x - d_{n-1}).$$

Hence, for almost all $x \in I_n$, we get

$$|\mathfrak{r}'(x)| = \Big| \mathfrak{g}'(x) - \frac{1}{2\ell_n} \int_{d_{n-1}}^{d_n} \mathfrak{g}'(t)\,dt \Big| \le \frac{1}{2\ell_n} \int_{d_{n-1}}^{d_n} |\mathfrak{g}'(x) - \mathfrak{g}'(t)|\,dt \le C. \tag{2.15}$$ ∎

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Lemma 2.8 *Suppose $\mathfrak{r}$ is a bounded Lipschitz function on $\mathbb{R}$, i.e., for each $x_1, x_2 \in \mathbb{R}$,*

$$|\mathfrak{r}(x_2) - \mathfrak{r}(x_1)| \le \mathrm{Lip}_{\mathfrak{r}} |x_2 - x_1|.$$

Then $\tilde{\mathfrak{r}}(x)$ is continuous and

$$\tilde{\mathfrak{r}}(x) = O(\log |x|),$$

as $|x| \to \infty$.

Proof Since $\mathfrak{r}$ is Lipschitz, $\tilde{\mathfrak{r}}$ is at least continuous. Indeed, its modulus of continuity behaves like $O(\Delta \log 1/\Delta)$ as $\Delta \to 0^+$ [13, p. 146].

Without loss of generality assume that $\mathfrak{r}(0) = 0$. Now

$$\pi \tilde{\mathfrak{r}}(x) = \fint_{x-1}^{x+1} \frac{\mathfrak{r}(t)}{x-t}\,dt + \int_{x-1}^{x+1} \frac{\mathfrak{r}(t)}{t}\,dt + \int_{|t-x|>1} \frac{x\mathfrak{r}(t)}{t(x-t)}\,dt = \mathfrak{I} + \mathfrak{I} + \mathfrak{I}.$$

Clearly,

$$|\mathfrak{I}| = \left| \int_{x-1}^{x+1} \frac{\mathfrak{r}(t) - \mathfrak{r}(x)}{t-x}\,dt \right| \le 2\,\mathrm{Lip}_{\mathfrak{r}},$$

and

$$|\mathfrak{I}| \le \|\mathfrak{r}\|_\infty \log \left| \frac{x+1}{x-1} \right| = o(1),$$

as $|x| \to \infty$. From now on we assume $x > 0$. Then, breaking $\mathfrak{I}$ into four more integrals, we get

$$\begin{aligned} |\mathfrak{I}| &\le \|\mathfrak{r}\|_\infty \int_{-\infty}^{-1} \frac{x}{t(t-x)}\,dt + \mathrm{Lip}_{\mathfrak{r}}\, x \int_{-1}^{1} \frac{dt}{x-t} \\ &\qquad + \|\mathfrak{r}\|_\infty \int_{1}^{x-1} \frac{x}{t(x-t)}\,dt + \|\mathfrak{r}\|_\infty \int_{x+1}^{\infty} \frac{x}{t(x-t)}\,dt \\ &= 2\|\mathfrak{r}\|_\infty \log(x^2-1) + \mathrm{Lip}_{\mathfrak{r}}\, x \log\left(\frac{x+1}{x-1}\right) \\ &= 4\|\mathfrak{r}\|_\infty \log x + O(1), \end{aligned}$$

as $x \to \infty$. The estimate for $x \to -\infty$ is now clear. Therefore,

$$|\tilde{\mathfrak{r}}(x)| \le \frac{4\|\mathfrak{r}\|_\infty}{\pi} \log x + O(1), \tag{2.16}$$

as $|x| \to \infty$. ■

Remark Our estimate of $\tilde{\mathfrak{r}}$ in Lemma 2.8 follows also from the inclusion $\tilde{\mathfrak{r}} \in \mathrm{BMO}$.

If $\mathfrak{r} = \mathfrak{g} - \mathfrak{u}$, where $\mathfrak{g}$ satisfies (2.13), then by (2.14), (2.15) and (2.16),

$$|\tilde{\mathfrak{r}}(x)| \le \frac{4L\,\mathrm{Lip}_{\mathfrak{g}}}{\pi} \log x + O(1),$$

as $|x| \to \infty$.

Corollary 2.9 *Under the conditions of Lemma 2.7, the ratio*

$$\frac{e^{-\tilde{\mathfrak{r}}(x)}}{(1+|x|)^{\alpha_{\mathfrak{g}}}},$$

where $\alpha_{\mathfrak{g}} = 4L\,\mathrm{Lip}_{\mathfrak{g}}/\pi$, is a bounded continuous function on $\mathbb{R}$.

Corollary 2.10 *Let $\mathfrak{g}$ be as in Lemma 2.7. Let $m \in \mathbb{N}$ such that $2m \geq \alpha_{\mathfrak{g}} + 2$, where $\alpha_{\mathfrak{g}} = 4L\,\mathrm{Lip}_{\mathfrak{g}}/\pi$. Then, for any choice of different d_{j_k}, $1 \leq k \leq m$, the ratio*

$$\frac{\exp\bigl(-\tilde{\mathfrak{g}}(x)\bigr)}{\prod_{k=1}^{m}(x-d_{j_k})^2}$$

is a bounded continuous and integrable function on $\mathbb{R}$.

Proof Since $\tilde{\mathfrak{g}} = \tilde{\mathfrak{r}} + \tilde{\mathfrak{u}}$, we have

$$\frac{e^{-\tilde{\mathfrak{g}}(x)}}{\prod_{k=1}^{m}(x-d_{j_k})^2} = \frac{e^{-\tilde{\mathfrak{r}}(x)}e^{-\tilde{\mathfrak{u}}(x)}}{\prod_{k=1}^{m}(x-d_{j_k})^2}.$$

By Lemma 2.6, the function

$$e^{-\tilde{\mathfrak{r}}(x)} \cdot \frac{e^{-\tilde{\mathfrak{u}}(x)}}{\prod_{k=1}^{m}(x-d_{j_k})^2}$$

is bounded and continuous on any bounded interval. For large values of $|x|$, by Lemma 2.6 and Corollary 2.9,

$$e^{-\tilde{\mathfrak{u}}(x)} \cdot \frac{e^{-\tilde{\mathfrak{r}}(x)}}{|x|^{\alpha_{\mathfrak{g}}}}$$

is bounded. Thus we get

$$\left|\frac{e^{-\tilde{\mathfrak{g}}(x)}}{\prod_{k=1}^{m}(x-d_{j_k})^2}\right| \leq \frac{\text{Const}}{x^2}$$

for large values of $|x|$. Therefore this ratio represents a bounded continuous and integrable function on $\mathbb{R}$. ■

2.6 Representation of a Mainly Increasing Function

We are now ready to complete the proof of Theorem 1.2.

Theorem 2.11 *Let $\mathfrak{f}$ be a mainly increasing function. Then there are a non-decreasing step function $\mathfrak{K}(x)$ with values all equal to integral multiples of 2π and also a function $\mathfrak{m} \geq 0$ with $\mathfrak{m} \in L^\infty(dt) \cap L^2(dt)$ and $\log \mathfrak{m} \in L^1(\frac{dt}{1+t^2})$, such that*

$$\mathfrak{f} = 2\widetilde{\log \mathfrak{m}} + \mathfrak{K}.$$

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Proof Put $\mathfrak{L}(x) = (2n+1)\pi$ for $x \in (d_n, d_{n+1})$. Then $\mathfrak{g} = \mathfrak{f} - \mathfrak{L}$ is a sawtooth function like the one examined in Section 2.4; it satisfies the conditions of Lemma 2.7. Choose m according to the recipe in the Corollary 2.10, and any m different points d_{j_k} from among the d_n. Put

$$\mathfrak{m}(x) = \frac{e^{-\frac{1}{2}\tilde{\mathfrak{g}}(x)}}{\prod_{k=1}^{m} |x - d_{j_k}|}.$$

Then

$$2 \log \mathfrak{m}(x) = -\tilde{\mathfrak{g}}(x) - 2 \sum_{k=1}^{m} \log |x - d_{j_k}|.$$

By Corollary 2.10, $\mathfrak{m} \in L^\infty(dt) \cap L^2(dt)$. Since $\mathfrak{g} = \mathfrak{u} + \mathfrak{r} \in L^\infty(dt)$,

$$\log \mathfrak{m}(x) \in L^1\Big(\frac{dt}{1+t^2}\Big).$$

If we substitute $u(t) = \log |t|$ into (4.1) of [6] we obtain a Hilbert transform equal to $-\frac{\pi}{2} \operatorname{sgn} t$. Here, however, it is convenient to subtract $\frac{\pi}{2}$ from the latter and we take

$$-\frac{\pi}{2} \operatorname{sgn} t - \frac{\pi}{2} = \begin{cases} 0, & t < 0 \\ -\pi, & t > 0 \end{cases}$$

as our Hilbert transform of $\log |t|$. Referring to Theorem 4.1 of [6], we thus get

$$\begin{aligned} 2\widetilde{\log \mathfrak{m}}(x) &= -\tilde{\tilde{\mathfrak{g}}}(x) - 2 \sum_{k=1}^{m} \Big(-\frac{\pi}{2} - \frac{\pi}{2} \operatorname{sgn}(x - d_{j_k})\Big) \\ &= \mathfrak{g}(x) + \pi \sum_{k=1}^{m} \Big(1 + \operatorname{sgn}(x - d_{j_k})\Big). \end{aligned}$$

Hence, for each $x \in \mathbb{R} \setminus \{d_n\}_{n\in\mathbb{Z}}$, we finally have

$$\mathfrak{f}(x) = 2\widetilde{\log \mathfrak{m}}(x) - \pi \sum_{k=1}^{m} \big(1 + \operatorname{sgn}(x - d_{j_k})\big) + \mathfrak{L}(x).$$

The function $\mathfrak{K}(x) = -\pi \sum_{k=1}^{m} \big(1 + \operatorname{sgn}(x - d_{j_k})\big) + \mathfrak{L}(x)$ is an increasing step function with values all equal to integral multiples of 2π. ■

The proof of Theorem 1.2, and thus Theorem 1.1, is now complete.

