Admissible rules of Łukasiewicz logic

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Abstract

We investigate admissible rules of Łukasiewicz multi-valued propositional logic. We show that admissibility of multiple-conclusion rules in Łukasiewicz logic, as well as validity of universal sentences in free MV-algebras, is decidable (in $PSPACE$).

1 Introduction

Investigation of nonclassical logics usually revolves around provability of formulas. When we generalize the problem from formulas to inference rules, there arises an important distinction between *derivable* and *admissible* rules, introduced by Lorenzen [16]. A rule

$$\varphi_1, \dots, \varphi_n \;/\; \psi$$

is derivable if it belongs to the consequence relation of the logic (defined semantically, or by a proof system using a set of axioms and rules); and it is admissible if the set of theorems of the logic is closed under the rule. These two notions coincide for the standard consequence relation of classical logic, but nonclassical logics often admit rules which are not derivable. (A logic whose admissible rules are all derivable is called *structurally complete*.) For example, all superintuitionistic (si) logics admit the Kreisel–Putnam rule

$$\neg p \to q \vee r \;/\; (\neg p \to q) \vee (\neg p \to r)$$

(Prucnal [21]), whereas many of these logics (such as IPC itself) do not derive this rule.

The research of admissible rules was stimulated by a question of H. Friedman [5], asking whether admissibility of rules in IPC is decidable. The problem was extensively investigated in a series of papers by Rybakov, who has shown that admissibility is decidable for a large class of modal and si logics, found semantic criteria for admissibility, and obtained other results on various aspects of admissibility. His results on admissible rules in transitive modal and si logics are summarized in the monograph [23]. He also applied his method to tense logics [24, 25, 26].

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Ghilardi [7, 8] discovered the connection of admissibility to projective formulas and unification, which provided another criteria for admissibility in certain modal and si logics, and new decision procedures for admissibility in some modal and si systems. Ghilardi's results were utilized by Iemhoff [10, 11, 12] to construct an explicit basis of admissible rules for *IPC* and some other si logics, and to develop Kripke semantics for admissible rules. These results were extended to modal logics by Jeřábek [14]. We note that decidability of admissibility is by no means automatic. An artificial decidable modal logic with undecidable admissibility problem was constructed by Chagrov [1], and natural examples of bimodal logics with undecidable admissibility (or even unification) problem were found by Wolter and Zakharyaschev [29]. In terms of computational complexity, admissibility in basic transitive logics is *coNE*-complete[1] by Jeřábek [15], whereas derivability in these logics is *PSPACE*-complete.

In contrast to the situation in modal and superintuitionistic logics, only very little is known about admissibility in other nonclassical logics. Here we are particularly interested in substructural and fuzzy logics. Structural completeness of various substructural logics was investigated by Olson et al. [20] and by Cintula and Metcalfe [3]. Dzik [4] studied unification in n-contractive extensions of Hájek's Basic Logic (**BL**).

In this paper we study admissible rules of Łukasiewicz logic (**Ł**). We chose this logic because it is one of the three fundamental t-norm fuzzy logics, and among the three it is the only one with a nontrivial admissibility problem: Gödel–Dummett logic is a si logic and it is well-known to be structurally complete, and product logic was shown to be structurally complete too by Cintula and Metcalfe [3]; in contrast, Łukasiewicz logic is structurally incomplete[2]. For more generality, we work with multiple-conclusion rules (cf. Shoesmith and Smiley [27]). We describe a criterion for admissibility of multiple-conclusion rules in **Ł**, and we show that admissibility in **Ł** is decidable. We also compute explicit bounds on the size of counterexamples to inadmissible rules, and use them to provide a *PSPACE*-algorithm for admissibility in **Ł**. Our results can be restated algebraically, namely we obtain that the universal theory of free *MV*-algebras is decidable (in *PSPACE*). We also show that **Ł** is 1-reducible wrt admissible rules (i.e., inadmissibility of any rule can be witnessed by a substitution using only one variable), or in algebraic terms, all free *MV*-algebras over nonempty sets of generators have the same universal theory.

For completeness, we also briefly consider the case of finite-valued Łukasiewicz logics $\mathbf{Ł}_n$. Being tabular extensions of **BL**, these logics are n-contractive, hence we easily derive from Dzik's results [4] that admissibility in $\mathbf{Ł}_n$ is decidable. We provide an explicit basis of admissible rules for $\mathbf{Ł}_n$ (and more generally, for any n-contractive extension of **BL**).

2 Preliminaries

The language of Hájek's Basic Logic (**BL**) [9] consists of *propositional formulas* built from variables p_n, $n \in \omega$, using connectives $\to$, $\cdot$, and $\bot$. A *substitution* is a mapping of proposi-

[1] *NE* is nondeterministic exponential time.

[2] Its $\{\to, \cdot, \wedge, \vee\}$-fragment is also structurally incomplete ([3]); on the other hand, the $\{\to, \wedge, \vee\}$-fragment is structurally complete (Wojtylak [28]).

tional formulas to propositional formulas which commutes with all connectives. A formula φ is *derivable* from a set of formulas Γ, written as $\Gamma \vdash_{\mathbf{BL}} \varphi$, is there exists a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$, and each φ_i is a member of Γ, an instance of one of the axioms (cf. Cintula [2])

$$\begin{gathered}
(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)), \\
\varphi \to (\psi \to \varphi), \\
(\varphi \to (\psi \to \chi)) \to (\varphi \cdot \psi \to \chi), \\
(\varphi \cdot \psi \to \chi) \to (\varphi \to (\psi \to \chi)), \\
\varphi \cdot (\varphi \to \psi) \to \psi \cdot (\psi \to \varphi), \\
((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi), \\
\bot \to \varphi,
\end{gathered}$$

or it is derived from some φ_j, φ_k, $j, k < i$ by an instance of the rule of modus ponens

$$\text{(MP)} \qquad \varphi, \varphi \to \psi \;/\; \psi.$$

We can introduce other connectives as abbreviations

$$\begin{aligned}
\neg\varphi &\equiv \varphi \to \bot, \\
\varphi \wedge \psi &\equiv \varphi \cdot (\varphi \to \psi), \\
\varphi \vee \psi &\equiv ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\
\varphi \leftrightarrow \psi &\equiv (\varphi \to \psi) \wedge (\psi \to \varphi), \\
\top &\equiv \neg\bot,
\end{aligned}$$

and we write $\varphi^n = \varphi \cdot \ldots \cdot \varphi$ with n occurrences of φ (if $n = 0$, we put $\varphi^0 = \top$). An *extension* of **BL** is a consequence relation $\vdash_L$ defined as $\vdash_{\mathbf{BL}}$ above, except that we allow an extra set of additional axioms, which is required to be closed under substitution. An extension of **BL** is *n-contractive* if it proves the schema $\varphi^n \to \varphi^{n+1}$.

Łukasiewicz logic (**Ł**) is the extension of **BL** by the axiom schema

$$\neg\neg\varphi \to \varphi.$$

In **Ł** we can introduce another connective

$$\varphi \oplus \psi \equiv \neg\varphi \to \psi \equiv \neg(\neg\varphi \cdot \neg\psi).$$

Connectives in **Ł** are interdefinable, we have

$$\begin{aligned}
\varphi \to \psi &\equiv \neg\varphi \oplus \psi \equiv \neg(\varphi \cdot \neg\psi), \\
\varphi \cdot \psi &\equiv \neg(\neg\varphi \oplus \neg\psi) \equiv \neg(\varphi \to \neg\psi),
\end{aligned}$$

hence we can take any of the sets $\{\to, \neg\}$, $\{\to, \bot\}$, $\{\cdot, \neg\}$, $\{\oplus, \neg\}$ as the set of basic connectives, and define the rest as abbreviations.

If L is a logic, an *L-unifier* of a formula φ is a substitution σ such that $\vdash_L \sigma\varphi$. A formula which has an L-unifier is called *L-unifiable.* A unifier is *ground* if its range consists of constant (i.e., variable-free) formulas. An L-unifier σ of φ is *projective* (Ghilardi [7], cf. [6]), if

$$\varphi \vdash_L \psi \leftrightarrow \sigma\psi$$

for every formula ψ. A formula is *L-projective*, if it has a projective L-unifier.

A *single-conclusion rule* is an expression of the form

$$\frac{\Gamma}{\varphi},$$

also written as $\Gamma \;/\; \varphi$, where φ is a formula, and Γ is a finite set of formulas. A single-conclusion rule $\Gamma \;/\; \varphi$ is *derivable* in a logic L, if $\Gamma \vdash_L \varphi$. The rule $\Gamma \;/\; \varphi$ is *L-admissible*, written as $\Gamma \mathrel{|\!\!\sim}_L \varphi$, if every common L-unifier of Γ is also an L-unifier of φ. A set B of L-admissible rules is a *basis* of L-admissible rules, if for every L-admissible rule $\Gamma \;/\; \varphi$, the formula φ is derivable from Γ using axioms and rules of L, and substitution instances of rules from B. We will often omit the prefix L- when it is clear from the context.

More generally, a *multiple-conclusion rule* is an expression of the form

$$\frac{\Gamma}{\Delta},$$

also written as $\Gamma \;/\; \Delta$, where Γ and Δ are finite sets of formulas. (Note that Γ or Δ or both can be empty.) We will often write just *rule* instead of "multiple-conclusion rule". A rule $\Gamma \;/\; \Delta$ is derivable in L if $\Gamma \vdash_L \varphi$ for some $\varphi \in \Delta$, and it is L-admissible if every common L-unifier of Γ is also an L-unifier of some formula $\varphi \in \Delta$. A set B of L-admissible rules is a *basis* of L-admissible rules, if every L-admissible rule can be inferred from L-derivable rules and instances of rules from B using weakening (from $\Gamma \;/\; \Delta$ infer $\Gamma, \Gamma' \;/\; \Delta, \Delta'$) and cut (from $\Gamma \;/\; \Delta, \varphi$ and $\Gamma, \varphi \;/\; \Delta$ infer $\Gamma \;/\; \Delta$). $\Gamma \;/\; \Delta$ is a *passive admissible rule* if Γ has no common unifier, i.e., if $\Gamma \mathrel{|\!\!\sim}_L$.

