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ADOMIAN DECOMPOSITION METHOD APPLIED TO LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: The paper deals with an application of Adomian Decomposition Method for solving linear stochastic differential equations. We derive new formulas such as the analytical approximate solution which convergces rapidely to the exact solution. The numerical experiments which are obtained show the efficiency of this method in the field of stochastic differential equations.

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Key Words: Adomian Decomposition Method, Brownian motion, Linear stochastic differential equations, Approximate solution.

1. Introduction

Adomian Decomposition Method (ADM) is widely used to solve linear and nonlinear equations of various kind [3, 2, 17]. The solution is found as an infinite series wich converges quickly towards an accurate solution, and its terms can be easily determined . The convergence of the ADM has been discussed by many authors, in particular Cherruault [8] is the first who proved the convergence, after that Abbaoui and Cherruault [1, 2], Himoun, et al. [11, 12], Hosseini and Nasabzadeh [13], Lesnic [14, 15], and Rach [17, 18]. El-Kalla [9] gave the

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error analysis of the method, Babolian and Biazar [5], Boumenir and Gordon [6] discussed the order and rate of convergence for ADM respectively.

The aim of this work is to extend the ADM for solving stochastic differential (SDE) which play an important role in various field such as finance, automatic control and physics, chemistry, astronomy, engineering, biology and others, the SDE are the fundamental tool to solve some problems in mathematics and finance stochastic modeling. The continuous-time models can be interpreted as a solution of SDE, these models are often assumed to be linear and may be Gaussian and continue to gain a growing interest of researchers (see for instance Brockwell [7] and the references therein). On the other hand, SDE are the more general framework of study and analysis of continuous-time random process. The rest of paper is organized as follows. Having defined the linear SDE and their properties. ADM is given in Section 3. The numerical experiments are provided in Section 4 and conclusion is in Section 5.

2. Linear Stochastic Differential Equations

We consider the process $(X(t))_{t\in R}$ generated by the following linear SDE

$$dX(t) = (\alpha_1 X(t) + \alpha_0)dt + \sigma dw(t) \tag{1}$$

where $(w(t))_{t\geq 0}$ is the standard Brownian motion (Bm) in $\mathbb{R}$ defined on some basic probability space $(\Omega, \mathcal{A}, P)$, α_1, α_0 and σ are constants, with $\sigma > 0$. The initial state $X(0) = X_0$ is a random variable, defined on $(\Omega, \mathcal{A}, P)$, independent of w such that $E\{X(0)\} = m(0)$ and $Var\{X(0)\} = V(0)$. The existence and uniqueness of the solution process $(X(t))_{t\geq 0}$ of equation (1) is ensured by the general results on stochastic differential equations [4] and under the above conditions. So, the solution of equation (1) is interpreted as satisfying the following integral equation

$$X(t) - X(0) = \int_0^t (\alpha_1 X(s) + \alpha_0)ds + \sigma w(t) \tag{2}$$

The term $(\alpha_1 X(s) + \alpha_0)$ is referred to as instantaneous mean of the process, and σ the instantaneous standard deviation .The solution of (1) can be written as

$$X(t) = e^{\alpha_1 t}X(0) + \alpha_0 \int_0^t e^{\alpha_1(t-s)}ds + \sigma \int_0^t e^{\alpha_1(t-s)}dw(s) \tag{3}$$

Let the mean of $X(t)$ be denoted by $m(t)$. It satisfies the equation

$$m(t) = \frac{\alpha_0}{\alpha_1}(e^{\alpha_1 t} - 1) + e^{\alpha_1 t}m(0). \tag{4}$$

$R(t,s)$ is the covariance of $X(t)$ given by

$$R(t,s) = e^{\alpha_1(t-s)}V(s), \text{ for all } t \geq s \tag{5}$$

where V(s) denotes the varians of $X(s)$ given by

$$V(s) = e^{2\alpha_1 s}V(0) + \sigma^2 \int_0^s e^{2\alpha_1(s-u)}du, \text{ for all } s \geq 0. \tag{6}$$