3 A Multiplier Theorem and Some of its Applications

One can look at the question of characterizing admissible majorants of K_Θ from a different point of view. If we consider the reciprocals $W(x) = \frac{1}{\omega(x)}$, we are interested in describing the W for which nonzero functions $f \in K_\Theta$ with

$$|W(x) f(x)| \le 1 \quad \text{for } x \in \mathbb{R} \tag{3.1}$$

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exist. One can think of such a W as a weight that can be *multiplied down* by the "multiplier" f and thus a *multiplier theorem* is any result describing conditions on W, or equivalently on ω, from which (3.1) follows for appropriate functions f. In those circumstances, we say that W *admits multipliers* or that ω is an *admissible majorant*.

3.1 An Admissibility Criterion

Here we find some admissible majorants. Using tools developed above, we deduce a multiplier theorem applying to all model subspaces K_Θ of H^2 formed from inner functions Θ having smooth arguments on the real line. Application of that result to meromorphic Blaschke products $\Theta(z) = B(z)$ will give concrete results involving the zeros of B.

Theorem 3.1 *If $\Omega(x) = -\log\omega(x)$ is bounded below and $\arg\Theta + 2\tilde{\Omega}$ is mainly increasing, then $\omega \in \operatorname{Adm}\Theta$.*

Proof By Theorem 2.11, there is an $\mathfrak{m} \in L^\infty(dt) \cap L^2(dt)$, $\mathfrak{m} \geq 0$, such that

$$\log\mathfrak{m} \in L^1\Big(\frac{dt}{1+t^2}\Big)$$

and

$$\arg\Theta + 2\tilde{\Omega} = 2\widetilde{\log\mathfrak{m}} + \mathfrak{K},$$

where $\mathfrak{K}$ is a step function with values all equal to integral multiples of 2π. Since $\omega = e^{-\Omega}$ is now bounded, we also have $\mathfrak{m}\omega \in L^2(dt)$. The inclusion $\omega \in \operatorname{Adm}\Theta$ follows now from Theorem 4.7 of [6] which was deduced from Dyakonov's description of the moduli of functions in K_Θ [4]. ■

Corollary 3.2 *Let $\Omega(x)$ be bounded below and $\varphi = \arg\Theta + 2\tilde{\Omega}$ be a uniformly Lipschitz function on $\mathbb{R}$, i.e.,*

$$|\varphi(x_1) - \varphi(x_2)| \leq C|x_1 - x_2| \quad \text{for } x_1, x_2 \in \mathbb{R},$$

where C is a positive constant, and suppose that

$$\lim_{x\to-\infty}\varphi(x) = -\infty, \quad \lim_{x\to\infty}\varphi(x) = \infty.$$

Suppose, moreover, that for x_1 and x_2 both either sufficiently large or sufficiently negative,

$$|\varphi(x_1) - \varphi(x_2)| \geq c|x_1 - x_2|,$$

where c is a positive constant. Then $\omega \in \operatorname{Adm}\Theta$.

Proof For each $n \in \mathbb{Z}$, let

$$d_n = \sup\{x \in \mathbb{R}; \varphi(x) = 2n\pi\}.$$

The point $(d_n, 2n\pi)$ is the last point where the horizontal line $y = 2n\pi$ meets the graph of $y = \varphi(x)$. By the intermediate value theorem for continuous functions and the limit relations in the hypothesis, we have $d_n < d_{n+1}$ for each $n \in \mathbb{Z}$. Moreover, for large values of $|n|$, we have

$$d_{n+1} - d_n \leq \frac{2\pi}{c},$$

and therefore such inequalities (with different constants on the right) hold for *all* n. Hence φ is a mainly increasing Lipschitz function and thus, by Theorem 3.1, $\omega = \exp(-\Omega)$ is an admissible majorant. ■

3.2 The Classical Case $\Theta(x) = e^{i\sigma x}$

Let $\varphi(x) = \sigma x + 2\tilde{\Omega}(x)$. Suppose $\tilde{\Omega}$ is Lipschitz. Then

$$c|x_1 - x_2)| \leq |\varphi(x_1) - \varphi(x_2)| \leq C|x_1 - x_2|,$$

with $c = (\sigma - 2\|\tilde{\Omega}'\|_\infty)$ and $C = (\sigma + 2\|\tilde{\Omega}'\|_\infty)$.

Referring to Corollary 3.2, we have, with the preceding,

Corollary 3.3 *If $\tilde{\Omega}$ is Lipschitz with*

$$\|\tilde{\Omega}'\|_\infty < \frac{\sigma}{2},$$

then $\omega \in \operatorname{Adm} e^{i\sigma x}$.

3.3 A Blaschke Product With Zeros in a Horizontal Strip

Here we turn to $\Theta = B$, a meromorphic Blaschke product with zeros almost uniformly distributed in a horizontal strip. To apply Theorem 1.1 we need some information on $\arg B$, the continuous argument of B.

Let $\{z_k = x_k + i y_k\}_{k\in\mathbb{Z}}$ be a Blaschke sequence in the upper half plane such that

$$0 < a \leq y_k \leq b < \infty \quad \text{for } k \in \mathbb{Z}.$$

Since

$$\frac{a}{x_k^2 + b^2} \leq \frac{y_k}{|z_k|^2} \leq \frac{b}{x_k^2 + a^2} \quad \text{for } k \in \mathbb{Z},$$

a necessary and sufficient condition for the uniform convergence of B_K to B on compact sets disjoint from $\{\bar{z}_k; k \in \mathbb{Z}\}$ is that

$$\sum_{k=-\infty}^{\infty} \frac{1}{1 + x_k^2} < \infty. \tag{3.2}$$

Suppose that the x_k are indexed in increasing order: $x_k \le x_{k+1}$, $k \in \mathbb{Z}$. Let $n(t)$ denote the counting function of the sequence $\{x_k\}_{k\in\mathbb{Z}}$. Then we have

$$\int_{-\infty}^{\infty} \frac{dn(t)}{1+t^2} = \sum_{k=-\infty}^{\infty} \frac{1}{1+x_k^2} < \infty.$$

This identity implies that $n(t) = o(t^2)$ as $|t| \to \infty$. Furthermore, by using integration by parts, one can show that $n(t) = O(|t|^\alpha)$ for $|t| \to \infty$, where $0 < \alpha < 2$, is sufficient for the convergence of $\int_{-\infty}^{\infty} \frac{dn(t)}{1+t^2}$.

Theorem 3.4 *Let a and b, $0 < a < b$, be fixed. Let B be a meromorphic Blaschke product with zeros in the horizontal strip $0 < a \le \Im z \le b < \infty$. Then:*

(a) $\frac{d \arg B(x)}{dx}$ *is uniformly bounded from above on $\mathbb{R}$ if and only if there exist a real number $d > 0$ and an integer $N > 0$ such that any rectangle*

$$R_{x,d} = [x, x+d] \times [a, b] \quad \textit{with } x \in \mathbb{R},$$

contains at most N zeros of z_k.

(b) $\frac{d \arg B(x)}{dx}$ *is uniformly bounded away from zero on $\mathbb{R}$ if there exists a real number $d > 0$ such that any rectangle $R_{x,d}$ contains at least one zero z_k.*

(c) $\frac{d \arg B(x)}{dx}$ *is uniformly bounded from above on $\mathbb{R}$ and uniformly bounded away from zero on $\mathbb{R}$ if and only if the zeros z_k satisfy both conditions of* (a) *and* (b) *simultaneously.*

Proof (a) Suppose d and N exist. Then the segments $[\ell d, (\ell+1)d]$, $\ell \in \mathbb{Z}$, cover $\mathbb{R}$. For simplicity, put

$$R_\ell = R_{\ell d, d} \text{ and } R_\ell^* = R_{\ell-1} \cup R_\ell \cup R_{\ell+1} \quad \text{for } \ell \in \mathbb{Z}.$$

Then, if $x \in R_\ell$,

$$\sum_{z_k \in R_\ell^*} \frac{2y_k}{(x-x_k)^2 + y_k^2} \le \sum_{z_k \in R_\ell^*} \frac{2}{y_k} \le \frac{6N}{a}.$$

For $x \in R_\ell$, we also have

$$\sum_{z_k \notin R_\ell^*} \frac{2y_k}{(x-x_k)^2 + y_k^2} \le \sum_{z_k \notin R_\ell^*} \frac{2b}{(x-x_k)^2} \le \sum_{m=1}^{\infty} \frac{4bN}{m^2 d^2}.$$

Therefore,

$$\frac{d \arg B(x)}{dx} \le \frac{6N}{a} + \frac{4bN}{d^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \quad \text{for } x \in \mathbb{R}.$$