An *MV-algebra* is a structure $\langle A, \oplus, \neg, 0\rangle$ which satisfies the identities

$$\begin{aligned}
(x \oplus y) \oplus z &= x \oplus (y \oplus z),\\
x \oplus 0 &= x,\\
x \oplus y &= y \oplus x,\\
\neg\neg x &= x,\\
x \oplus \neg 0 &= \neg 0,\\
\neg(\neg x \oplus y) \oplus y &= \neg(\neg y \oplus x) \oplus x.
\end{aligned}$$

We can define other operations on an MV-algebra by

$$\begin{aligned}
x \to y &= \neg x \oplus y, \\
x \cdot y &= \neg(\neg x \oplus \neg y), \\
x \wedge y &= x \cdot (x \to y), \\
x \vee y &= (x \to y) \to y, \\
x \leftrightarrow y &= (x \to y) \wedge (y \to x), \\
1 &= \neg 0.
\end{aligned}$$

The operations $\wedge, \vee$ turn A into a distributive lattice with bounds $0, 1$, which induces a partial order $\leq$ on A. We can identify propositional formulas with terms in the language of MV-algebras in a natural way. A *valuation* in an MV-algebra A is a homomorphism v from the term algebra to A. If φ is a formula in the first k variables, $a \in A^k$, and v is the valuation such that $v(p_i) = a_i$, we also write $\varphi(a) = v(\varphi)$. A valuation v satisfies a formula φ if $v(\varphi) = 1$, and it satisfies a rule $\Gamma \mathbin{/} \Delta$ if $v(\varphi) \neq 1$ for some $\varphi \in \Gamma$, or $v(\varphi) = 1$ for some $\varphi \in \Delta$. A rule $\Gamma \mathbin{/} \Delta$ is *valid* in an MV-algebra A, written as $A \vDash \Gamma \mathbin{/} \Delta$, if the rule is satisfied by every valuation in A. In other words, $A \vDash \Gamma \mathbin{/} \Delta$ if and only if the open first-order formula

$$\bigwedge_{\varphi \in \Gamma} (\varphi = 1) \to \bigvee_{\varphi \in \Delta} (\varphi = 1)$$

is valid in A. Conversely, validity of open formulas (or equivalently, universal sentences) in A can be reduced to validity of rules. Any open formula Φ can be expressed in the conjunctive normal form as $\Phi = \bigwedge_{i<k} \Phi_i$, where each Φ_i is a clause: a disjunction of atomic formulas (i.e., equations) and their negations. Then $A \vDash \Phi$ iff $A \vDash \Phi_i$ for each $i < k$, and a clause

$$\bigvee_{i<n} (\varphi_i = \psi_i) \vee \bigvee_{i<m} (\varphi'_i \neq \psi'_i)$$

is valid in A iff A validates the rule

$$\{\varphi'_i \leftrightarrow \psi'_i \mid i < m\} \mathbin{/} \{\varphi_i \leftrightarrow \psi_i \mid i < n\}.$$

Łukasiewicz logic is algebraizable, and the variety of MV-algebras is its equivalent algebraic semantics, using the translation between propositional formulas and equations described above. We thus have (cf. [9]):

Fact 2.1 *A rule $\Gamma \mathbin{/} \Delta$ is valid in all MV-algebras if and only if it is derivable in* **Ł**. □

A *free MV-algebra* over a set X of generators is an MV-algebra $F \supseteq X$ such that every mapping from X to an MV-algebra A can be uniquely extended to a homomorphism from F to A. As another corollary to algebraizability of **Ł**, free MV-algebras can be described as Lindenbaum–Tarski algebras of **Ł**: F consists of equivalence classes of formulas using elements of X as propositional variables modulo the equivalence relation $\varphi \sim \psi$ iff $\vdash_{\text{Ł}} \varphi \leftrightarrow \psi$, with operations defined in the natural way. Note that valuations in F correspond to substitutions whose range consists of formulas using variables from X, and a formula φ is satisfied under a valuation given by such a substitution σ if and only if $\vdash_{\text{Ł}} \sigma\varphi$. We obtain the following characterization of admissibility:

Fact 2.2 *For any rule $\Gamma\ /\ \Delta$, the following are equivalent.*

(i) $\Gamma \mathrel{|\!\!\sim}_{\text{Ł}} \Delta$.

(ii) $\Gamma\ /\ \Delta$ *is valid in all free MV-algebras.*

(iii) $\Gamma\ /\ \Delta$ *is valid in all free MV-algebras over finite sets of generators.*

(iv) $\Gamma\ /\ \Delta$ *is valid in some free MV-algebra over an infinite set of generators.*

□

The *standard MV-algebra* $[0,1]_{\text{Ł}}$ is the algebra $\langle [0,1]_{\mathbb{R}}, \oplus, \neg, 0\rangle$, where

$$x \oplus y = \min\{x + y, 1\},$$
$$\neg x = 1 - x.$$

Notice that the rational interval $[0,1]_{\mathbb{Q}}$ is a subalgebra of $[0,1]_{\text{Ł}}$. Both $[0,1]_{\text{Ł}}$ and $[0,1]_{\mathbb{Q}}$ generate the variety of MV-algebras, hence we have the following strengthening of Fact 2.1: a formula is derivable in **Ł** iff it is valid in $[0,1]_{\text{Ł}}$ iff it is valid in $[0,1]_{\mathbb{Q}}$. (In fact, $[0,1]_{\text{Ł}}$ and $[0,1]_{\mathbb{Q}}$ generate all MV-algebras as a quasivariety, hence the same characterization also holds for derivability of single-conclusion rules.)

For any integer $n > 0$, the set $\{0, 1/n, \ldots, (n-1)/n, 1\}$ is also a subalgebra of $[0,1]_{\text{Ł}}$. The extension of **BL** given by all formulas valid in this subalgebra is the *finite-valued Łukasiewicz logic* Ł_{n+1}.

A description of free MV-algebras over finite sets of generators was given by McNaughton. Let $n \in \omega$. A function $f\colon [0,1]^n_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ is called *piecewise linear with integer coefficients*, if there are finitely many functions $L_j\colon [0,1]^n_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ such that for every $x \in [0,1]^n_{\mathbb{Q}}$ there exists j such that $f(x) = L_j(x)$, and each $L_j(x_0, \ldots, x_{n-1})$ is of the form $\sum_{i<n} a_i x_i + b$ for some $\vec{a}, b \in \mathbb{Z}$. Let F_n be the MV-algebra of continuous piecewise linear functions $f\colon [0,1]^n_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ with integer coefficients, with operations defined pointwise (i.e., F_n is a subalgebra of the Cartesian power $[0,1]_{\text{Ł}}^{[0,1]^n_{\mathbb{Q}}}$).

Theorem 2.3 (McNaughton [18]) *F_n is the free n-generated MV-algebra. The projection functions $\pi_i(x_0, \ldots, x_{n-1}) = x_i$ for $i < n$ are its free generators.* □

Note that F_n are usually defined to consist of *real* functions $f\colon [0,1]^n_{\mathbb{R}} \to [0,1]_{\mathbb{R}}$ which are continuous and piecewise linear with integer coefficients. It is easy to see that both definitions lead to isomorphic algebras, and it will be more convenient for us to work with the rational version.

In general, we will denote by F_κ the free MV-algebra over κ generators for every cardinal number κ.

If $f = \langle f_0, \ldots, f_{k-1}\rangle$ is a k-tuple of functions $f_i \in F_n$, we will identify f with the corresponding function $f\colon [0,1]^n_{\mathbb{Q}} \to [0,1]^k_{\mathbb{Q}}$.

Since F_n is isomorphic to a Lindenbaum–Tarski algebra of **Ł**, elements $f \in F_n$ represent formulas in n variables (up to **Ł**-provable equivalence). As we identify propositional formulas with terms, we may evaluate them in every MV-algebra. We can therefore define $f(a) \in A$ for every $f \in F_n$ and $a \in A^n$, where A is an MV-algebra. (In algebraic terms, $f(a) = \bar{a}(f)$, where $\bar{a}\colon F_n \to A$ is the unique homomorphism such that $\bar{a}(\pi_i) = a_i$ for each $i < n$.) Notice that this notation agrees with the literal usage of f as a function $f\colon [0,1]_{\mathbb{Q}}^n \to [0,1]_{\mathbb{Q}}$ in the case $A = [0,1]_{\mathbb{Q}}$.

We assume the reader is familiar with basic linear algebra. We identify vectors $v \in \mathbb{Q}^n$ with n-by-1 matrices, i.e., we view them as column vectors. In particular, if $v, w \in \mathbb{Q}^m$, then $v^{\mathsf{T}} w$ coincides with the inner product of v and w. We number coordinates of vectors, as well as rows and columns of matrices, starting from 0. If $v \in \mathbb{Q}^m$ and $i < m$, we denote the ith coordinate of v by v_i (though we will also use subscript indices for many other purposes).

3 The finite case

We first consider the easy case of finite-valued Łukasiewicz logics. The results in fact apply more generally to all n-contractive extensions of **BL**. We rely on the following key result.

Theorem 3.1 (Dzik [4]) *Let L be an n-contractive extension of* **BL** *for some $n \in \omega$. Then every L-unifiable formula is L-projective.* □

Corollary 3.2 *Let L be an n-contractive extension of* **BL** *for some $n \in \omega$. A rule $\Gamma \,/\, \Delta$ is L-admissible if and only if $\bigwedge \Gamma$ is not L-unifiable or there exists $\delta \in \Delta$ such that $\Gamma \vdash_L \delta$.*

Proof: Assume that $\Gamma \mathrel{|\!\!\sim}_L \Delta$, and let σ be a projective unifier of $\bigwedge \Gamma$. We have $\vdash_L \sigma\Gamma$, hence $\vdash_L \sigma\delta$ for some $\delta \in \Delta$, which implies $\Gamma \vdash_L \delta$. □

Lemma 3.3 *Let L be a consistent extension of* **BL**. *A formula φ is L-unifiable if and only if it is satisfiable in classical logic.*

Proof: Right-to-left: let e be a classical assignment such that $e(\varphi) = 1$, and let σ be the ground substitution such that $\sigma p_i = e(p_i)$. As **BL** can evaluate constant formulas, we have $\vdash_L \sigma\varphi$.

Left-to-right: let σ be a substitution such that $\vdash_L \sigma\varphi$. We may assume that σ is ground. As **BL** can evaluate constant formulas, we have $\vdash_L e(\varphi)$, where e is the classical assignment such that $e(p_i)$ is the value of the sentence σp_i. As L is consistent, we must have $e(\varphi) = 1$. □

Notice that the last lemma already holds for extensions of $\mathbf{FL_w}$ in place of **BL**. We do not have any use for this observation.

Corollary 3.4 *Let L be a decidable n-contractive extension of* **BL**. *Then admissibility in L is decidable.* □

Our main contribution in this section is a description of an explicit basis of admissible rules for n-contractive extensions of **BL**. By Corollary 3.2, this amounts to axiomatization of passive admissible rules.

Definition 3.5 We introduce the rules

$$(CC_n) \qquad \frac{\neg(p \vee \neg p)^n}{}$$

and their variants

$$(CC_n^1) \qquad \frac{\neg(p \vee \neg p)^n}{\bot}$$

for $n \in \omega$. Let $CC = \{CC_n \mid n \in \omega\}$, $CC^1 = \{CC_n^1 \mid n \in \omega\}$.