2.1. Euler Approximation

We can approach

$$X(t+h) = X(t) + \int_t^{t+h} (\alpha_1 X(t) + \alpha_0)ds + \sigma \int_t^{t+h} dw(s), \tag{7}$$

by

$$X(t_{j+1}) \approx X(t_j) + (\alpha_1 X(t_j) + \alpha_0)h + \sigma(w(t_{j+1}) - w(t_j)), \tag{8}$$

where $h = t_{j+1} - t_j$. The Euler approximation suppose that the integrals are constant over the interval of integration. Since the variables $(w(t_{j+1}) - w(t_j))$are independent and normal with variance h, we obtain the scheme

$$X_{(j+1)h} = X_{jh} + (\alpha_1 X_{jh} + \alpha_0)h + \sigma\sqrt{h}Z_j \tag{9}$$

where $(Z_j)_{j\in\mathbb{N}}$ is a Gaussian white noise process with 0 mean and finite variance. Phillips [16] showed that the exact discrete model corresponding to (1) is given by

$$X_{(j+1)h} = e^{\alpha_1 h}X_{jh} + \frac{\alpha_0}{\alpha_1}(e^{\alpha_1 h} - 1) + \sigma\sqrt{\frac{1 - e^{2\alpha_1 h}}{-2\alpha_1}}Z_j. \tag{10}$$

3. First Application of Adomian Decomposition Method

Adomian's method consists in calculating the solution $X(t)$ of (1) in a series form

$$X(t) = \sum_{j=0}^{\infty} X_j(t) \tag{11}$$

Putting equation (11) into (1) gives

$$\sum_{j=0}^{\infty} X_j(t) = X(0) + \alpha_0 t + L^{-1}\left(\alpha_1 \sum_{j=0}^{\infty} X_j(t)\right) + L^{-1}\left(\sigma \frac{dw(t)}{dt}\right), \tag{12}$$

where $L = \frac{d}{dt}$ is the derivative of 1-order, then the corresponding L^{-1} operator can be written in the form $L^{-1}[.] = \int_0^t (.)ds$. The iterates are determined by following recursive way

$$X_0(t) = X(0) + \alpha_0 t + \sigma w(t) \tag{13}$$

and $\forall n \geq 0$,

$$X_{n+1}(t) = \alpha_1 L^{-1}[X_n(t)], \tag{14}$$

we get

$$\begin{aligned}
X_1(t) &= \alpha_1 t X(0) + \frac{\alpha_1 \alpha_0}{2} t^2 + \sigma \alpha_1 \int_0^t w(t)dt \\
X_2(t) &= \alpha_1^2 \frac{t^2}{2} X(0) + \frac{\alpha_1^2 \alpha_0}{6} t^3 + \sigma \alpha_1^2 \int_0^t \int_0^t w(t)dtdt \\
X_3(t) &= \alpha_1^3 \frac{t^3}{6} X(0) + \frac{\alpha_1^3 \alpha_0}{24} t^4 + \sigma \alpha_1^3 \int_0^t \int_0^t \int_0^t w(t)dtdtdt \\
&\vdots \\
X_n(t) &= \alpha_1^n \frac{t^n}{n!} X(0) + \frac{\alpha_1^n \alpha_0}{(n+1)!} t^{n+1} + \sigma \alpha_1^n \int_0^t \underbrace{\cdots}_{n-fold} \int_0^t w(t)dt \underbrace{\cdots}_{n-fold} dt
\end{aligned}$$

Finally, we approximate the solution by the truncated series

$$\forall N \geq 1, \Phi_N(t) = \sum_{n=0}^{N-1} X_n(t). \tag{15}$$

In order to show the convergence almost sure (a.s) of the series to the exact solution we need the following Theorem which is given and proved by I. El-Kalla [10].

Theorem 3.1 (Theorem 1, I. El-Kalla [10]). *If $q(t)$ is integrable and at least the derivative of 1-order of $r(t)$ exists then we can prove that*

$$e^{-r(t)} \int e^{r(t)} q(t)dt = \sum_{k=0}^{\infty} (-1)^k \int \frac{dr(t)}{dt} \underbrace{\cdots}_{k-fold} \int \frac{dr(t)}{dt} \int q(t)dtdt \underbrace{\cdots}_{(k+1)-fold} dt.$$

In particular, if $\frac{dq}{dt}$ exists then we have

$$e^{-r(t)} \int e^{r(t)} dq(t) = \sum_{k=0}^{\infty} (-1)^k \int \frac{dr(t)}{dt} \underbrace{\cdots}_{k-fold} \int \frac{dr(t)}{dt} \int q(t)dtdt \underbrace{\cdots}_{k-fold} dt.$$