Now suppose that $\frac{d \arg B(x)}{dx}$ is bounded from above. Put $d = 1$, and let N_ℓ denote the number of zeros in R_ℓ. Then, for $x \in [\ell, \ell+1]$,

$$\frac{d \arg B(x)}{dx} \ge \sum_{z_k \in R_\ell} \frac{2a}{1+b^2} \ge N_\ell \frac{2a}{1+b^2}.$$

Therefore,

$$N_\ell \le \frac{1+b^2}{2a}\|(\arg B)'\|_\infty \quad \text{for } \ell \in \mathbb{Z}.$$

(b) If $x \in [\ell d, (\ell+1)d]$, with $\ell \in \mathbb{Z}$ and R_ℓ contains at least one zero, we have

$$\frac{d \arg B(x)}{dx} = \sum_k \frac{2y_k}{(x-x_k)^2 + y_k^2} \ge \frac{2a}{d^2+b^2}.$$

(c) Suppose $d \arg B(x)/dx$ is uniformly bounded from above and uniformly bounded away from zero. Then z_k have the property stated in (a). We only have to check the property stated in (b). Suppose for any $\Delta > 0$ there is a rectangle $R = R_{x,\Delta}$ free of z_k. Take a big integer M and put $\Delta = (2M+1)d$, where d is the number from (a). Let c be the center of $[x, x+\Delta]$. We have

$$(\arg B)'(c) = \sum_{z_k \notin R} \frac{2y_k}{(c-x_k)^2 + y_k^2} \le \sum_{m \ge M} \frac{2bN}{m^2\, d^2} = O(1/M). \qquad \blacksquare$$

The condition stated in part (b) is not necessary. As a counter example, consider the Blaschke product with k^2 repeated zeros at points $\pm k^2 + i$, $k \ge 1$. Then $\frac{d \arg B(x)}{dx}$ is an even function, and for $x \in [\ell^2 - \ell - 1, \ell^2 + \ell + 1]$ with $\ell \ge 2$,

$$\frac{d \arg B(x)}{dx} \ge \ell^2 \cdot \frac{2}{(x-\ell^2)^2 + 1} \ge \frac{2\ell^2}{\ell^2 + 2\ell + 2} \ge \frac{4}{5}.$$

Hence

$$\frac{d \arg B(x)}{dx} \ge \frac{4}{5} \quad \text{for } x \ge 1.$$

Moreover, for $0 \le x \le 1$,

$$\frac{d \arg B(x)}{dx} \ge \frac{2}{(x-1)^2 + 1} \ge 1.$$

Therefore, $\arg B$ is bounded away from zero, but the number d mentioned in part (b) does not exists.

Let B be a meromorphic Blaschke product with zeros in the horizontal strip $0 < a \le \Im z_k \le b < \infty$. We say that the zeros of B are *almost uniformly distributed* if there exist $d > 0$ and an integer $N > 0$ such that any rectangle $R_{x,d} = [x, x+d] \times [a, b]$, $x \in \mathbb{R}$, contains at most N zeros, and at least one zero, of B.

Corollary 3.5 *Let B be a meromorphic Blaschke product with zeros in the horizontal strip $0 < a \le \Im z_k \le b < \infty$. Then there are positive constants c and C such that for all $x_1, x_2 \in \mathbb{R}$*

$$c|x_2 - x_1| \le |\arg B(x_2) - \arg B(x_1)| \le C|x_2 - x_1|. \tag{3.3}$$

if and only if the zeros of B are almost uniformly distributed in that strip.

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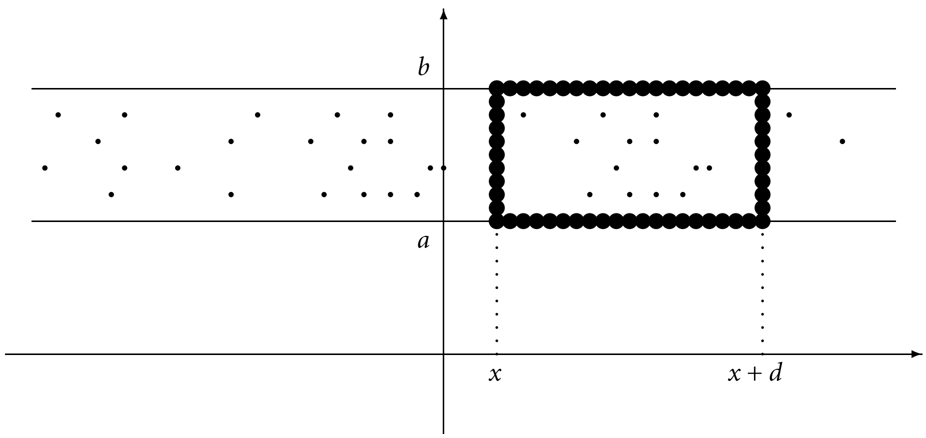

Now we are ready to prove our result mentioned as Theorem 1.4 in the Section 1.2.

Theorem 3.6 *Let B be a meromorphic Blaschke product whose zeros are almost uniformly distributed in a horizontal strip way from $\mathbb{R}$. Suppose that $\tilde{\Omega}$ is uniformly Lipschitz on $\mathbb{R}$, and that for*

$$L < \frac{1}{2}\inf_{x\in\mathbb{R}}\left|\frac{d\arg B(x)}{dx}\right|,$$

we have

$$|\tilde{\Omega}(x_2) - \tilde{\Omega}(x_1)| \leq L|x_2 - x_1|$$

for x_1 and x_2 both either sufficiently large or sufficiently negative. Then $\omega \in \operatorname{Adm} B$.

Proof By Corollary 3.5,

$$c = \inf_{x\in\mathbb{R}}\left|\frac{d\arg B(x)}{dx}\right| > 0$$

and (3.3) holds. Here we assumed that $L < c/2$. Hence, by (3.3), the function $\varphi(x) = \arg B(x) + 2\tilde{\Omega}(x)$ is uniformly Lipschitz on $\mathbb{R}$. Moreover,

$$|\varphi(x_2) - \varphi(x_1)| \geq (c - 2L)|x_2 - x_1|$$

for x_1 and x_2 both either sufficiently large or sufficiently negative. Thus $\omega(x)$ is an admissible majorant for K_B by Corollary 3.2. ∎

Remark Note that Theorem 3.6 is applicable to any Blaschke product satisfying (3.3).

3.4 A Slight Generalization of the Admissibility Criterion

The results of Sections 3.2 and 3.3 can be given a stronger form. Suppose $\tilde{\Omega}$ is in $C^1(\mathbb{R})$ and satisfies the following conditions:

(a) $\tilde{\Omega}'(x) \geq -\delta$ where $\delta < \sigma/2$ (in Section 3.2), or $\delta < c/2$ (in Section 3.3).

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(b) $\operatorname{osc}_I \tilde{\Omega}'$ is bounded uniformly with respect to intervals I of length at most one.

So we do not even need to assume $\tilde{\Omega}$ is Lipschitz; we only need the lower *Lipschitz estimate* $\tilde{\Omega}'(x) \geq -\delta$ plus a mild control of the oscillations of $\tilde{\Omega}$ on intervals of length at most one.

To see this, note that the functions $\varphi_1(x) = \sigma x + 2\tilde{\Omega}(x)$ and $\varphi_2(x) = \arg B(x) + 2\tilde{\Omega}(x)$ are mainly increasing if (a) and (b) hold; their derivatives are bounded away from zero, so that they tend monotonically to $\pm\infty$ as x tends to $\pm\infty$; (a) clearly implies the uniform boundedness of ℓ_n, and

$$\frac{1}{2\ell_n} \int_{d_{n-1}}^{d_n} |\varphi_j(t) - \varphi_j(x)|\, dt, \qquad j = 1, 2,$$

are uniformly bounded with respect to $n \in \mathbb{Z}$ and $x \in I_n$, because of (b) and the boundedness of ℓ_n.

4 Further Applications of Our Multiplier Theorem

4.1 The Relation Between Convolution and Hilbert Transform

We want now to modify the admissibility criteria of Corollary 3.3 and Theorem 3.4 so as to describe some majorants in $\bigcap_{\sigma>0} \operatorname{Adm} e^{i\sigma x}$ and $\cap \operatorname{Adm} B$ where the intersection is taken over *all* meromorphic Blaschke products B with zeros almost uniformly distributed in a horizontal strip. This will be done by simultaneous regularization of Ω and $\tilde{\Omega}$, *i.e.* by means of convolutions $\Omega * \Phi$, $\tilde{\Omega} * \Phi$ with mollifier Φ. The results we get will be stated first in terms of $\tilde{\Omega}$ in subsection 4.3, but then in subsections 4.4 and 4.5 we also obtain some admissibility theorems in terms of Ω itself.

Here we assume Ω to be non-negative continuous function on $\mathbb{R}$ satisfying, as always,

$$\mathcal{L}(\Omega) = \int_{-\infty}^{\infty} \frac{\Omega(t)}{1+t^2}\, dt < \infty. \tag{4.1}$$

These assumptions imply the *local summability* of $\tilde{\Omega}$. (In our final result $\tilde{\Omega}$ will be in fact *continuous*, so that the following simple proof could be omitted.) To prove $\tilde{\Omega} \in L^1_{\text{loc}}(dt)$ note that for any $x \in \mathbb{R}$

$$\Omega = \Omega_1^x + \Omega_2^x,$$

where $\Omega_1^x = \chi_{(x-1,x+1)}\Omega$, so that Ω_2^x is analytic in $(x-1, x+1)$ whereas is summable on any compact interval by the M. Riesz theorem [8, p. 88]. (In fact $\Omega_1^x + \Omega_2^x$ is the boundary trace of an $H^p(\mathbb{C}_+)$ function for any $p < \infty$.)