The theorem below actually holds for all extensions of **MTL**.

Theorem 3.6 *If L is an extension of* **BL**, *then CC^1 is a basis of single-conclusion passive L-admissible rules. If L is consistent, then CC is a basis of multiple-conclusion passive L-admissible rules.*

Proof: Clearly, CC_n are passive admissible rules by Lemma 3.3. On the other hand, assume that the rule $\Gamma \,/\, \Delta$ is passive, i.e., the formula $\bigwedge \Gamma$ is not unifiable. Then $\neg \bigwedge \Gamma$ is a classical tautology. As $\mathbf{CPC} = \mathbf{BL} + p \vee \neg p$, there are formulas ψ_i such that

$$\vdash_L \prod_{i<n} (\psi_i \vee \neg \psi_i) \to \neg \bigwedge \Gamma$$

by the deduction theorem, where $\prod$ is to $\cdot$ as $\bigwedge$ is to $\wedge$. Put $\psi = \bigwedge_{i<n} (\psi_i \vee \neg \psi_i)$. Since **BL** proves De Morgan laws and $\neg(p \vee \neg p) \to q \vee \neg q$, we have $\vdash_L \neg \psi \to \psi$, hence $\vdash_L \psi \vee \neg \psi \to \psi_i \vee \neg \psi_i$ for every i. Thus

$$\vdash_L (\psi \vee \neg \psi)^n \to \neg \bigwedge \Gamma$$

and

$$\Gamma \vdash_L \neg(\psi \vee \neg \psi)^n,$$

hence $\Gamma \,/\, \Delta$ follows from an instance of CC_n. If $\Delta \neq \varnothing$, we can use CC_n^1 instead of CC_n. □

Note that in n-contractive logics, CC is equivalent to CC_n, and CC^1 is equivalent to CC_n^1.

Corollary 3.7 *If L is an n-contractive extension of* **BL** *for some $n \in \omega$, then CC_n^1 is a basis of single-conclusion L-admissible rules. If L is consistent, then CC_n is a basis of multiple-conclusion L-admissible rules.* □

4 The infinite case

In this section we are going to prove our main result: a characterization of admissible rules of the infinite-valued Łukasiewicz logic which establishes their decidability. Note that admissibility in **Ł** is a more complex problem than in the finite-valued case, it is no longer sufficient to characterize passive admissible rules: for example, the rule

$$\frac{p \vee \neg p \vee \neg\big(q^2 \vee (\neg q)^2\big) \qquad (q \vee \neg q)^4 \to p}{p}$$

from Example 4.20 is Ł-admissible, but it is neither derivable nor passive.

Recall that $\Gamma \mathrel{|\!\!\sim_{\text{Ł}}} \Delta$ if and only if $F_n \vDash \Gamma \;/\; \Delta$ for all $n \in \omega$. We first show that it suffices to consider only the case $n = 1$.

Theorem 4.1 *Ł is 1-reducible wrt admissible rules, i.e., for every inadmissible rule $\Gamma \;/\; \Delta$, there exists a substitution σ in only one variable such that $\vdash_{\text{Ł}} \sigma\Gamma$ and $\nvdash_{\text{Ł}} \sigma\delta$ for each $\delta \in \Delta$. In algebraic terms, $\Gamma \mathrel{|\!\!\sim_{\text{Ł}}} \Delta$ iff $F_1 \vDash \Gamma \;/\; \Delta$.*

Proof: If $\Gamma \mathrel{|\!\!\not\sim_{\text{Ł}}} \Delta$, there exists an $n \in \omega$ such that $F_n \nvDash \Gamma \;/\; \Delta$. Let e be a valuation in F_n such that $\Gamma(e) = 1$ and $\delta(e) \neq 1$ for all $\delta \in \Delta$. We can represent e by a piecewise linear function with integer coefficients $e\colon [0,1]^n_{\mathbb{Q}} \to [0,1]^k_{\mathbb{Q}}$, where k is such that $\Gamma \cup \Delta$ uses only variables $p_0, \dots, p_{k-1}$. We enumerate $\Delta = \{\delta_i \mid i < r\}$. We may assume $r > 0$ without loss of generality.

Consider an $i < r$. There exists an $x^i \in [0,1]^n_{\mathbb{Q}}$ such that $\delta_i(e(x^i)) < 1$. We can write $x^i = \langle p_0/q, \dots, p_{n-1}/q\rangle$ for some natural numbers $\vec{p}, q$ such that $q \geq 2$, and we define a function $f^i\colon [0,1]_{\mathbb{Q}} \to [0,1]^n_{\mathbb{Q}}$ by

$$f^i_j(t) = \begin{cases} \min\{p_j t, 1\} & t \leq 1/2, \\ \min\{p_j(1-t), 1\} & t \geq 1/2. \end{cases}$$

Then $f^i \in F_1^n$, $f^i(0) = f^i(1) = \vec{0}$, and $f^i(1/q) = x^i$.

Let us define $f\colon [0,1]_{\mathbb{Q}} \to [0,1]^n_{\mathbb{Q}}$ by

$$f(t) = f^i(rt - i), \qquad i/r \leq t \leq (i+1)/r.$$

Notice that $f(i/r) = \vec{0}$ is well-defined. By the construction, $f \in F_1^n$, and for each $\delta \in \Delta$ there exists $t \in [0,1]_{\mathbb{Q}}$ such that $\delta(e(f(t))) < 1$. Trivially $\gamma(e(f(t))) = 1$ for each t and $\gamma \in \Gamma$, hence the valuation $e \circ f\colon [0,1]_{\mathbb{Q}} \to [0,1]^k_{\mathbb{Q}}$ witnesses that $F_1 \nvDash \Gamma \;/\; \Delta$. □

Corollary 4.2 *All free MV-algebras F_κ, $\kappa \neq 0$, have the same universal theory.* □

We can equivalently restate the last corollary as follows: every finite partial subalgebra of F_κ can be embedded in any F_λ ($\lambda > 0$). Nevertheless, it is not possible to embed all of F_κ in F_λ at once unless $\kappa \leq \lambda$:

Theorem 4.3 *If $\kappa > \lambda$ are cardinals, then F_κ is not embeddable in F_λ.*

Proof: It is easy to see that it suffices to show the result for finite κ and λ. Assume for contradiction that $\varphi\colon F_\kappa \to F_\lambda$ is an embedding, and consider the continuous piecewise linear function $f\colon [0,1]^\lambda_{\mathbb{Q}} \to [0,1]^\kappa_{\mathbb{Q}}$ such that $f_i = \varphi(x_i)$, where $\{x_i \mid i < \kappa\}$ are the free generators of F_κ. We can extend f to a continuous piecewise linear function $\hat{f}\colon [0,1]^\lambda_{\mathbb{R}} \to [0,1]^\kappa_{\mathbb{R}}$. As piecewise linear functions do not increase topological dimension, we have $\dim(\operatorname{rng}(\hat{f})) \leq \dim([0,1]^\lambda_{\mathbb{R}}) = \lambda < \kappa = \dim([0,1]^\kappa_{\mathbb{R}})$, hence $\hat{f}$ is not onto. Being a continuous image of a compact space, $\operatorname{rng}(\hat{f})$ is closed, hence there exists a point $v \in [0,1]^\kappa_{\mathbb{Q}}$ and an $\varepsilon > 0$ such that

$\mathrm{rng}(\hat{f}) \cap \prod_{i<\kappa}[v_i - \varepsilon, v_i + \varepsilon] = \varnothing$, where $\prod$ denotes Cartesian product. For each $i < \kappa$, we can write $v_i = p_i/q_i$ for some integers p_i, q_i such that $q_i > 1/\varepsilon$, and define $g^i \colon [0,1]_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ by

$$g^i(t) = \min\{1, |p_i - q_i t|\}.$$

We have $g^i(v_i) = 0$, and $g^i(u) = 1$ for all $u \notin [v_i - \varepsilon, v_i + \varepsilon]$. We define $g \colon [0,1]^\kappa_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ by

$$g(t_0, \dots, t_{\kappa-1}) = \max\{g^i(t_i) \mid i < \kappa\}.$$

By the construction, $g \in F_\kappa$, $g(v) = 0$, and $g(x) = 1$ for all $x \in \mathrm{rng}(f)$. Thus $\varphi(g) = g \circ f = 1 = \varphi(1)$, but $g \neq 1$, which contradicts φ being an embedding. $\square$

Since we will work a lot with F_1, we introduce convenient notation for elements of F_1^m, as well as other continuous piecewise linear functions in one variable.

Definition 4.4 If $t_0 < t_1 < \cdots < t_k$ are rational numbers and $x_0, \dots, x_k \in \mathbb{Q}^m$, then $L(t_0, x_0; t_1, x_1; \dots; t_k, x_k)$ is the continuous piecewise linear function $f \colon [t_0, t_k]_{\mathbb{Q}} \to \mathbb{Q}^m$ such that $f(t_i) = x_i$, and f is linear on each interval $[t_i, t_{i+1}]$.

The concept of a rational piecewise linear function is straightforward enough, however the definition of F_1 also involves the condition of coefficients being integers, hence we have to understand what that means. Moreover, if $\Gamma \,/\, \Delta$ is a rule in m variables, $0 < t_1 < \cdots < t_k < 1$ and $x_0, \dots, x_{k+1} \in [0,1]^m_{\mathbb{Q}}$, then the validity of $\Gamma \,/\, \Delta$ under the valuation $e = L(0, x_0; t_1, x_1; \dots; 1, x_{k+1})$ is completely determined by the vectors x_i, it does not depend on the parametrization of the function: if we reparametrize it as $e' = L(0, x_0; t'_1, x_1; \dots; 1, x_{k+1})$ for some $0 < t'_1 < \cdots < t'_k < 1$, then e satisfies $\Gamma \,/\, \Delta$ if and only if e' does. For this reason, we will investigate the following question: given $x_0, \dots, x_k \in \mathbb{Q}^m$, when do there exist rational $t_0 < \cdots < t_k$ such that $L(t_0, x_0; \dots; t_k, x_k)$ has integer coefficients?

The answer given by Lemma 4.10 involves the concept of anchoredness:

Definition 4.5 Let $X \subseteq \mathbb{Q}^m$. The *convex hull* $C(X)$ is the smallest convex subset of $\mathbb{Q}^m$ which includes X, and the *affine hull* $A(X)$ is the smallest affine subspace of $\mathbb{Q}^m$ which includes X. Notice that

$$C(X) = \bigcup_{\substack{Y \subseteq X \\ Y \text{ finite}}} C(Y),$$

$$A(X) = \bigcup_{\substack{Y \subseteq X \\ Y \text{ finite}}} A(Y),$$

$$C(x_0, \dots, x_{k-1}) = \Big\{ \sum_{i<k} \alpha_i x_i \;\Big|\; \alpha_i \in \mathbb{Q}, \alpha_i \geq 0, \sum_{i<k} \alpha_i = 1 \Big\},$$

$$A(x_0, \dots, x_{k-1}) = \Big\{ \sum_{i<k} \alpha_i x_i \;\Big|\; \alpha_i \in \mathbb{Q}, \sum_{i<k} \alpha_i = 1 \Big\}.$$

A set $X \subseteq \mathbb{Q}^m$ is *anchored* if $A(X) \cap \mathbb{Z}^m \neq \varnothing$.