Proof. See the proof of Theorem 1 in [10]. □

Then the convergence of the approximate solution $\Phi_N(t)$ to the exact solution process is given by the folowing theorem :

Theorem 3.2. *Let $\Phi_N(t)$ is the approximate solution given by Adomian decomposition method of the linear SDE (1), then we have with probability 1*

$$\lim_{N\to+\infty} \Phi_N(t) = X(t) \tag{16}$$

Proof. First, $\Phi_N(t)$ can be written as follow

$$\Phi_N(t) = X(0)\sum_{k=0}^{N-1}\frac{(\alpha_1 t)^k}{k!} + \frac{\alpha_0}{\alpha_1}\sum_{k=1}^{N}\frac{(\alpha_1 t)^k}{k!} + \sigma\sum_{k=0}^{N-1}\alpha_1^k\int_0^t \underbrace{\cdots}_{k-fold}\int_0^t w(t)dt \underbrace{\cdots}_{k-fold} dt \tag{17}$$

and apply Theorem 3.1 on the third term of (17) we get

$$\sum_{k=0}^{\infty}\alpha_1^k\int_0^t \underbrace{\cdots}_{k-fold}\int_0^t w(t)dt \underbrace{\cdots}_{k-fold} dt = \int_0^t e^{\alpha_1(t-u)}dw(u) \tag{18}$$

Then we obtain

$$\lim_{N\to+\infty}\Phi_N(t) = X(0)\sum_{k=0}^{\infty}\frac{(\alpha_1 t)^k}{k!} + \frac{\alpha_0}{\alpha_1}\sum_{k=1}^{\infty}\frac{(\alpha_1 t)^k}{k!} + \sigma\int_0^t e^{\alpha_1(t-u)}dw(u)$$

$$= X(0)e^{\alpha_1 t} + \frac{\alpha_0}{\alpha_1}(e^{\alpha_1 t}-1) + \sigma\int_0^t e^{\alpha_1(t-u)}dw(u) = X(t).$$

and the proof is complete. □

4. Second Application of Adomian Decomposition Method

In this section we give another approximate solution of the linear SDE (1). This approach follows from the fact that a normalized Brownian motion $w(t)$ can be written in the form

$$w(t) = \frac{Z_0}{\sqrt{2\pi}}t + \sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{Z_k}{k}sin(kt) \text{ for } t\in[0,\pi]$$

where $Z_0, Z_1, ...$ be mutually independent random variables with identical normal distributions $\mathcal{N}(0,1)$. Then The iterates are determined by following recursive way

$$\chi_0(t) = X(0) + \alpha_0 t + \sigma \frac{Z_0}{\sqrt{2\pi}} t + \sigma \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{Z_k}{k} sin(kt) \tag{19}$$

and $\forall n \geq 0$,

$$\chi_{n+1}(t) = \alpha_1 L^{-1}[\chi_n(t)], \tag{20}$$

we obtain

$$\begin{aligned}
\chi_1(t) &= \alpha_1 t X(0) + \frac{\alpha_1\alpha_0}{2} t^2 + \alpha_1 \sigma \frac{Z_0}{\sqrt{2\pi}} \frac{t^2}{2} + \alpha_1 \sigma \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \frac{(-1)}{k^2} cos(kt) \\
\chi_2(t) &= \alpha_1^2 \frac{t^2}{2} X(0) + \frac{\alpha_1^2\alpha_0}{6} t^3 + \alpha_1^2 \sigma \frac{Z_0}{\sqrt{2\pi}} \frac{t^3}{6} + \alpha_1^2 \sigma \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \frac{(-1)}{k^3} sin(kt) \\
\chi_3(t) &= \alpha_1^3 \frac{t^3}{6} X(0) + \frac{\alpha_1^3\alpha_0}{24} t^4 + \alpha_1^3 \sigma \frac{Z_0}{\sqrt{2\pi}} \frac{t^4}{24} + \alpha_1^3 \sigma \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \frac{(-1)^2}{k^4} cos(kt) \\
&\vdots \\
\chi_n(t) &= \frac{(\alpha_1 t)^n}{n!} X(0) + \frac{\alpha_0}{\alpha_1} \frac{(\alpha_1 t)^{n+1}}{(n+1)!} + \frac{\sigma}{\alpha_1} \frac{Z_0}{\sqrt{2\pi}} \frac{(\alpha_1 t)^{n+1}}{(n+1)!} \\
&\quad + \alpha_1^n \sigma \left(\sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \frac{(-1)^{[\frac{n}{2}]}}{k^{n+1}} sin(kt) \right) \text{ if } n \text{ is even} \\
\chi_n(t) &= \frac{(\alpha_1 t)^n}{n!} X(0) + \frac{\alpha_0}{\alpha_1} \frac{(\alpha_1 t)^{n+1}}{(n+1)!} + \frac{\sigma}{\alpha_1} \frac{Z_0}{\sqrt{2\pi}} \frac{(\alpha_1 t)^{n+1}}{(n+1)!} \\
&\quad + \alpha_1^n \sigma \left(\sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \frac{(-1)^{[\frac{n}{2}]+1}}{k^{n+1}} cos(kt) \right) \text{ if } n \text{ is odd}
\end{aligned}$$