Fix now a positive $L > 0$ and consider a function $\Phi \in L^\infty(\mathbb{R})$ vanishing off $(-L, L)$. Then both convolutions $\Omega * \Phi$ and $\tilde{\Omega} * \Phi$ exists. Moreover,

$$\Omega * \Phi \in L^1\Big(\frac{dt}{1+t^2}\Big). \tag{4.2}$$

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It is enough to show that

$$\int_{3L}^{\infty} \frac{(\Omega * |\Phi|)(x)}{x^2}\, dx < \infty.$$

According to the definition of Φ and convolution we have

$$\begin{aligned}
\int_{3L}^{\infty} \frac{(\Omega * |\Phi|)(x)}{x^2}\, dx &= \int_{3L}^{\infty} \frac{dx}{x^2} \int_{x-L}^{x+L} \Omega(t)|\Phi(x-t)|\, dt \\
&\le \|\Phi\|_\infty \int_{3L}^{\infty} \frac{dx}{x^2} \int_{x-L}^{x+L} \Omega(t)\, dt \\
&\le \|\Phi\|_\infty \int_{2L}^{\infty} \Big(\int_{t-L}^{t+L} \frac{dx}{x^2} \Big) \Omega(t)\, dt \\
&= 2L\|\Phi\|_\infty \int_{2L}^{\infty} \frac{\Omega(t)}{(t-L)^2}\, dt < \infty.
\end{aligned}$$

In view of (4.2) the Poisson integral U of $\Omega * \Phi$ and $\tilde{\Omega} * \Phi$ are well defined (respectively in $\mathbb{C}_+$ and almost every where in $\mathbb{R}$).

Lemma 4.1 *Let Ω and Φ be as described in the preceding paragraphs. Then*

$$(\widetilde{\Omega * \Phi})(x) = (\tilde{\Omega} * \Phi)(x)$$

almost every where on $\mathbb{R}$.

Proof Denote by u the Poisson integral of Ω in $\mathbb{C}_+$, and let v be the harmonic conjugate of u, so that

$$\tilde{\Omega}(x) = \lim_{y \downarrow 0} v(x, y)$$

almost every where on $\mathbb{R}$. We also have

$$\lim_{y \downarrow 0} \int_I |\tilde{\Omega}(x) - v(x, y)|\, dx = 0 \tag{4.3}$$

for any compact interval $I \in \mathbb{R}$ (see the end of the proof for local summability of $\tilde{\Omega}$ at the beginning of this subsection).

Let V be the harmonic conjugate of U. By Fubini's theorem

$$\begin{aligned}
U(x, y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-s)^2 + y^2} \Big(\int_{-\infty}^{\infty} \Omega(s-t)\Phi(t)\, dt \Big)\, ds \\
&= \int_{-\infty}^{\infty} \Phi(t) u(x-t, y)\, dt
\end{aligned} \tag{4.4}$$

Using (4.4) a direct consideration of the Cauchy-Riemann system for U and V shows

$$V(x, y) = \int_{-L}^{L} \Phi(t) v(x-t, y)\, dt + \text{Const}, \quad x + iy \in \mathbb{C}_+. \tag{4.5}$$

The left side of (4.5) tends to $(\widetilde{\Omega * \Phi})(x)$ as $y \downarrow 0$, almost every where on $\mathbb{R}$, whereas by (4.3) its right side tends to $\tilde{\Omega} * \Phi$ in $L^1(I)$ on any compact interval I. ∎

4.2 Admissibility of Ω in Terms of Ω_1

Here we give a simple method to deduce the admissibility of a majorant ω from the admissibility of another majorant ω_1 obtained from ω by a regularization of Ω. But first we have to impose one more condition on Ω: *the oscillations of* Ω *on intervals of length one are uniformly bounded*, *i.e.*

$$c_\Omega = \sup\{|\Omega(x') - \Omega(x'')| : |x' - x''| \le 1\} < \infty. \tag{4.6}$$

Let $\varphi \in L^\infty(\mathbb{R})$ be a non-negative function vanishing off $(-1, 1)$ and such that $\int_{-1}^{1} \varphi = 1$. Fix a number $L > 0$ and put

$$\Phi(t) = \frac{1}{L}\varphi\Big(\frac{t}{L}\Big).$$

Then, for any $x \in \mathbb{R}$,

$$|\Omega(x) - (\Omega * \Phi)(x)| \le \int_{-L}^{L} |\Omega(x) - \Omega(x-t)|\Phi(t)\,dt \tag{4.7}$$

$$\le (L+1)c_\Omega = c(\Omega, L). \tag{4.8}$$

Put

$$\Omega_1 = \Omega * \Phi + c(\Omega, L), \quad \omega_1 = e^{-\Omega_1}. \tag{4.9}$$

Clearly, $\Omega_1 \in L^1\big(dt/(1+t^2)\big)$ (by (4.2) or simply because $\Omega - \Omega_1$ is bounded by (4.7), and $\Omega_1 \ge \Omega$). We arrive at the following conclusion.

Theorem 4.2 *Given an inner function* Θ *in* $\mathbb{C}_+$, *if* $\omega_1 \in \operatorname{Adm}\Theta$, *then* $\omega \in \operatorname{Adm}\Theta$ *(recall that* $\Omega = \log 1/\omega$*).*

Proof If $f \in K_\Theta$, $f \not\equiv 0$ and $|f| \le \omega_1$ almost every where on $\mathbb{R}$, then $e^{-c(\Omega,L)}|f| \le \omega$ almost every where on $\mathbb{R}$. ■

Remark This assertion shows that studying the admissibility of an ω satisfying (4.6) there is no loss of generality to assume $\Omega \in \mathcal{C}^\infty(\mathbb{R})$, since Ω_1 is in $\mathcal{C}^\infty(\mathbb{R})$ whenever Φ is. Note that $\tilde{\Omega} \in \mathcal{C}^\infty(\mathbb{R})$ if $\Omega \in \mathcal{C}^\infty(\mathbb{R})$.

4.3 Classical Case Revisited

Now we turn again to the classical case ($\Theta = e^{i\sigma z}$, $\sigma > 0$) and obtain two admissibility criteria applicable to *all* $\sigma > 0$ (see Theorems 4.3 and 4.4 below). As before we assume that $\Omega \ge 0$ is continuous and satisfies (4.1) and (4.6). We denote by h_F the continuity modulus of a function F defined on $\mathbb{R}$:

$$h_F(t) = \sup\{|F(x') - F(x'')| : |x' - x''| \le t\}, \quad t > 0. \tag{4.10}$$

We avoid the usual notation ω_F, since ω is for us a generic notation of a majorant. Note also that unlike in the traditional situations we are mainly interested in the behavior of $h_F(t)$ for *large* rather than small values of t. Thanks to (4.6), $h_\Omega(t) < \infty$ for all $t > 0$.

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Theorem 4.3 *If $h_{\tilde\Omega}(t) < \infty$ for all $t > 0$, and*

$$\lim_{t\to\infty} \frac{h_{\tilde\Omega}(t)}{t} = 0, \tag{4.11}$$

then $\omega \in \bigcap_{\sigma>0} \operatorname{Adm} e^{i\sigma z}$.

Proof Here we use the Steklov mollifier $\varphi = s = \frac{1}{2}\chi_{[-1,1]}$ and apply Theorem 4.2 with $\Phi(t) = \frac{1}{L}s(t/L)$ where L is large depending on σ.

For a fixed $\sigma > 0$, we show that

$$\omega_1 = e^{-\Omega_1} \in \operatorname{Adm} e^{i\sigma z} \tag{4.12}$$

where $\Omega_1 = \Omega * \Phi + \text{Const}$ (see (4.9)). By Lemma 4.1 and assuming Ω and Ω_1 are smooth (see the remark after Theorem 4.2, for any $x \in \mathbb{R}$, we have

$$\begin{aligned} |\Omega_1'(x)| &= \Big|\frac{d}{dx}(\tilde\Omega * \Phi)(x)\Big| = \Big|\frac{d}{dx}\Big(\frac{1}{2L}\int_{x-L}^{x+L}\tilde\Omega(t)\,dt\Big)\Big| \\ &= \frac{|\tilde\Omega(x+L) - \tilde\Omega(x-L)|}{2L} \le \frac{h_{\tilde\Omega}(2L)}{2L} \le \frac{\sigma}{2}, \end{aligned} \tag{4.13}$$

if L is large. By Corollary 3.3 we conclude that $\omega_1 \in \operatorname{Adm} e^{i\sigma z}$, and by Theorem 4.2, $\omega \in \operatorname{Adm} e^{i\sigma z}$. ■

Remark Our condition (4.11) is also sufficient for ω to be in Adm B for *any* meromorphic Blaschke product B with zeros almost uniformly distributed in a horizontal strip. The proof remains the same (reduction to Theorem 3.4 instead of Corollary 3.3).

We can immediately illustrate Theorem 4.3 and the preceding remarks on Blaschke products by an example where the admissibility condition is stated explicitly in terms of Ω.

Example Suppose $\Omega \in \operatorname{Lip}_\alpha$, $0 < \alpha < 1$, *i.e.*

$$|\Omega(x') - \Omega(x'')| \le \text{Const}\,|x' - x''|^\alpha, \quad x', x'' \in \mathbb{R}. \tag{4.14}$$

Then $\omega \in \bigcap_{\sigma>0} \operatorname{Adm} e^{i\sigma z}$. An analogous result is also applicable to the Blaschke products with zeros almost uniformly distributed in a horizontal strip.