Example 4.6 Let $x = \langle \frac{3}{2}, \frac{3}{2}, 2\rangle$, $y = \langle \frac{4}{3}, 2, \frac{4}{3}\rangle$, and $z = \langle 1, \frac{3}{2}, \frac{3}{4}\rangle$. Then $\{x, y, z\} \subseteq \mathbb{Q}^3$ is anchored, since $A(x, y, z)$ contains the point $\frac{2}{3}x - y + \frac{4}{3}z = \langle 1, 1, 1\rangle \in \mathbb{Z}^3$. The set $\{\langle 1, 1, -\frac{3}{2}\rangle, \langle \frac{1}{3}, \frac{1}{6}, 0\rangle\}$ is not anchored: its affine hull is the line defined be the system of linear equations $\{x_0 + x_1 + x_2 = \frac{1}{2}, 5x_0 - 4x_1 = 1\}$, which has no integer solution.

We first provide a characterization (Lemma 4.8) of the condition of anchoredness, which also entails its decidability. Strictly speaking, we do not need the characterization, we could show decidability of anchoredness in another way (cf. the proof of Theorem 4.22). Nevertheless, we decided to include it because we feel that it provides a useful insight into the notion of anchoredness, which allows us to understand it better.

Recall that a matrix is in *row-echelon form* if all nonzero rows are above any rows of all zeros, and the left-most nonzero entry in any row is strictly to the right of the left-most nonzero entry of any row above it.

Lemma 4.7 *For any $X \in \mathbb{Q}^{m\times k}$, there exists an $M \in GL(m, \mathbb{Z})$ such that MX is in row-echelon form.*

Proof: We may assume without loss of generality that $X \in \mathbb{Z}^{m\times k}$. The effect of multiplication by M from left is to perform certain operations on the rows of X. In particular, $GL(m, \mathbb{Z})$ contains all permutation matrices, whose effect is to permute the rows of X, and integer matrices which differ from the identity matrix only in one element which is not on the diagonal, whose effect is to add an integer multiple of some row of X to another row. It thus suffices to show that we can transform X into row-echelon form by a sequence of these row operations.

The proof goes by induction on m. If X is the zero matrix, there is nothing to do. Otherwise let j be the first nonzero column of X. If $x_{i,j}$ and $x_{i',j}$ are two nonzero elements of the jth column and $|x_{i,j}| \le |x_{i',j}|$, we may add a suitable multiple of the ith row to the i'th row to reduce $x_{i',j}$ modulo $x_{i,j}$. This operation makes the quantity $\sum_i |x_{i,j}|$ strictly smaller, hence after a finite number of similar steps we reach the situation that the jth column contains only one nonzero entry $x_{i,j}$. We may permute the rows to ensure $i = 0$, and using the induction hypothesis we apply some operations on the remaining rows to bring the rest of the matrix to row-echelon form. □

Lemma 4.8 *The following are equivalent for any $X \subseteq \mathbb{Q}^m$.*

(i) X is anchored.

(ii) For every $u \in \mathbb{Z}^m$ and $a \in \mathbb{Q}$, if $u^\mathsf{T} x = a$ for all $x \in X$, then $a \in \mathbb{Z}$.

Proof: (i) → (ii): $\{x \in \mathbb{Q}^m \mid u^\mathsf{T} x = a\}$ is an affine space containing X, hence it includes $A(X)$, which contains an integer point x. Then $a = u^\mathsf{T} x \in \mathbb{Z}$.

(ii) → (i): Clearly $X \neq \varnothing$. Since the affine space $A(X)$ is finitely dimensional, we can find finitely many points $x_0, \dots, x_k \in X$ such that $A(X) = A(x_0, \dots, x_k)$. Put $y_i = x_i - x_k$ for $i < k$, and let Y be the m-by-k matrix whose ith column is y_i. We have

$$A(X) = \{x_k + Yv \mid v \in \mathbb{Q}^k\}.$$

The condition (ii) then states that for every $u \in \mathbb{Z}^m$, if $u^\top Y = 0$, then $u^\top x_k \in \mathbb{Z}$.

By Lemma 4.7, there exists a matrix $M \in GL(m, \mathbb{Z})$ such that $Y' = MY$ is in row-echelon form. Put $x' = Mx_k$, and let r be the number of nonzero rows of Y'. Consider any $i \geq r$. If $u = M^\top e_i$, where e_i is the ith basis vector, we have $u \in \mathbb{Z}^m$ and $u^\top Y = e_i^\top Y' = 0$, hence $e_i^\top x' = u^\top x_k \in \mathbb{Z}$. Thus, $x'_i \in \mathbb{Z}$ for all $i \geq r$. As the first r rows of Y' are linearly independent, there exists $v \in \mathbb{Q}^k$ such that $Y'v = \langle -x'_0, \dots, -x'_{r-1}, 0, \dots, 0\rangle$. Then $x' + Y'v = \langle 0, \dots, 0, x'_r, \dots, x'_{m-1}\rangle \in \mathbb{Z}^m$, hence $A(X) \ni x_k + Yv = M^{-1}(x' + Y'v) \in \mathbb{Z}^m$. □

Corollary 4.9 *Given $x_0, \dots, x_k \in \mathbb{Q}^m$, it is decidable whether $\{x_0, \dots, x_k\}$ is anchored.*

Proof: Being anchored is r.e. by definition, and co-r.e. by Lemma 4.8, hence it is decidable. In fact, the proofs of Lemmas 4.8 and 4.7 provide a more efficient explicit algorithm. □

Lemma 4.10 *If $x_0, \dots, x_k \in \mathbb{Q}^m$, then the following are equivalent.*

(*i*) *There exist rationals $t_0 < \cdots < t_k$ such that $L(t_0, x_0; \dots; t_k, x_k)$ has integer coefficients.*

(*ii*) *$\{x_i, x_{i+1}\}$ is anchored for each $i < k$.*

Proof: (i) → (ii): If $f = L(t_i, x_i; t_{i+1}, x_{i+1})$ has integer coefficients, then $f(0) \in \mathbb{Z}^m \cap A(x_i, x_{i+1})$.

(ii) → (i): By induction on k. If $k = 0$, any t_0 will do, as the condition on coefficients is vacuously true. Assume $k > 0$, and let $t_1 < \cdots < t_k$ be such that $L(t_1, x_1; \dots; t_k, x_k)$ has integer coefficients by the induction hypothesis. We may add a sufficiently large integer to all t_i, hence we may assume $t_1 > 0$ without loss of generality. Choose an integer $e > 0$ such that $e(x_1 - x_0) \in \mathbb{Z}^m$, and an $\alpha \in \mathbb{Q}$ such that $b = \alpha x_0 + (1 - \alpha)x_1 \in \mathbb{Z}^m$. By adding an integer multiple of e to α we can ensure that $\alpha > 0$. We write $\alpha = p/q$, $t_1 = r/s$ for some natural numbers p, q, r, s. Since we may divide all t_i by eqr, we may assume that $r = 1$ and $eq \mid s$. Let

$$a = \frac{\alpha}{t_1}(x_1 - x_0) = p\frac{s}{eq}(e(x_1 - x_0)) \in \mathbb{Z}^m,$$
$$t_0 = (1 - \alpha^{-1})t_1.$$

We have $t_0 < t_1$, and

$$t_1 a + b = \alpha(x_1 - x_0) + \alpha x_0 + (1 - \alpha)x_1 = x_1,$$
$$t_0 a + b = (\alpha - 1)(x_1 - x_0) + \alpha x_0 + (1 - \alpha)x_1 = x_0,$$

hence $L(t_0, x_0; t_1, x_1; \dots; t_k, x_k)$ has integer coefficients. □

A remarkable feature of Lemma 4.10 is that the linear segments do not interact with each other: in order to find a parametrization so that $L(t_0, x_0; \dots; t_k, x_k)$ has integer coefficients, it is sufficient to parametrize individually each $L(t, x_i; t', x_{i+1})$ to have integer coefficients.

Example 4.11 Let $x_0 = \langle 1, 1\rangle$, $x_1 = \langle \frac{1}{4}, \frac{1}{4}\rangle$, $x_2 = \langle \frac{1}{3}, \frac{1}{5}\rangle$, $x_3 = \langle \frac{1}{6}, \frac{1}{2}\rangle$, and $x_4 = \langle 0, 1\rangle$. We can parametrize the line segments $C(x_i, x_{i+1})$ to have integer coefficients as follows:

$L(0,x_0;\frac{3}{4},x_1)=\langle 1-t,1-t\rangle$, $L(\frac{1}{4},x_1;\frac{4}{15},x_2)=\langle 5t-1,1-3t\rangle$, $L(\frac{2}{15},x_2;\frac{1}{6},x_3)=\langle 1-5t,9t-1\rangle$, and $L(-\frac{1}{6},x_3;0,x_4)=\langle -t,1+3t\rangle$. Notice that we cannot directly put these parametrizations together, since they do not match at the endpoints. Nevertheless, by Lemma 4.10 there exist $t_0<t_1<t_2<t_3<t_4$ such that $L(t_0,x_0;\dots;t_4,x_4)$ has integer coefficients. Indeed, we can take for example $f=L(0,x_0;\frac{1}{8},x_1;\frac{2}{15},x_2;\frac{1}{6},x_3;\frac{1}{5},x_4)$, since

$$f(t)=\begin{cases}\langle 1-6t,1-6t\rangle, & 0\le t\le \frac{1}{8},\\ \langle 10t-1,1-6t\rangle, & \frac{1}{8}\le t\le \frac{2}{15},\\ \langle 1-5t,9t-1\rangle, & \frac{2}{15}\le t\le \frac{1}{6},\\ \langle 1-5t,15t-2\rangle, & \frac{1}{6}\le t\le \frac{1}{5}.\end{cases}$$

Definition 4.12 A (convex) *polytope* is a set of the form $\{x\in\mathbb{Q}^m \mid \forall i<k\, L_i(x)\ge 0\}$, where L_i are linear (more precisely, affine) functions with integer (or rational, it makes no difference) coefficients.

The following is an effective version of the easy part of Theorem 2.3. Let us define the L^1-norm $\|f\|_1$ of a linear function f to be the sum of absolute values of its coefficients.

Lemma 4.13 *Let Γ be a finite set of formulas in m variables closed under subformulas, and $n=|\Gamma|$. For all $j<2^n$, $i<n$, and $\varphi\in\Gamma$, we can compute linear functions $L_{j,i}$ and $L_{j,\varphi}$ with integer coefficients and L^1-norm at most n such that the polytopes*

$$C_j=\{x\in[0,1]^m_{\mathbb{Q}} \mid \forall i<n\, L_{j,i}(x)\ge 0\}$$

satisfy

$$\bigcup_{j<2^n} C_j=[0,1]^m_{\mathbb{Q}},$$

and

$$L_{j,\varphi}(x)=\varphi(x)$$

for each $x\in C_j$ and $\varphi\in\Gamma$.

Proof: By induction on n. We assume for simplicity that all formulas are expressed in the basis $\{\to,\bot\}$. If Γ consists only of variables or $\bot$, we can take $L_{j,i}=0$, $L_{j,p_i}=x_i$, and $L_{j,\bot}=0$. Otherwise we pick a formula $\psi\to\chi\in\Gamma$ which is not a proper subformula of any formula from Γ. We can apply the induction hypothesis to obtain $\{L_{j,i}\mid j<2^{n-1}, i<n-1\}$ and $\{L_{j,\varphi}\mid j<2^{n-1},\varphi\in\Gamma\smallsetminus\{\psi\to\chi\}\}$. We put $L'_{2j,i}=L'_{2j+1,i}=L_{j,i}$ for $i<n-1$, $L'_{2j,n-1}=-L'_{2j+1,n-1}=L_{j,\psi}-L_{j,\chi}$, $L'_{2j,\varphi}=L'_{2j+1,\varphi}=L_{j,\varphi}$ for $\varphi\in\Gamma\smallsetminus\{\psi\to\chi\}$, $L'_{2j,\psi\to\chi}=1-L_{j,\psi}+L_{j,\chi}$, $L'_{2j+1,\psi\to\chi}=1$. By induction on the complexity of φ, it is easy to see that $\|L_{j,\varphi}\|_1\le|\varphi|\le n$, hence also $\|L_{j,i}\|_1\le n$. It is straightforward to verify the other required properties using the definition of $\to$ in $[0,1]_{\text{Ł}}$. $\square$

Example 4.14 The exponential number of polytopes in Lemma 4.13 cannot be significantly reduced. For example, let Γ be the set of all subformulas of the formula

$$\varphi=x_0^2\oplus x_1^2\oplus\cdots\oplus x_{n-1}^2,$$

which has length $O(n)$ (and thus $|\Gamma| = O(n)$). For any subset $I \subseteq \{0, \dots, n-1\}$, we have

$$\varphi = 2 \sum_{i \in I} x_i - |I|$$

if $x_i \leq \frac{1}{2}$ for all $i \notin I$, $x_i \geq \frac{1}{2}$ for all $i \in I$, and $2\sum_{i\in I} x_i \leq 1 + |I|$. Since all these linear functions are distinct, we need at least 2^n polytopes to express Γ as in Lemma 4.13.

The main problem in showing decidability of admissibility in **Ł** is that potential counterexamples $L(t_0, x_0; t_1, x_1; \dots; t_k, x_k)$ to a rule $\Gamma \ / \ \Delta$ in F_1 may have arbitrary length, hence we must find a way how to shorten them. The basic idea is as follows. The formulas from Γ define piecewise linear functions, and the domain of each piece is a polytope, let thus consider a particular polytope C such that all formulas from Γ are linear on C, and a part $L(t_i, x_i; t_{i+1}, x_{i+1}; \dots; t_j, x_j)$ of the valuation such that $x_i, \dots, x_j \in C$. We could try to simply replace this part with $L(t_i, x_i; t_j, x_j)$: since $\Gamma(x_i) = \Gamma(x_j) = 1$, and $\mathrm{rng}(L(t_i, x_i; t_j, x_j)) = C(x_i, x_j) \subseteq C$, we have $\Gamma(x) = 1$ for each x in the range of such a function, which is the main thing we have to preserve. However, there is no guarantee that we can reparametrize $L(t_i, x_i; t_j, x_j)$ to have integer coefficients. For example, consider the case $i = 1$, $j = 3$, and $x_2 = \vec{0}$: then $\{x_1, x_2\}$ and $\{x_2, x_3\}$ are anchored for any x_1 and x_3, but $\{x_1, x_3\}$ need not be anchored. Fortunately, it cannot get any worse, it turns out that we can do in just two steps what we cannot do in one step: there is $x \in C$ such that $\{x_i, x\}$ and $\{x, x_j\}$ are anchored. This will follow from Lemma 4.16.

Lemma 4.15 *Let X be a nonempty convex subset of $\mathbb{Q}^m$. There exists a point $x \in X$ and an open neighbourhood $U \ni x$ such that $A(X) \cap U \subseteq X$.*

Proof: Since $A(X)$ has finite dimension, we can find points $x_0, \dots, x_k \in X$ such that $A(X) = A(x_0, \dots, x_k)$, and the x_i's are affinely independent (i.e., $x_i \notin A(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$ for each i). Put

$$x = \frac{1}{k+1} \sum_{i \leq k} x_i.$$

Let $A = \{\alpha \in \mathbb{Q}^{k+1} \mid \sum_i \alpha_i = 1\}$, and define a mapping $\varphi\colon A \to \mathbb{Q}^m$ by

$$\varphi(\alpha) = \sum_{i \leq k} \alpha_i x_i.$$

Then φ is a homeomorphism of A onto $A(X)$ which maps the subset $B = \{\alpha \in A \mid \forall i \, \alpha_i \geq 0\}$ onto $C(x_0, \dots, x_k)$. The point $a = \langle 1/(k+1), \dots, 1/(k+1) \rangle$ is in the interior of B (in A), hence $\varphi(a) = x$ is in the interior of $C(x_0, \dots, x_k) \subseteq X$ relative to $A(X)$. □

Lemma 4.16 *Let X be an anchored subset of $\mathbb{Q}^m$, and $x_0, \dots, x_k \in \mathbb{Q}^m$. Then there exists $w \in C(X)$ such that $\{x_i, w\}$ is anchored for each $i \leq k$.*

Proof: We may assume that X is convex without loss of generality. Fix $c \in A(X) \cap \mathbb{Z}^m$, and let $x \in X$ and U be as in Lemma 4.15. If $c \in X$, we can take $w = c$. Otherwise $c \neq x$, hence

$A(c,x)$ is a line, and its intersection with U must contain two distinct points y,z. We have $y,z \in A(X) \cap U \subseteq X$, and $c \in A(y,z) \cap \mathbb{Z}^m$.

We write $w_\alpha = (1-\alpha)y + \alpha z$. Fix α such that $w_\alpha = c$, and let $\beta > 0$ be an integer such that $\beta(z-y) \in \mathbb{Z}^m$. For every $i \le k$, we can find an integer $\gamma_i > 0$ such that $\gamma_i(w_\alpha - x_i) \in \mathbb{Z}^m$. Then $w_{\alpha+p\beta} + q\gamma_i(w_\alpha - x_i) \in \mathbb{Z}^m$ for every $p,q \in \mathbb{Z}$. If $q \ge 0$, we have

$$\begin{aligned} w_{\alpha+p\beta} + q\gamma_i(w_\alpha - x_i) &= (1+q\gamma_i)w_\alpha - q\gamma_i x_i + p\beta(z-y) \\ &= (1+q\gamma_i)w_{\alpha+p\beta/(1+q\gamma_i)} - q\gamma_i x_i \in A\bigl(x_i, w_{\alpha+p\beta/(1+q\gamma_i)}\bigr), \end{aligned}$$

hence $\bigl\{x_i, w_{\alpha+p\beta/(1+q\gamma_i)}\bigr\}$ is anchored. Let $\gamma > \beta$ be a common multiple of all γ_i. There exists an integer p such that $\delta = \alpha + p\beta/(1+\gamma) \in [0,1]$. Then $w_\delta \in C(y,z) \subseteq X$, and $\{x_i, w_\delta\}$ is anchored for all $i \le k$. □

Now we have all tools in order, and we can proceed to our main characterization.

Theorem 4.17 *Let Γ, Δ be finite sets of formulas in variables $\{p_i \mid i < m\}$, and $\{C_j \mid j < r\}$ a set of polytopes in $\mathbb{Q}^m$ such that*

$$\bigcup_{j<r} C_j = \{x \in [0,1]^m_{\mathbb{Q}} \mid \Gamma(x) = 1\}.$$

The following are equivalent.

- (i) $\Gamma \not\vdash_{\text{Ł}} \Delta$.
- (ii) *There exists $a \in \{0,1\}^m$ such that $\Gamma(a) = 1$ and for every $\delta \in \Delta$ there exists a sequence $\{j_i \mid i \le k\}$ of indices $j_i < r$ such that*
  - (α) *j_i are pairwise distinct, in particular, $k < r$,*
  - (β) *$a \in C_{j_0}$,*
  - (γ) *C_{j_i} is anchored for each $i \le k$,*
  - (δ) *$C_{j_i} \cap C_{j_{i+1}} \neq \varnothing$ for each $i < k$,*
  - (ε) *there exists $x \in C_{j_k}$ such that $\delta(x) < 1$.*

Proof: (i) → (ii): We may pick $f \in F^m_1$ such that $\Gamma(f) = 1$ and $\delta(f) \neq 1$ for each $\delta \in \Delta$ by Theorem 4.1. We can represent f as $L(t_0, x_0; \dots; t_s, x_s)$ with integer coefficient for some $0 = t_0 < t_1 < \dots < t_s = 1$ and $x_0, \dots, x_s \in [0,1]^m_{\mathbb{Q}}$. Since $\operatorname{rng}(f) \subseteq \bigcup_j C_j$ and the intersection of $[t_i, t_{i+1}]$ with any $f^{-1}[C_j]$ is a (possibly empty or degenerate) interval, we can refine the sequence of t_i's and x_i's to ensure that the range of each $L(t_i, x_i; t_{i+1}, x_{i+1})$ is included in some C_j. Put $a = x_0$. We have $a \in \mathbb{Z}^m$ since a is the constant coefficient of $L(0, x_0; t_1, x_1)$, and as $a \in [0,1]^m$, we must have $a \in \{0,1\}^m$.

Consider any $\delta \in \Delta$. There exists $t \in [0,1]_{\mathbb{Q}}$ such that $\delta(f(t)) < 1$. Let $k < s$ be such that $t \in [t_k, t_{k+1}]$. For every $i \le k$, pick $j_i < r$ such that $C(x_i, x_{i+1}) = \operatorname{rng}(L(t_i, x_i; t_{i+1}, x_{i+1})) \subseteq C_{j_i}$. The construction immediately implies conditions (β) and (ε). Since $L(t_i, x_i; t_{i+1}, x_{i+1})$ has integer coefficients, $C_{j_i} \supseteq \{x_i, x_{i+1}\}$ is anchored by Lemma 4.10, hence (γ) holds. Clearly,

$x_{i+1} \in C_{j_i} \cap C_{j_{i+1}}$, thus ($\delta$). It remains to satisfy (α). If $j_i = j_{i'}$ for some $i < i'$, we may modify the sequence by replacing the subsequence $j_i, j_{i+1}, \dots, j_{i'}$ with just j_i. This makes the sequence shorter, and conditions (β)–(ε) remain true, hence after finitely many steps we obtain an injective sequence $\{j_i \mid i \le k\}$.