Indeed, (4.14) implies (4.1) and (4.6). Then by a classical Privalov theorem (see [12] and [13, p. 144]) $\tilde\Omega \in \operatorname{Lip}_\alpha$ whence $h_{\tilde\Omega}(t) = O(t^\alpha)$ as $t \to \infty$, and (4.11) holds.

The Privalov theorem is not applicable if $\alpha = 1$ (it may happen that for an $\Omega \in \operatorname{Lip} 1$ satisfying (4.1) the function $\tilde\Omega(x+1) - \tilde\Omega(x)$ is unbounded). Thus the above argument cannot be applied to the Lip 1 case and does not lead to Beurling-Malliavin theorem in its full strength. However, this example will be generalized in an essential way below in subsection 4.5.

Remark In the proof of Theorem 4.3, we actually used a formally weaker assumption than (4.11), *i.e. for any $\sigma > 0$ there are $A, L > 0$ such that $|\tilde\Omega(a+2L) - \tilde\Omega(a)| < \sigma L$ for any $a \in \mathbb{R}\setminus[-A, A]$*. Indeed, putting $a = x - L$ and following (4.13) we get $|\Omega_1'(x)| < \sigma/2$ for $|x| > L + A$ which is sufficient for $\omega_1 \in \operatorname{Adm} e^{i\sigma z}$ (see Corollary 3.3).

4.4 Universal Admissibility

Our next theorem gives a BMO type condition for the universal (*i.e.* for all $\sigma > 0$) admissibility in the classical case and for all Blaschke product B with zeros almost uniformly distributed in a horizontal strip.

Denote by f_I the average of a function f over a bounded interval I:

$$f_I = \frac{1}{|I|}\int_I f,$$

and put

$$b(f, I) = \inf\{|f - c|_I : c \in \mathbb{R}\}$$

and

$$B(f, L) = \sup\{b(f, I) : |I| = L\}.$$

Theorem 4.4 *Let Ω be as in Theorem 4.3 and suppose that*

$$\lim_{L\to\infty} \frac{B(\tilde{\Omega}, L)}{L} = 0.$$

Then $\omega \in \bigcap_{\sigma>0} \operatorname{Adm} e^{i\sigma z}$ and $\omega \in \operatorname{Adm} B$ for any Blaschke product B with zeros almost uniformly distributed in a horizontal strip.

Proof We again use Lemma 4.1 and define Ω_1 and ω_1 as in (4.9). This time $\Phi(t) = \frac{1}{L}\varphi(t/L)$ where $\varphi \geq 0$ is in $\mathcal{C}^\infty(\mathbb{R})$, vanishes off $(-1, 1)$ and $\int_{\mathbb{R}} \varphi = 1$. We prove $\|\tilde{\Omega}_1'\|_\infty < \sigma/2$. In the case of a Blaschke product B, $\sigma/2$ is to be replaced by the respective constant

$$\frac{1}{2}\inf_{x\in\mathbb{R}}\left|\frac{d\arg B(x)}{dx}\right|.$$

We have

$$\tilde{\Omega}_1(x) = \int_{\mathbb{R}} \tilde{\Omega}(t)\Phi(x-t)\,dt,$$

hence

$$\begin{aligned}(\tilde{\Omega}_1)'(x) &= \int_{\mathbb{R}} \tilde{\Omega}(t)\Phi'(x-t)\,dt = \int_{-L}^{L} \tilde{\Omega}(x-t)\Phi'(x-t)\,dt \\ &= \int_{-L}^{L} \big(\tilde{\Omega}(x-t) - c\big)\,\Phi'(x-t)\,dt\end{aligned}$$

where $c \in \mathbb{R}$ is arbitrary. Therefore,

$$|(\tilde{\Omega}_1)'(x)| \leq \frac{1}{L^2}\|\varphi'\|_\infty \int_{-L}^{L} |\tilde{\Omega}(x-t) - c|\,dt, \qquad I + (x-L, x+L).$$

Choosing an appropriate c we get

$$|(\tilde{\Omega}_1)'(x)| \leq \frac{3}{L}\|\varphi'\|_\infty b(\tilde{\Omega}, I) \leq \frac{3}{L}\|\varphi'\|_\infty B(\tilde{\Omega}, 2L) < \frac{\sigma}{2},$$

where L is large. ■

4.5 Admissibility in Terms of Ω (Not $\tilde{\Omega}$)

Now we give conditions stated in terms of Ω sufficient for admissibility. The first theorem is a generalization of the example in subsection 4.3; its proof uses the admissibility test stated in Theorem 4.2. The second is deduced directly from Corollary 3.3.

Here we need the following condition on a positive increasing function h defined on $(0, \infty)$:

$$\int_0^1 \frac{h(t)}{t}\,dt + \int_1^\infty \frac{h(t)}{t^2}\,dt < \infty. \tag{4.15}$$

It is well known in connection with conjugate functions and polynomial best approximation [2], [14, p. 121].

Suppose $h = h_\Omega$ satisfies (4.15). Then Ω is continuous, satisfies (4.6) and (4.1), since $\Omega(t) \le \Omega(0) + h(|t|)$, $t \in \mathbb{R}$, so that $\tilde{\Omega}$ exists. Moreover, it exists at any point $x \in \mathbb{R}$. We are going to show that $\tilde{\Omega}$ is even continuous and estimate its continuity modulus $h_{\tilde{\Omega}}$. This estimate is analogous to a known estimate of the continuity modulus of conjugate function on the circle. The case of $\mathbb{R}$ is somewhat special and we have the following result.

Lemma 4.5 *If $h = h_\Omega$ satisfies (4.15), then*

$$h_{\tilde{\Omega}}(t) \le \frac{2}{\pi}\int_0^\infty \left(\frac{2}{\tau(\tau^2+1)} + \frac{1}{\tau^2+1}\right) h(t\tau)\,d\tau. \tag{4.16}$$

Note that the right side of (4.16), say $R(t)$, is comparable with

$$\int_0^t \frac{h(s)}{s}\,ds + t\int_t^\infty \frac{h(s)}{s^2}\,ds.$$

Therefore, if $h(t) = t^\alpha$, $0 < \alpha < 1$, *i.e.* if $\Omega \in \mathrm{Lip}_\alpha$, then $R(t) \asymp t^\alpha$, so that $\tilde{\Omega} \in \mathrm{Lip}_\alpha$ in accordance with Privalov's theorem.

Now if we just combine Theorem 4.3 and the estimate

$$\lim_{t\to\infty} \frac{h_{\tilde{\Omega}}(t)}{t} = 0$$

we arrive at the following result.

Theorem 4.6 *If $h = h_\Omega$ satisfies (4.15), then $\omega \in \bigcap_{\sigma>0} \mathrm{Adm}\, e^{i\sigma z}$ and $\omega \in \mathrm{Adm}\, B$ for any Blaschke product B with zeros almost uniformly distributed in a horizontal strip.*

4.6 Ω Even and Increasing

This subsection is a technical preparation to the next one where the admissibility of regular ω's with $\mathcal{L}(\omega) < \infty$ is proved.

Let ω be even, decreasing for $x > 0$, and such that $0 < \omega(x) \le 1$ for $x \in \mathbb{R}$ with, however, $\int_{-\infty}^{\infty} \frac{|\log \omega(x)|}{1+x^2}\,dx < \infty$. Moreover, we assume, without loss of generality, that $\omega(x) \equiv 1$ for $-e^2 \le x \le e^2$. Then $\Omega(x) = -\log\bigl(\omega(x)\bigr)$ is even, positive and increasing for $x > 0$ with $\Omega(x) \equiv 0$ for $-e^2 \le x \le e^2$ and $\int_{-\infty}^{\infty} \frac{\Omega(x)}{1+x^2}\,dx < \infty$. Put

$$\Omega_1(x) = \int_0^{|x|} \frac{\Omega(et)}{t}\,dt \quad \text{for } x \in \mathbb{R}. \tag{4.17}$$

Obviously $\Omega_1(x) \equiv 0$ for $-e \le x \le e$. Then, since $\Omega(t)$ is positive, even and increasing for $t > 0$,

$$\Omega_1(x) \ge \int_{|x|/e}^{|x|} \frac{\Omega(et)}{t}\,dt \ge \Omega(x) \int_{|x|/e}^{|x|} \frac{1}{t}\,dt = \Omega(x) \quad \text{for } x \in \mathbb{R}.$$

Furthermore, we have

$$\begin{aligned}\int_0^{\infty} \frac{\Omega_1(x)}{x^2}\,dx &= \int_0^{\infty}\int_0^{x} \frac{\Omega(et)}{tx^2}\,dt\,dx = \int_0^{\infty} \Bigl(\int_t^{\infty} \frac{dx}{x^2}\Bigr) \frac{\Omega(et)}{t}\,dt \\ &= \int_0^{\infty} \frac{\Omega(et)}{t^2}\,dt = e\int_0^{\infty} \frac{\Omega(\tau)}{\tau^2}\,d\tau < \infty.\end{aligned}$$

Now put

$$\Omega_2(x) = \int_0^{|x|} \frac{\Omega_1(et)}{t}\,dt \quad \text{for } x \in \mathbb{R}. \tag{4.18}$$

By the preceding argument, $\Omega_2(x)$ is also positive, even and increasing for $x > 0$, and identically zero for $-1 \le x \le 1$. Moreover,

$$\Omega_2(x) \ge \Omega_1(x) \ge \Omega(x) \quad \text{for } x \in \mathbb{R}, \tag{4.19}$$

and

$$\int_0^{\infty} \frac{\Omega_2(x)}{x^2}\,dx = e\int_0^{\infty} \frac{\Omega_1(x)}{x^2}\,dx = e^2\int_0^{\infty} \frac{\Omega(x)}{x^2}\,dx < \infty. \tag{4.20}$$

An immediate consequence of (4.20) is that

$$\Omega_2(x) = o(x),\ \Omega_1(x) = o(x),\ \Omega(x) = o(x) \quad \text{for } |x| \to \infty. \tag{4.21}$$

Lemma 4.7 *The function $\tilde{\Omega}_2(x)$ is differentiable on $\mathbb{R}$ and*

$$\frac{d\tilde{\Omega}_2(x)}{dx} = -\frac{1}{\pi}\int_0^{\infty} \log\Bigl|\frac{1+\tau}{1-\tau}\Bigr| \frac{\Omega(e^2\,x\,\tau)}{|x|\,\tau}\,d\tau$$

for each $x \in \mathbb{R} \setminus \{0\}$.