(ii) $\to$ (i): We fix $a \in \{0,1\}^m$ as in (ii). If $\Delta = \varnothing$, then the constant function a is a valuation in F_1 which refutes $\Gamma \mathbin{/} \Delta$. Otherwise we enumerate $\Delta = \{\delta_p \mid p \le s\}$. Consider any $p \le s$, and let $\{j_i \mid i \le k_p\}$ be a sequence as in (ii). We put $x^p_0 = a$, and find $x^p_{2k_p+2} \in C_{j_{k_p}}$ such that $\delta_p(x^p_{2k_p+2}) < 1$. For every $i < k_p$ we pick $x^p_{2(i+1)} \in C_{j_i} \cap C_{j_{i+1}}$. Since each C_{j_i} is convex and anchored, we can find $x^p_{2i+1} \in C_{j_i}$ such that $\{x^p_{2i}, x^p_{2i+1}\}$ and $\{x^p_{2i+2}, x^p_{2i+1}\}$ are anchored by Lemma 4.16. We form the sequence

$$x^0_0, x^0_1, \dots, x^0_{2k_0+2}, x^0_{2k_0+1}, \dots, x^0_1, x^0_0 = x^1_0, x^1_1, \dots,$$
$$\dots, x^{s-1}_0 = x^s_0, x^s_1 \dots, x^s_{2k_s+2}, x^s_{2k_s+1}, \dots, x^s_1, x^s_0,$$

and relabel it as $x_0, \dots, x_k$ to simplify the notation, where $k = 4\sum_{p \le s}(k_p + 1)$. By the construction, we have the following properties for each applicable i:

- $x_i \in [0,1]^m_{\mathbb{Q}}$, $x_0, x_k \in \{0,1\}^m$,
- $\{x_i, x_{i+1}\}$ is anchored,
- for each $\delta \in \Delta$ there exists i such that $\delta(x_i) < 1$,
- $\{x_i, x_{i+1}\}$ is included in some C_j, in particular, $\Gamma(x) = 1$ for every $x \in C(x_i, x_{i+1})$.

By Lemma 4.10, there exists a sequence of rational numbers $t_0 < t_1 < \cdots < t_k$ such that $L(t_0, x_0; \dots; t_k, x_k)$ has integer coefficients. As in the proof of Lemma 4.10, we may add a sufficiently large integer to all t_i to ensure $t_0 > 0$, and we may divide all t_i by a sufficiently large integer to ensure $t_k < 1$. Since $x_0, x_k \in \mathbb{Z}^m$, $f = L(0, x_0; t_0, x_0; t_1, x_1; \dots; t_k, x_k; 1, x_k)$ also has integer coefficients. Thus $f \in F^m_1$, and the aforementioned properties ensure that $\Gamma(f) = 1$ and $\delta(f) \ne 1$ for every $\delta \in \Delta$, hence $F_1 \nvDash \Gamma \mathbin{/} \Delta$, and $\Gamma \not\mathrel{|\!\!\sim}_{\text{Ł}} \Delta$. □

Theorem 4.18 *Given a rule $\Gamma \mathbin{/} \Delta$, it is decidable whether $\Gamma \mathrel{|\!\!\sim}_{\text{Ł}} \Delta$.*

Proof: Using Lemma 4.13 we compute a description of a sequence of polytopes $\{C'_j \mid j < r\}$ such that $\bigcup_j C'_j = [0,1]^m_{\mathbb{Q}}$, and every formula $\varphi \in \Gamma \cup \Delta$ is defined by a linear function on any C'_j. We put $C_j = \{x \in C'_j \mid \forall \gamma \in \Gamma\, \gamma(x) = 1\}$. Then the assumptions of Theorem 4.17 hold, it thus suffices to check whether the condition (ii) is true. We can do it by a brute-force search for possible $a \in \{0,1\}^m$ and $\{j_i \mid i \le k\}$, $j_i < r$, $k < r$. We only need to check that conditions (α)–(ε) can be algorithmically verified. Conditions (α) and (β) are immediate. We can verify (δ) and (ε) by any linear programming algorithm, as we can express δ by a linear function on C_{j_k}.

Finally, we need to verify whether C_{j_i} is anchored. This can be done as follows. We know from the theory of linear programming that C_{j_i} is the convex hull of its vertices, and each vertex can be described as the unique solution of a system of linear equations obtained from a subset of the defining inequalities of C_{j_i} by changing $\le$ to $=$. We can systematically list all

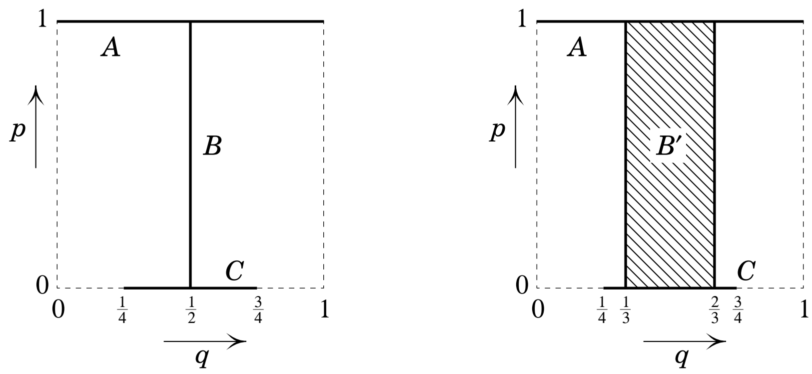

Figure 1: Truth sets of assumptions of the rules in Example 4.20

such linear systems, use Gaussian elimination to check whether it has a unique solution, and if so, to compute the solution, and verify whether it satisfies the remaining inequalities from C_{j_i}. In this way we obtain the list $x_0, \dots, x_k$ of all vertices of C_{j_i}, and then we can check whether $\{x_0, \dots, x_k\}$ is anchored by Corollary 4.9. □

Corollary 4.19 *The universal theory of free MV-algebras is decidable.* □

Example 4.20 Let us show that the rule

$$(*) \qquad \frac{p \lor \neg p \lor \neg\big(q^2 \lor (\neg q)^2\big) \qquad (q \lor \neg q)^4 \to p}{p}$$

is Ł-admissible using the criterion of Theorem 4.17. The set of points of $[0,1]^2$ where the assumptions of $(*)$ have value 1 is depicted in Figure 1 (left). It consists of three polytopes (line segments, in this case), denoted A, B, and C in the picture. Notice that B is not anchored, as its affine hull is the line $q = 1/2$, which does not hit any integer point.

Assume for contradiction that $(*)$ is not admissible. Then by Theorem 4.17, we can arrange some of the polytopes A, B, C into a sequence such that its first member contains an integer point, its last member contains a point where $\delta(p) = p < 1$, each member is anchored, and adjacent members intersect. Since B is not anchored, the sequence can only contain A and C. Moreover, A and C are disjoint, hence the sequence can only contain one of them; and this sole element cannot be C, because $C \cap \{0,1\}^2 = \varnothing$. The sequence thus contains only A, however $p = 1$ on A, a contradiction.

On the other hand, consider the rule

$$(**) \qquad \frac{p \lor \neg p \lor \neg\big(q^3 \lor (\neg q)^3\big) \qquad (q \lor \neg q)^4 \to p}{p}.$$

The set of points where its assumptions are true, depicted in Figure 1 (right), differs from $(*)$ in that the line segment B is expanded to a rectangle B'. Now, B' is anchored (as its affine hull is the whole plane), hence the sequence of polytopes A, B' witnesses that $(**)$ satisfies condition (ii) of Theorem 4.17, which means that $(**)$ is not Ł-admissible. Indeed, the reader can verify that the substitution $p/(\neg q \to q^2) \vee (q \to (\neg q)^2)$ unifies the assumptions of $(**)$, but not the conclusion.

4.1 Complexity

In this section we take a look on complexity issues concerning admissibility in **Ł**. First, Theorem 4.18 implies that there exists a computable bound on the size (number of bits) of a counterexample to an inadmissible rule. We provide explicit estimates below. Second, we give an upper bound on the computational complexity of $\mathrel|\joinrel\sim_{\text{Ł}}$, namely we show that it is computable in polynomial space (and therefore in exponential time).

Definition 4.21 The *height* $H(x)$ of a rational number x is $\max\{|p|,|q|\}$, where p, q are coprime integers such that $x = p/q$. More generally, if $x \in \mathbb{Q}^m$, its height $H(x)$ is

$$\max\{|qx_i| \bigm| i < m\} \cup \{q\},$$

where q is the smallest nonzero natural number such that $qx \in \mathbb{Z}^m$. Notice that the natural representation of x in binary takes $O(m \log H(x))$ bits.

Theorem 4.22 *Let $\Gamma \mathbin{/} \Delta$ be a rule of length $n = \sum_{\varphi \in \Gamma \cup \Delta} |\varphi|$ in m variables, and put $d = m \log(n/m + 1) \le n$. If $\Gamma \not\mathrel|\joinrel\sim_{\text{Ł}} \Delta$, there exists a valuation*

$$f = L(t_0 = 0, x_0; t_1, x_1; \ldots; t_k = 1, x_k) \in F_1^m$$

such that

- $k = 1 + O(|\Delta| 2^n) = O(n 2^n)$,
- $\gamma(f) = 1$ *for every* $\gamma \in \Gamma$,
- *for each* $\delta \in \Delta$ *there exists* $r \le k$ *such that* $\delta(x_r) < 1$,
- *for each* $r \le k$*:* $H(x_r) = 2^{O(md)}$, $H(t_r) = 2^{O(mdk)} = 2^{O(n^3 2^n)}$, *and there exists* $a_r \in A(x_r, x_{r+1}) \cap \mathbb{Z}^m$ *of height* $H(a_r) = 2^{O(md)}$.

Proof: Note that $t_r \in [0,1]$ and $x_r \in [0,1]^m$, hence their height is just a bound on the denominator q, whereas the height of a_r coincides with its L^∞-norm $\|a_r\|_\infty$. Let Σ be the set of all subformulas of $\Gamma \cup \Delta$. As $n \ge |\Sigma|$, by Lemma 4.13 we can write

$$\{x \in [0,1]^m_{\mathbb{Q}} \mid \Gamma(x) = 1\} = \bigcup_{j < 2^n} C_j,$$

where each C_j is described by a set of linear inequalities with L^1-norm at most n and integer coefficients. Also, every $\delta \in \Delta$ is defined on each C_j by a linear function with L^1-norm at most n and integer coefficients. We assume that condition (ii) of Theorem 4.17 is satisfied, and we will extract explicit bounds from the proof of the implication (ii) $\to$ (i). Recall that in the proof, we first chose the even-numbered vectors x_{2r} in a suitable way as an outline of the resulting valuation f, and then we fixed it up with the odd-numbered vectors x_{2r+1} so that $\{x_{2r}, x_{2r+1}\}$ and $\{x_{2r+1}, x_{2r+2}\}$ are anchored.

We first bound x_{2r}. By the construction, we have $x_{2r} = a \in \{0,1\}^m$, which has height 1, or we choose an arbitrary x_{2r} in some $C = C_j \cap C_{j'}$, or we choose $x_{2r} \in C = C_j$ such that $\delta(x_{2r}) < 1$ for some $\delta \in \Delta$. In the latter two cases, we may take for x_{2r} a vertex of the polytope C. Such a vertex is the unique solution of a system of linear equations obtained from a subset of the defining inequalities of C by replacing $\geq$ with $=$. If we take a minimal set of these equations, then their number must be m (a larger number of equations would be linearly dependent, and a smaller number cannot have a unique solution). We can thus write $Ax_{2r} = b$, where A is a regular m-by-m matrix, and the coefficients of A and b are integers such that each row of A has L^1-norm at most n. Then $x_{2r} = A^{-1}b$, and by Cramer's rule the height (i.e., common denominator) of x_{2r} is a divisor of $\det(A)$. We thus have $H(x_{2r}) \leq |\det(A)| \leq n^m = 2^{O(m \log n)}$.

In order to get a better bound, we exploit the structure of the defining inequalities of C using extension variables. Let us denote the original variables of $L_{j,i}$ and $L_{j,\varphi}$ by $\{u_i \mid i < m\}$ to avoid clashes with the vectors x_r. We assume for simplicity $C = C_j$, the case of $C = C_j \cap C_{j'}$ is similar. We know from above that x_{2r} is the unique solution of a regular system $\{L_i = 0 \mid i < m\}$ of linear equations, where each L_i is of the form $L_{j,i'}$ or $u_{i'}$ or $1 - u_{i'}$. In order to keep the notation manageable, we will assume that the first case always applies, we thus have $L_i = L_{j,\varphi_{2i}} - L_{j,\varphi_{2i+1}}$ for some formula $(\varphi_{2i} \to \varphi_{2i+1}) \in \Sigma$. We introduce new variables v_i corresponding to each φ_i, and extend the linear system by the equations $L_{j,\varphi_i} = v_i$, $i < 2m$. The new system has a unique solution $x'_{2r} = \langle x_{2r}, \varphi_0(x_{2r}), \ldots, \varphi_{2m-1}(x_{2r})\rangle \in [0,1]^{3m}_{\mathbb{Q}}$. In order to bound $H(x_{2r}) \leq H(x'_{2r})$, we express x'_{2r} as the unique solution of a different linear system as follows.

For each $i < m$ we fix an occurrence of $\varphi_{2i} \to \varphi_{2i+1}$ (and thus occurrences of φ_{2i}, φ_{2i+1}) as a subformula in $\Gamma \cup \Delta$. For each $i < 2m$, let φ'_i be the formula obtained from the fixed occurrence of φ_i by replacing topmost subformulas which are fixed occurrences of some $\varphi_{i'}$, $i' \neq i$, with the corresponding variable $v_{i'}$. For $i < m$, let $L_{\varphi'_i}$ be the linear function which defines φ'_i on C. Put $L'_i = L_{\varphi'_i} - v_i$ for $i < 2m$, and $L'_{2m+i} = v_{2i} - v_{2i+1}$ for $i < m$. Then it is easy to see that the system $\{L'_i = 0 \mid i < 3m\}$ uniquely defines x'_{2r}. As in Lemma 4.13, we have $\|L'_i\|_1 \leq 1 + |\varphi'_i|$ for $i < 2m$. By the construction, the formulas φ'_i with the $v_{i'}$ variables removed occur as *disjoint* parts of subformulas of $\Gamma \cup \Delta$, and each variable $v_{i'}$ occurs in at most one formula φ'_i, hence

$$\sum_{i<2m} |\varphi'_i| \leq n + 2m.$$

Thus, we have $Ax'_{2r} = b$, where A is a regular integer $3m$-by-$3m$ matrix, b is an integer vector, and if a_i denotes the L^1-norm of the ith row of A (which is the vector of coefficients of L'_i except for the constant coefficient), then $a_{2m+i} \leq 2$ for $i < m$, and

$$\sum_{i<2m} a_i \leq n + 4m.$$

Using Cramer's rule,

$$H(x_{2r}) \le H(x'_{2r}) \le |\det(A)| \le \prod_{i<3m} a_i \le 2^m \prod_{i<2m} a_i.$$

It is not hard to show that the product $\prod_{i<2m} a_i$ is maximized on the compact set

$$\Big\{\vec{a} \in [0,\infty)^{2m} \;\Big|\; \sum_{i<2m} a_i \le s\Big\}$$

when $a_i = s/2m$ for every i, hence

$$H(x_{2r}) \le 2^m \left(\frac{n+4m}{2m}\right)^{2m} = \left(\frac{n}{m}+1\right)^{O(m)} = 2^{O(d)}.$$

Now we proceed to bound x_{2r+1}. Recall that this point is chosen so that $\{x_{2r}, x_{2r+1}\}$ and $\{x_{2r+2}, x_{2r+1}\}$ are anchored, and $x_{2r+1} \in C_j$ for certain j, where we know that C_j is anchored. Let $\{u_0, \dots, u_l\}$ be a maximal affinely independent set of vertices of C_j. Clearly $l \le m$, and by the same reasoning as above, we have $H(u_i) = 2^{O(d)}$. Since C_j is the convex hull of its vertices, we must have $A(C_j) = A(u_0, \dots, u_l)$. For each $i < l$, let v_i be a multiple of $u_i - u_l$ such that $v_i \in \mathbb{Z}^m$ and $H(v_i) = 2^{O(d)}$. We have $A(C_j) = \{u_l + Vz \mid z \in \mathbb{Q}^l\}$, where V is the m-by-l integer matrix whose ith column is v_i. There exists a point $c = u_l + Vz \in A(C_j) \cap \mathbb{Z}^m$. We may add any integer vector to z, hence we may assume that $\|z\|_\infty \le 1$, thus $\|Vz\|_\infty \le \sum_i \|v_i\|_\infty = 2^{O(d)}$ and $H(c) = \|c\|_\infty = 2^{O(d)}$.

Put $x = \frac{1}{l+1}\sum_{i\le l} u_i$. Note that the denominator of x is at most the product of the denominators of the u_i's times $l+1$, hence $H(x) = 2^{O(md)}$. We need to find an $\varepsilon > 0$ such that $A(C_j) \cap U \subseteq C_j$, where U is the Euclidean ball around x with radius ε. The interior of $C(u_0, \dots, u_l)$ relative to $A(C_j)$ is the set $\big\{\sum_i \alpha_i u_i \mid \alpha_i > 0, \sum_i \alpha_i = 1\big\}$, hence a suitable ε is

$$\varepsilon = \min_{i\le l} \varrho(x, A(u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_l)) = \frac{1}{l+1} \min_{i\le l} \varrho(u_i, A(u_0, \dots, u_{i-1}, u_{i+1}, \dots, u_l)),$$

where ϱ denotes the Euclidean (L^2) distance. By symmetry, it suffices to find a lower bound on

$$\varrho(u_l, A(u_0, \dots, u_{l-1})) = \varrho(u_l - u_0, S),$$

where S is the linear span of $\{u_1 - u_0, \dots, u_{l-1} - u_0\}$. For each $i = 1, \dots, l-1$, let w_i be a multiple of $u_i - u_0$ such that $w_i \in \mathbb{Z}^m$ and $H(w_i) = 2^{O(d)}$. Let W be the m-by-$(l-1)$ integer matrix whose columns are the vectors w_i. As w_i are linearly independent, $W^\intercal W$ is regular, and we may define $P = W(W^\intercal W)^{-1}W^\intercal$. We have $P = P^\intercal$, $PW = W$, and $P = WX$ for some matrix X, hence P is the orthogonal projection on S. Thus

$$\begin{aligned}\varrho(u_l - u_0, S)^2 = \|(u_l - u_0) - P(u_l - u_0)\|_2^2 &= (u_l - u_0)^\intercal (I - P^\intercal)(I - P)(u_l - u_0) \\ &= (u_l - u_0)^\intercal (I - P)(u_l - u_0).\end{aligned}$$

Since W is an integer matrix, we have $DP \in \mathbb{Z}^{m\times m}$ for $D = |\det(W^\intercal W)| = 2^{O(md)}$ by Cramer's rule. Since also $H(u_l - u_0) = 2^{O(d)}$, the denominator of $\varrho(u_l - u_0, S)^2$ is $2^{O(md)}$. As u_i are affinely independent, we must have $\varrho(u_l - u_0, S) > 0$, hence we obtain

$$\varepsilon = 2^{-O(md)}.$$

Therefore we can find $y \neq z$, $y, z \in C_j \cap A(c,x)$ such that $H(y), H(z) = 2^{O(md)}$. Then, using the notation of Lemma 4.16, we have $H(\alpha) = 2^{O(md)}$, $\beta = 2^{O(md)}$, $\gamma_i = 2^{O(d)}$, $\gamma = 2^{O(md)}$, and $H(\delta) = 2^{O(md)}$, hence the height of $x_{2r+1} = w_\delta$ is $2^{O(md)}$.

Also $p = 2^{O(md)}$, hence the height of the integer point $a_{2r} = w_{\alpha+p\beta} + \gamma(w_\alpha - x_{2r}) \in A(x_{2r}, x_{2r+1})$ is $2^{O(md)}$, and similarly for a_{2r+1}.