Proof In the first place, $\tilde{\Omega}_2(x)$ is odd (since $\Omega_2(x)$ is even). Moreover, $\tilde{\Omega}_2(x)$ is C^∞ near the origin where $\Omega_2(x)$ vanishes. The differentiability of $\tilde{\Omega}_2(x)$ and the formula in question need therefore only be verified at points $x > 0$, $\tilde{\Omega}_2$ being odd. In the latter circumstance we have, by the evenness of Ω_2,

$$\begin{aligned}\tilde{\Omega}_2(x) &= \frac{1}{\pi} ⨍_{\mathbb{R}} \Big(\frac{1}{x-t} + \frac{t}{1+t^2} \Big) \Omega_2(t)\, dt = \frac{1}{\pi} ⨍_0^\infty \Big(\frac{1}{x-t} + \frac{1}{x+t} \Big) \Omega_2(t)\, dt \\ &= \frac{1}{\pi} \Omega_2(t) \log\Big| \frac{x+t}{x-t} \Big| \, \Big|_{t=0}^{t\to\infty} - \frac{1}{\pi} \int_0^\infty \log\Big| \frac{x+t}{x-t} \Big| \frac{\Omega_1(et)}{t}\, dt \\ &= -\frac{1}{\pi} \int_0^\infty \log\Big| \frac{1+\tau}{1-\tau} \Big| \frac{\Omega_1(ex\tau)}{\tau}\, d\tau\end{aligned}$$

Thus, fixing $x > 0$,

$$\frac{\tilde{\Omega}_2(x+\Delta x) - \tilde{\Omega}_2(x)}{\Delta x} = -\frac{e}{\pi} \int_0^\infty \log\Big| \frac{1+\tau}{1-\tau} \Big| \frac{\Omega_1(ex\tau + e\tau\Delta x) - \Omega_1(ex\tau)}{e\tau\Delta x}\, d\tau.$$

But, by (4.17), we have, as long as $x + \Delta x > 0$,

$$\frac{\Omega_1(ex\tau + e\tau\Delta x) - \Omega_1(ex\tau)}{e\tau\Delta x} = \frac{1}{e\tau\Delta x} \int_{ex\tau}^{ex\tau + e\tau\Delta x} \frac{\Omega(e,s)}{s}\, ds = \frac{\Omega(e^2 x'\tau)}{ex'\tau}$$

for some x' between x and $x + \Delta x$. Thus, for $|\Delta x| < \frac{x}{2}$, Ω being increasing on $[0,\infty)$,

$$0 \le \frac{\Omega_1(ex\tau + e\tau\Delta x) - \Omega_1(ex\tau)}{e\tau\Delta x} \le \frac{2\Omega(2e^2 x\tau)}{ex\tau}$$

For $\tau > 2$, the Taylor series expansion of $\log(1+s)$ around the origin yields

$$\begin{aligned}\log\Big| \frac{1+\tau}{1-\tau} \Big| &= \log\Big(1 + \frac{1}{\tau}\Big) - \log\Big(1 - \frac{1}{\tau}\Big) = 2\Big(\frac{1}{\tau} + \frac{1}{3\tau^3} + \frac{1}{5\tau^5} + \cdots \Big) \\ &\le \frac{2}{\tau}\Big(1 + \frac{1}{3\times 2^2} + \cdots\Big) = \frac{2\log 3}{\tau}.\end{aligned} \tag{4.22}$$

By this inequality and (4.20) $\log\big|\frac{1+\tau}{1-\tau}\big| \frac{\Omega(2e^2x\tau)}{ex\tau}$ is in $L^1(d\tau)$ for $2 \le \tau < \infty$. Also, $\log\big|\frac{1+\tau}{1-\tau}\big|$ is in $L^1(d\tau)$ for $0 \le \tau \le 2$ and $\frac{\Omega(2e^2x\tau)}{ex\tau}$ is bounded there. Hence $\log\big|\frac{1+\tau}{1-\tau}\big| \frac{\Omega(2e^2x\tau)}{ex\tau}$ is in $L^1(d\tau)$ for $0 \le \tau < \infty$. Thus, by the dominated convergence theorem, we have

$$\begin{aligned}\frac{d\tilde{\Omega}_2(x)}{dx} &= \lim_{\Delta x\to 0} \frac{\Omega_2(x+\Delta x) - \Omega_2(x)}{\Delta x} \\ &= -\frac{e}{\pi} \int_0^\infty \log\Big| \frac{1+\tau}{1-\tau} \Big| \lim_{\Delta x\to 0} \frac{\Omega_1(ex\tau + e\tau\Delta x) - \Omega_1(ex\tau)}{e\tau\Delta x}\, d\tau \\ &= -\frac{e}{\pi} \int_0^\infty \log\Big| \frac{1+\tau}{1-\tau} \Big| \frac{d\Omega_1(ex\tau)}{d\tau}\, d\tau \\ &= -\frac{1}{\pi} \int_0^\infty \log\Big| \frac{1+\tau}{1-\tau} \Big| \frac{\Omega(e^2x\tau)}{x\tau}\, d\tau.\end{aligned}$$

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This shows that $\tilde{\Omega}_2'(x)$ exists and is given by the asserted formula for $x > 0$, all that remained to be checked. ■

Theorem 4.8 *Let $\varepsilon > 0$. Then $\tilde{\Omega}_2$ is uniformly Lipschitz on $\mathbb{R}$ and*

$$|\tilde{\Omega}_2(x_2) - \tilde{\Omega}_2(x_1)| \le \varepsilon |x_2 - x_1|$$

for x_1 and x_2 both either sufficiently large or sufficiently negative.

Proof Referring to Lemma 4.7, we have, for $x > 0$,

$$\frac{d\tilde{\Omega}_2(x)}{dx} = -\frac{1}{\pi}\int_0^{\infty} \log\Big|\frac{1+\tau}{1-\tau}\Big| \frac{\Omega(e^2 x\tau)}{x\tau}\, d\tau$$

$$(4.23) \qquad = -\frac{1}{\pi}\int_0^{2} \log\Big|\frac{1+\tau}{1-\tau}\Big| \frac{\Omega(e^2 x\tau)}{x\tau}\, d\tau - \frac{1}{\pi}\int_2^{\infty} \log\Big|\frac{1+\tau}{1-\tau}\Big| \frac{\Omega(e^2 x\tau)}{x\tau}\, d\tau$$

Since $\Omega(t)$ vanishes for t near the origin, and is increasing for $t > 0$, $\frac{\Omega(t)}{t}$ is bounded by (4.21). Therefore, the first integral on the right in (4.23) is bounded for $x > 0$, since $\int_0^2 \log|\frac{1+\tau}{1-\tau}|\, d\tau$ is finite. By (4.22), the second integral on the right in (4.23) is in absolute value

$$\le 2\log 3 \int_2^{\infty} \frac{\Omega(e^2 x\tau)}{|x|\tau^2}\, d\tau = 2e^2 \log 3 \int_{2e^2 x}^{\infty} \frac{\Omega(s)}{s^2}\, ds.$$

and this is bounded for $x > 0$ by (4.20). Thus, $|\tilde{\Omega}_2'(x)|$ is bounded for $x > 0$, and, since $\tilde{\Omega}_2(x)$ is odd, for $x \ne 0$, so, finally for all x. The uniform Lipschitz character of $\tilde{\Omega}_2(x)$ now follows from the mean value theorem.

When $x \to \infty$, both right hand integrals in (4.23) tend to zero, the first by dominated convergence, since then, by (4.21), $\frac{\Omega(e^2 x\tau)}{x\tau} \to 0$ for $\tau > 0$, and the second by the preceding estimate and (4.20), since then $2e^2 x \to \infty$. The relation affirmed by the theorem now follows from this by the mean value theorem and the oddness of $\tilde{\Omega}_2(x)$. ■

4.7 A New Proof of Admissibility of Regular Majorants

The following classical result has several proofs. Some constructive methods are available in [5, pp. 276, 393], [9, p. 97] and [10, p. 159]. We give a new proof.

Theorem 4.9 *Suppose that ω is even, decreasing for $x > 0$, and that $0 < \omega(x) \le 1$ for $x \in \mathbb{R}$, with $\int_{-\infty}^{\infty} \frac{|\log \omega(x)|}{1+x^2}\, dx < \infty$. Then there is a nonzero entire function f of arbitrarily small exponential type > 0, with*

$$|f(x)| \le \omega(x) \quad \text{for } x \in \mathbb{R}.$$

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Proof By Theorem 2.5 of [6] and Theorem 4.8 and Corollary 3.2, $\omega_2(x) = e^{-\Omega_2(x)}$, with Ω_2 defined by (4.18), is an admissible majorant for $K_{e^{i\sigma x}}$ for any $\sigma > 0$. By (4.19) we have $\omega_2(x) \leq \omega(x)$ and thus $\omega(x)$ is also an admissible majorant for $K_{e^{i\sigma x}}$ for any $\sigma > 0$. Hence, by Theorem 2.5 of [6], there is a nonzero entire function f of arbitrarily small exponential type > 0 with

$$|f(x)| \leq \omega(x) \quad \text{for } x \in \mathbb{R}. \qquad \blacksquare$$

4.8 Admissibility Depends on the Distribution of Zeros

Now we are able to generalize Theorem 4.9 for more classes of model subspaces.

Theorem 4.10 *Let B be a meromorphic Blaschke product whose zeros are almost uniformly distributed in a horizontal strip separated from $\mathbb{R}$. Suppose that ω is even, decreasing for $x > 0$ and that $0 < \omega(x) \leq 1$ for $x \in \mathbb{R}$, with*

$$\int_{-\infty}^{\infty} \frac{|\log \omega(x)|}{1 + x^2}\, dx < \infty.$$

Then $\omega \in \operatorname{Adm} B$.

Proof By Theorems 3.6 and 4.8, $\omega_2(x) = \exp\bigl(-\Omega_2(x)\bigr)$ is an admissible majorant for K_B. By (4.19) we have $\omega_2(x) \leq \omega(x)$ and thus $\omega(x)$ is also an admissible majorant for K_B. $\blacksquare$

Corollary 4.11 *Under the conditions of Theorem 4.10, any function of the form*

$$\omega(x) = \exp(-c|x|^{\alpha})$$

with $c > 0$ and $0 < \alpha < 1$ is an admissible majorant for K_B.

The preceding corollary implies that $\omega(x) = \exp(-c|x|^{\alpha})$, $0 < \alpha < 1$, is an admissible majorant for K_B, where $B(z)$ is any meromorphic Blaschke product whose zeros are almost uniformly distributed in a horizontal strip. On the other hand, by (3.10) of [6], $\omega(x)$ ceases, as soon as $\alpha > 1/2$, to be an admissible majorant for K_B when $B(z)$ is the Blaschke product

$$\prod_{k=1}^{\infty} \Bigl(\frac{ik^2 - z}{ik^2 + z}\Bigr).$$

Therefore, the *admissibility* of a given $\omega(x) \geq 0$ for K_B depends on the *distribution of the zeros* of the Blaschke product B.

5 Admissible Majorants For Some Bs With Rapidly Growing $\arg B$

In this section we consider the Blaschke products

$$B_\alpha(z) = \prod_{k=-\infty}^{\infty} \frac{1 - z/(x_k + i)}{1 - z/(x_k - i)},$$

where $x_k = |k|^\alpha \operatorname{sgn} k$ and $1/2 < \alpha < 1$. The restriction $\alpha > 1/2$ comes from the Blaschke condition. If $\alpha = 1$, then we return to a very particular case of Section 2, *i.e.*, the continuous argument of B_1 behaves *almost linearly*: $(\arg B_1)'(x) \asymp 1$, $x \in \mathbb{R}$. As to the case $\alpha > 1$, it is studied in [6], since in that case

$$B_\alpha(z) = \frac{E^*_\alpha(z)}{E_\alpha(z)},$$

where

$$E_\alpha(z) = \prod_{k=-\infty}^{\infty} \Big(1 - \frac{z}{x_k - i}\Big)$$

is an entire function of zero exponential type. Moreover,

$$\text{(5.1)} \quad \cot(\pi/2\alpha)|x|^{1/\alpha} + O(\log|x|) \le \log|E_\alpha(x)| \le \csc(\pi/2\alpha)|x|^{1/\alpha} + O(\log|x|),$$

as $|x| \to \infty$. Thus $E_\alpha(z)$ is in the Cartwright class and $1/E_\alpha(x)$ is in $L^2(\mathbb{R})$. Therefore, according to Theorem 3.5 of [6], $1/|E_\alpha(x)|$ is a minimal majorant for K_{B_α} if $\alpha > 1$. The estimate (5.1) follows from the general results in [11, p. 64]; a proof can be given following the argument on [5, pp. 146–151]. We shall see in subsection 5.1 that for $\alpha \in (1/2, 1)$ the continuous argument of $B_\alpha(x)$ behaves as $|x|^{1/\alpha} \operatorname{sgn} x$, a much faster rate of growth than for $\alpha \ge 1$.

5.1 Asymptotic Behavior of $(\arg B_\alpha)'(x)$

Here $\arg B_\alpha(x)$ denotes the continuous argument of B_α vanishing at the origin.

Lemma 5.1 *If $\alpha \in (1/2, 1)$, then*

$$\frac{d}{dx} \arg B_\alpha(x) = \frac{2\pi}{\alpha}|x|^{\frac{1}{\alpha}-1} + O(1), \quad x \in \mathbb{R}.$$

Proof According to Lemma 4.5 of [6]

$$\text{(5.2)} \qquad \frac{d}{dx} \arg B_\alpha(x) = 2 \sum_{k=-\infty}^{\infty} \frac{1}{(x - |k|^\alpha \operatorname{sgn} k)^2 + 1} = 2\sigma(x), \quad x \in \mathbb{R}.$$

Suppose that x is a large positive number, say $\ell^\alpha \le x < (\ell+1)^\alpha$. Then

$$\sigma(x) = \Big(\sum_{k=-\infty}^{0} + \sum_{k=1}^{\ell} + \sum_{k=\ell+1}^{\infty}\Big) \frac{1}{(x - |k|^\alpha \operatorname{sgn} k)^2 + 1} = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

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Now,

$$\Sigma_1 = \frac{1}{1+x^2} + \sum_{k=1}^{\infty} \frac{1}{(x+k^\alpha)^2+1} \le 1 + \sum_{k=1}^{\infty} \frac{1}{k^{2\alpha}} = O(1).$$

To estimate Σ_2 and Σ_3 put

$$F(t) = \frac{1}{(x-t^\alpha)^2+1}, \quad t > 0,$$

so that $\Sigma_2 = \sum_{k=1}^{\ell} F(k)$ and $\Sigma_3 = \sum_{k=\ell+1}^{\infty} F(k)$. Clearly $F(t)$ increases as t grows from 0 to $x^{1/\alpha}$ whence

$$F(1) + \int_1^\ell F(t)\,dt \le \Sigma_2 \le \int_1^\ell F(t)\,dt + 1.$$

(note that $0 < F \le 1$). Therefore,

$$\Sigma_2 = \int_1^\ell F(t)\,dt + O(1) = \int_0^{x^{1/\alpha}} F(t)\,dt + O(1).$$

But F decreases on $(x^{1/\alpha}, \infty)$ whence

$$F(\ell+1) + \int_{\ell+1}^\infty F(t)\,dt \ge \Sigma_3 \ge \int_{\ell+1}^\infty F(t)\,dt,$$

so that

$$\Sigma_3 = \int_{x^{1/\alpha}}^\infty F(t)\,dt + O(1).$$

To estimate Σ_2 put $\tau = 1 - t^\alpha/x$ for $t \in [0, x^{1/\alpha}]$; then

$$\begin{aligned}
\int_0^{x^{1/\alpha}} \frac{1}{(x-t^\alpha)^2+1}\,dt &= \int_0^{x^{1/\alpha}} \frac{1}{(x-t^\alpha)^2+1}\,d(t-x^{1/\alpha}) \\
&= \frac{t-x^{1/\alpha}}{(x-t^\alpha)^2+1}\Bigg|_{t=0}^{t=x^{1/\alpha}} - \int_0^{x^{1/\alpha}} \frac{2\alpha t^{\alpha-1}(x-t^\alpha)(t-x^{1/\alpha})}{\big((x-t^\alpha)^2+1\big)^2}\,dt \\
&= \frac{x^{1/\alpha}}{x^2+1} + 2x^{2+1/\alpha} \int_0^1 \frac{\tau\big(1-(1-\tau)^{1/\alpha}\big)}{(x^2\tau^2+1)^2}\,d\tau \\
&= 2x^{-2+1/\alpha} \int_0^1 \frac{\tau}{(\tau^2+x^{-2})^2}\Big(\frac{\tau}{\alpha} + \tau^2\varphi(\tau)\Big)\,d\tau + O(1),
\end{aligned}$$

where φ is a bounded function on $[0,1]$. Put $s = x\tau$. Hence

$$\begin{aligned}
\Sigma_2 &= \frac{2}{\alpha}x^{-1+1/\alpha} \int_0^x \frac{s^2}{(s^2+1)^2}\,ds + \frac{2}{\alpha}x^{-2+1/\alpha} \int_0^x \frac{s^3\varphi(s/x)}{(s^2+1)^2}\,ds + O(1) \\
&= \frac{2}{\alpha}x^{-1+1/\alpha} \int_0^\infty \frac{s^2}{(s^2+1)^2}\,ds + O(x^{-1+1/\alpha}\cdot x^{-1}) + O(x^{-2+1/\alpha}\log x) + O(1) \\
&= \frac{\pi}{2\alpha}x^{-1+1/\alpha} + O(1).
\end{aligned}$$