It remains to analyze the inductive argument in Lemma 4.10. In order to get a decent bound, we modify the induction hypothesis: we require $0 < t_0 < \dots < t_k < 1$, and $1/t_0 \in \mathbb{Z}$. Consider the induction step. Using the notation of Lemma 4.10, we have $e = 2^{O(md)}$ and $H(\alpha) = 2^{O(md)}$. We may ensure $\alpha > 2$. What happens to the t_r's is that we first divide all t_r by $eq = 2^{O(md)}$, and we put $t_0 = (1-\alpha^{-1})t_1$. Then $t_0 \geq t_1/2$, hence the extra conditions $t_0 > 0$ and $t_k < 1$ are satisfied. Since $t_0 = (p-q)/ps$, it suffices to divide all t_r by $p - q = 2^{O(md)}$ in order to satisfy the condition $t_0^{-1} \in \mathbb{Z}$. All in all, we see that $\max_r H(t_r)$ is multiplied by at most $2^{O(md)}$ in each step, hence the final $H(t_r)$ is $2^{O(kmd)}$. (In fact, the same argument shows $H(\langle t_0, \dots, t_k \rangle) = 2^{O(kmd)}$.) □

Theorem 4.23 *Admissibility in* **Ł** *is in PSPACE. More precisely, we can test admissibility of multiple-conclusion rules in* **Ł** *using space* $O(nm^2 \log(n/m+1)) \subseteq O(n^2 m)$*, where* n *is the total size of the input, and* $m \leq n$ *is the number of variables.*

Proof: Let us first describe a nondeterministic polynomial-space algorithm. The basic idea is to nondeterministically search for a counterexample f as in Theorem 4.22, but there are several caveats. First, the numbers t_r have exponential size. Fortunately we do not really need them, it suffices to construct the x_r's and the integer points a_r witnessing that $\{x_r, x_{r+1}\}$ is anchored. Second, k is exponentially large, hence we cannot write down a full description of f. Instead we search for it piece by piece, we only need to remember two successive x_r at a time, as all the conditions are local. Third, we need somehow to certify that $\Gamma(x) = 1$ for all $x \in C(x_r, x_{r+1})$. By the proof of Theorem 4.22, we may require that $x_r, x_{r+1} \in C$ for a suitable polytope C from Lemma 4.13 such that all $\gamma \in \Gamma$ are linear on C; then it suffices to verify $\Gamma(x_r) = \Gamma(x_{r+1}) = 1$.

More precisely, let Σ be the closure of $\Gamma \cup \Delta$ under subformulas, and $n_\Sigma = |\Sigma| \leq n$. Let $\{C'_j \mid j < 2^{n_\Sigma}\}$ be the sequence of polytopes from Lemma 4.13 for Σ. An inspection of the proof shows that given $j < 2^{n_\Sigma}$ in binary, we can compute the sets of linear functions $\{L_{j,i} \mid i < n_\Sigma\}$ and $\{L_{j,\varphi} \mid \varphi \in \Sigma\}$ in space linear in the size of the output, i.e., $O(nd)$, where $d = m \log(n/m+1) \leq n$. Putting $C_j = \{x \in C'_j \mid \Gamma(x) = 1\}$, we see that we can compute inequalities defining C_j within the same space bound: they consist of

$$\{L_{j,i} \geq 0 \mid i < n_\Sigma\} \cup \{x_i \geq 0, 1 - x_i \geq 0 \mid i < m\} \cup \{L_{j,\gamma} - 1 \geq 0 \mid \gamma \in \Gamma\}.$$

Consider the algorithm in Figure 2. Notice that the test $a \in A(x,y)$ can be implemented as follows. If $x = y$, then it is equivalent to $a = x$. Otherwise we find the least i such that $x_i \neq y_i$, compute $\alpha = (a_i - x_i)/(y_i - x_i)$, and check whether $a_j = (1-\alpha)x_j + \alpha y_j$ for all j.

It follows from Theorems 4.22 and 4.17 that there exists an accepting computation path if and only if $\Gamma \not\vdash_{\mathbf{Ł}} \Delta$. The space requirements of the algorithm are dominated by $O(m^2 d)$ to

```
D ← Δ
nondeterministically guess x ∈ {0,1}^m
for i ← 1, ..., O(n2^n) do:
    nondeterministically guess j < 2^{n_Σ}
    if x ∉ C_j then REJECT
    if ∃γ ∈ Γ L_{j,γ}(x) < 1 then REJECT
    if L_{j,δ}(x) < 1 for some δ ∈ D, then D ← D ∖ {δ}
    if D = ∅ then ACCEPT
    nondeterministically guess y ∈ [0,1]^m_ℚ with H(y) = 2^{O(md)}
    if y ∉ C_j then REJECT
    nondeterministically guess a ∈ ℤ^m with H(a) = 2^{O(md)}
    if a ∉ A(x,y) then REJECT
    x ← y
REJECT
```

Figure 2: A nondeterministic algorithm for $\not\models_{\text{Ł}}$

store the vectors x, y, a, and $O(nd)$ to perform the operations with C_j. In total, the algorithm works in space $O((n+m^2)d) \subseteq O(n^3)$.

By Savitch's theorem, we can construct a deterministic algorithm working in space $O(n^6)$. We can obtain a better bound if we write down the deterministic algorithm explicitly and analyze it. We consider the recursive procedure in Figure 3. We see that $\mathit{Path}(x,y,k,D)$ accepts if and only if there exists a sequence $x_0, \dots, x_k \in [0,1]^m_{\mathbb{Q}}$ with $H(x_r) = 2^{O(md)}$ such that $x_0 = x$, $x_k = y$, for each $r < k$, $\{x_r, x_{r+1}\}$ has an anchor of height $2^{O(md)}$ and is contained in some C_j, $\Gamma(x_r) = 1$, and for each $\delta \in D$ there is $r < k$ such that $\delta(x_r) < 1$. Using Theorems 4.22 and 4.17 again, we have

$$\Gamma \not\models_{\text{Ł}} \Delta \Leftrightarrow \exists a \in \{0,1\}^m\, \mathit{Path}(a, a, O(n2^n), \Delta).$$

```
function Path(x, y, k, D):
if k = 1:
    for every j < 2^{n_Σ} and a ∈ ℤ^m such that H(a) = 2^{O(md)}:
        if x ∈ C_j ∧ y ∈ C_j ∧ a ∈ A(x,y)
                ∧ ∀γ ∈ Γ L_{j,γ}(x) = 1 ∧ ∀δ ∈ D L_{j,δ}(x) < 1
            then ACCEPT
    REJECT
for every D' ⊆ D and z ∈ [0,1]^m_ℚ such that H(z) = 2^{O(md)}:
    if Path(x, z, ⌊k/2⌋, D') ∧ Path(z, y, ⌈k/2⌉, D ∖ D') then ACCEPT
REJECT
```

Figure 3: A deterministic subprocedure for $\not\models_{\text{Ł}}$

Each recursive call of *Path* needs local storage $O(m^2 d)$, and the depth of recursion is $O(n)$, hence the recursion needs space $O(nm^2 d)$. The base case $k = 1$ needs $O(m^2 d)$ bits to store a, and $O(nd)$ to perform operations with C_j. Therefore the total space requirements of the algorithm are $O(nm^2 d)$.

We can reduce the space further by observing that in Theorem 4.22 we only need $H(x_r) = 2^{O(md)}$ for odd r, whereas for even r we have a better bound $H(x_r) = 2^{O(d)}$. We can modify *Path* so that all recursive calls except the deepest one are performed with z being an "even point". This brings down the space requirement of the recursive phase to $O(nmd)$. The base case also fits into this bound as $m \leq n$, hence the total space used by the algorithm is $O(nmd)$. □

Remark 4.24 We can also devise a *PSPACE* algorithm for $\mathrel{|\!\!\sim}_{\mathbf{Ł}}$ by an exhaustive search for the sequences $\{j_i \mid i \leq k\}$ from Theorem 4.17 instead of the sequence $\{x_r \mid r \leq k\}$. We can use the estimates of Theorem 4.22 to implement space-efficient tests for $C_{j_i} \cap C_{j_{i+1}} \neq \varnothing$ and anchoredness of C_{j_i}. If we further employ a log-space algorithm for undirected connectivity (Reingold [22]) and space-efficient formula evaluation similar to Lynch [17], we can obtain in this way an algorithm for $\mathrel{|\!\!\sim}_{\mathbf{Ł}}$ working in space $O(n + m^3 \log(n/m + 1)) \subseteq O(nm^2)$, which is slightly better than the bound of Theorem 4.23. We omit the details.

Corollary 4.25 *The universal theory of free MV-algebras is decidable in PSPACE.*

Proof: The case of F_0 is trivial, let thus $\kappa > 0$. If $\varphi(\vec{x})$ is an open formula in the language of *MV*-algebras, we can write φ as $\psi(t_0 = s_0, \ldots, t_{k-1} = s_{k-1})$ for some propositional formula ψ, and terms t_i, s_i in variables $\vec{x}$. We identify t_i, s_i with the corresponding formulas of **Ł**. Write $\alpha^0 = \alpha$, $\alpha^1 = \neg\alpha$. Using Corollary 4.2, we have

$$F_\kappa \vDash \forall \vec{x}\, \varphi(\vec{x}) \Leftrightarrow \forall e\colon k \to 2 \left(\psi(e) = 0 \Rightarrow F_\kappa \vDash \forall \vec{x} \bigvee_{i<k} (t_i = s_i)^{e(i)}\right)$$
$$\Leftrightarrow \forall e\colon k \to 2\, (\psi(e) = 0 \Rightarrow \{t_i \leftrightarrow s_i \mid e(i) = 1\} \mathrel{|\!\!\sim}_{\mathbf{Ł}} \{t_i \leftrightarrow s_i \mid e(i) = 0\}),$$

which is decidable in *PSPACE* by Theorem 4.23. □

5 Open problems

We have provided a solution to the most obvious question concerning admissibility in **Ł**, namely whether it is decidable. Nevertheless, there are many other problems in this area. First, we did not obtain any information on bases of **Ł**-admissible rules. We address this in a follow-up paper [13], where we construct an explicit basis of **Ł**-admissible rules, and prove that there is no finite basis.

Another set of questions concerns unification in **Ł**, and projectivity. In many superintuitionistic and modal logics, as well as n-contractive extensions of **BL**, every formula has a finite basis of unifiers, which are the most general unifiers of some projective formulas. Moreover, projective formulas have a transparent semantic characterization. Let us say that a *projective approximation* of a formula φ is a finite set Π of projective formulas such that $\varphi \mathrel{|\!\!\sim}_{\mathbf{Ł}} \Pi$, and $\pi \vdash_{\mathbf{Ł}} \varphi$ for each $\pi \in \Pi$.

Problem 5.1 *Does every formula have a projective approximation in* **Ł***? Give a description of* **Ł***-projective formulas. What is the unification type of* **Ł***?*

With regards to computational complexity of admissibility in **Ł**, we have shown a *PSPACE* upper bound, but we only have a trivial *coNP* lower bound given by the complexity of the set of **Ł**-tautologies (Mundici [19]).

Problem 5.2 *Is* $\mathrel{|\!\sim}_{\mathbf{Ł}}$ *PSPACE-complete? Is it in coNP, or at least in the polynomial hierarchy?*

Finally, our analysis of admissible rules in **Ł** heavily relied on the relatively transparent structure of free *MV*-algebras given by McNaughton's theorem. It seems much more difficult to generalize it to weaker fuzzy logics.

Problem 5.3 *Is admissibility decidable in* **BL** *or* **MTL***?*

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