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To estimate Σ_3 put $\tau = -1 + t^\alpha/x$ for $t \in [x^{1/\alpha}, \infty]$; then

$$\int_{x^{1/\alpha}}^{\infty} \frac{1}{(x-t^\alpha)^2+1}\,dt = \int_{x^{1/\alpha}}^{\infty} \frac{1}{(x-t^\alpha)^2+1}\,d(t-x^{1/\alpha})$$
$$= \frac{t-x^{1/\alpha}}{(x-t^\alpha)^2+1}\Big|_{t=x^{1/\alpha}}^{t\to\infty} - \int_{x^{1/\alpha}}^{\infty} \frac{2\alpha t^{\alpha-1}(x-t^\alpha)(t-x^{1/\alpha})}{\big((x-t^\alpha)^2+1\big)^2}\,dt$$
$$= 2x^{2+1/\alpha}\int_0^\infty \frac{\tau\big((1+\tau)^{1/\alpha}-1\big)}{(x^2\tau^2+1)^2}\,d\tau$$
$$= \frac{2}{\alpha}x^{2+1/\alpha}\int_0^1 \frac{\tau^2}{(\tau^2+x^{-2})^2}\,d\tau + \frac{2}{\alpha}x^{2+1/\alpha}\int_0^1 \frac{\tau^3\psi(\tau)}{(\tau^2+x^{-2})^2}\,d\tau$$
$$+\, 2x^{2+1/\alpha}O\Big(\int_1^\infty \frac{(1+\tau)^{1/\alpha}-1}{\tau^3}\,d\tau\cdot\frac{1}{x^4}\Big),$$

where ψ is a bounded function on $[0,1]$. As in the previous case, the first two integrals are correspondingly $\frac{\pi}{2\alpha}x^{-1+1/\alpha}\big(1+O(1)\big)$ and $O(x^{-2+1/\alpha}\log x)$. The third integral is $O(x^{-2+1/\alpha})$. Hence

$$\Sigma_3 = \frac{\pi}{2\alpha}x^{-1+1/\alpha} + O(1).$$

Putting all previous estimates together, we have

$$\sigma(x) = \frac{\pi}{\alpha}x^{-1+1/\alpha} + O(1). \qquad \blacksquare$$

5.2 Some Elements of $\operatorname{Adm} B_\alpha$ With $1/2 < \alpha < 1$

Let ω be a positive function on $\mathbb{R}$ and $\Omega = \log 1/\omega$. As usual we assume that $\int_{\mathbb{R}} |\Omega(x)|/(1+x^2)\,dx < \infty$, so that $\tilde\Omega$ exists. To ensure the admissibility of ω for K_{B_α} we consider the function $f(x) = \arg B_\alpha(x) + 2\tilde\Omega(x)$ and try to find conditions (to be imposed on Ω) making f mainly increasing (see Section 2.1). Put $c_\alpha = \pi/\alpha$. Suppose that $\tilde\Omega$ is in $C^1(\mathbb{R})$ and

$$-c_\alpha < \liminf_{|x|\to\infty} \frac{\tilde\Omega'(x)}{|x|^{\frac{1}{\alpha}-1}} \le \limsup_{|x|\to\infty} \frac{\tilde\Omega'(x)}{|x|^{\frac{1}{\alpha}-1}} < C_\alpha. \tag{5.3}$$

Then, according to Lemma 5.1,

$$f'(x) = 2c_\alpha|x|^{\frac{1}{\alpha}-1} + \varphi_\alpha(x) + 2\tilde\Omega'(x) \ge 2c_\alpha|x|^{\frac{1}{\alpha}-1} + \varphi_\alpha(x) + (\delta - 2c_\alpha)|x|^{\frac{1}{\alpha}-1}, \tag{5.4}$$

where δ is a positive constant, $\varphi_\alpha(x)$ is bounded on $\mathbb{R}$. Thus, by (5.3) and (5.4), and for sufficiently large values of $|x|$,

$$f'(x) \asymp |x|^{\frac{1}{\alpha}-1}. \tag{5.5}$$

We see that

$$f(x) \asymp |x|^{\frac{1}{\alpha}} \quad \text{as } |x| \to \infty,$$

and

$$\lim_{x\to\pm\infty} f(x) = \pm\infty,$$

and f is increasing on $\mathbb{R} \setminus [-\Delta, \Delta]$ for a large $\Delta > 0$. Thus there exists an increasing sequence $\{d_n\}_{n\in\mathbb{Z}}$ satisfying $f(d_n) = 2\pi n$, $n \in \mathbb{Z}$, and

$$|d_n| \asymp |n|^{\alpha}. \tag{5.6}$$

(See Section 2.1). To estimate ℓ_n, we have

$$2\pi = f(d_n) - f(d_{n-1}) = f'(\xi_n)2\ell_n,$$

whence, by (5.5) and (5.6),

$$\ell_n \le \frac{\pi}{\inf_{I_n} f'} \le \frac{\text{Const}}{(|n|^{\alpha})^{\frac{1}{\alpha}-1}} \le L|n|^{\alpha-1}.$$

We now have to show that

$$\mathcal{I}_n = \frac{1}{2\ell_n}\int_{I_n} |f'(t) - f'(x)|\, dt$$

is bounded uniformly with respect to n and $x \in I_n = (d_{n-1}, d_n)$. According to (5.4)

$$\mathcal{I}_n = \frac{c_\alpha}{\ell_n}\int_{I_n}\Big|\, |t|^{\frac{1}{\alpha}-1} - |x|^{\frac{1}{\alpha}-1}\Big|\, dt + \frac{1}{2\ell_n}\int_{I_n} |\varphi_\alpha(t) - \varphi_\alpha(x)|\, dt$$
$$+ \frac{1}{2\ell_n}\int_{I_n} |\tilde\Omega'(t) - \tilde\Omega'(x)|\, dt.$$

The first and the second integrals are uniformly bounded, since

$$\Big|\, |t|^{\frac{1}{\alpha}-1} - |x|^{\frac{1}{\alpha}-1}\Big| \le |t-x|^{\frac{1}{\alpha}-1} \le (2\ell_n)^{\frac{1}{\alpha}-1},$$

and φ_α is a bounded function on $\mathbb{R}$. To estimate the third integral, denote by λ_t the modulus of continuity of $\tilde\Omega'$ on $\mathbb{R} \setminus (-t, t)$, *i.e.*,

$$\lambda_t(\delta) = \sup\{\operatorname{osc}_I \tilde\Omega' : I \text{ an interval}, I \cap (-t,t) = \varnothing, |I| \le \delta\}, \quad t > 0,\ \delta > 0.$$

Recall that

$$\lambda_t(c\delta) \le (c+1)\lambda_t(\delta), \quad c > 0,\ \delta > 0.$$

Suppose that $\lambda_t(t^{1-\frac{1}{\alpha}})$ is bounded, *i.e.*, there exists a number $K > 0$ such that

$$\lambda_t(t^{1-\frac{1}{\alpha}}) \le K \tag{5.7}$$

for all $t > 0$. Clearly (5.7) is fulfilled if $\tilde{\Omega}''(x) = O(|x|^{\frac{1}{\alpha}-1})$ as $|x| \to \infty$. Then, for each $x \in I_n$,

$$\frac{1}{2\ell_n}\int_{I_n} |\tilde{\Omega}'(t) - \tilde{\Omega}'(x)|\,dt \le \lambda_{(c|n|)^\alpha}\big((c'|n|)^{1-\frac{1}{\alpha}}\big)$$
$$\le \big((c/c')^{1-\frac{1}{\alpha}} + 1\big)\,\lambda_{(c|n|)^\alpha}\big((c|n|)^{1-\frac{1}{\alpha}}\big)$$
$$\le \big((c/c')^{1-\frac{1}{\alpha}} + 1\big)\,K.$$

Hence f is mainly increasing. Now we can sum up this reasoning as stated in Theorem 1.10 in the Introduction.

Theorem 5.2 *Suppose* $\int_{\mathbb{R}} |\Omega(x)|/(1+x^2)\,dx < \infty$, $\tilde{\Omega} \in C^1(\mathbb{R})$ *and (5.3) and (5.7) hold. Then* $\omega \in \operatorname{Adm} B_\alpha$.

Comparing this result with the result of Section 3.4, we see that the restrictions imposed there on $\tilde{\Omega}$ are essentially stronger than in Theorem 5.2. The conditions moderating the speed of descent of $\tilde{\Omega}$, *i.e.*, lower estimates of $\tilde{\Omega}'$, are

$$\tilde{\Omega}'(x) > -k$$

in Section 3.4, and

$$\tilde{\Omega}'(x) > -k|x|^{\frac{1}{\alpha}-1}$$

in Theorem 5.2; restrictions on general oscillations are respectively

$$\operatorname{osc}_I \tilde{\Omega}'(x) \le K$$

for all intervals I of length at most one, and

$$\operatorname{osc}_I \tilde{\Omega}'(x) \le K$$

if $I \subset \mathbb{R}\backslash(-t,t)$ and $|I| < t^{1-\frac{1}{\alpha}}$. This liberalization is natural, since the density of singularities of functions from the class K_{B_α}, *i.e.*, the poles at $\pm k^\alpha - i$, $k = 1, 2, \dots$ grows as α moves from 1 to $1/2$. A generic element of K_{B_α} for $\alpha \in (1/2, 1)$ is *less analytic* in $\mathbb{C}_+ \cup \mathbb{R}$ than an element of B_1. Hence the mere convergence of $\int_{\mathbb{R}} |\Omega(x)|/(1+x^2)\,dx$ gets closer to a condition sufficient for the admissibility of ω for K_{B_α}.

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