Finally, we approximate the solution by the truncated series

$$\forall N \geq 1, \quad \Psi_N(t) = \sum_{n=0}^{N-1} \chi_n(t). \tag{21}$$

5. Numerical Experiments

The obtained results for various values of the parameters α_0, α_1 and σ where compared with the exact solution X and where shown graphicaly. Assume the process $X(t)$ is observed in the interval $[0, T]$ at points $(t_1, t_2, ..., t_m)$, where $h = t_{i+1} - t_i$ is the fixed space scale. Then we have the sequence of m observations $X_h, X_{2h}, ..., X_{mh}$. In our experiment we choose $T = 1$. We use two kinds of approximated solution $\Phi_1(t) = X_0$ which is an approximation with one term, and $\Phi_2(t) = X_0 + X_1$ using a two terms of approximation.

We can prove that the discrete models corresponding to $\Phi_1(t)$ and $\Phi_2(t)$ are given respectively by

$$\Phi_1(t_{j+1}) = \Phi_1(t_j) + \alpha_0 h + \sigma\sqrt{h}Z_j,$$

and

$$\Phi_2(t_{j+1}) = \Phi_2(t_j) + (\alpha_0 + \alpha_1 X(0))\, h + \frac{\alpha_0\alpha_1}{2}(2j+1)h^2 + \sigma(\alpha_1 j + 1)\sqrt{h}Z_j,$$

for all $j \geq 0$ with $t_0 = 0$ and $(Z_j)_{j\in\mathbb{N}}$ is a Gaussian white noise process with 0 mean and finite variance equal to 1. The table 1 shows the results from $t = 0.1$ to $t = 1$ with an increment $h = 0.1$ where we compute the absolute error of the both approximate solutions $\Phi_1(t)$ and $\Phi_2(t)$ with the parameters $\alpha_0 = 0.02$, $\alpha_1 = -0.05$ and $\sigma = 1$.

t	$X(t)$	$\Phi_1(t)$	$\Phi_2(t)$	$\mid \Phi_1(t) - X(t) \mid$	$\mid \Phi_2(t) - X(t) \mid$
0.1	-0.1348	-0.1348	-0.1348	0	0.0000
0.2	-0.6588	-0.6595	-0.6588	0.0007	0.0000
0.3	-0.6139	-0.6179	-0.6139	0.0040	0.0000
0.4	-0.5179	-0.5249	-0.5179	0.0070	0.0000
0.5	-0.8758	-0.8854	-0.8758	0.0096	0.0000
0.6	-0.4928	-0.5068	-0.4928	0.0140	0.0000
0.7	-0.1123	-0.1288	-0.1122	0.0165	0.0001
0.8	-0.1217	-0.1387	-0.1215	0.0170	0.0001
0.9	-0.0155	-0.0332	-0.0153	0.0176	0.0002
1	0.0418	0.0240	0.0420	0.0177	0.0003

Table 1: Absolut error of Φ_1 and Φ_2, with $\alpha_0 = 0.02$, $\alpha_1 = -0.05$, $\sigma = 1$, $h = 0.1$.

The table 2 shows the results from $t = 0.1$ to $t = 1$ but with an increment $h = 0.01$, we compute the absolute error of the approximate solutions $\Phi_1(t)$ and $\Phi_2(t)$ with the parameters $\alpha_0 = 0$, $\alpha_1 = -0.1$ and $\sigma = 1$.

Finally, in oredr to show that Adomian series is a rapidly converging to the exact solution process $X(t)$. Figure 1, 2 present the plots of $\Phi_1(t)$ and $\Phi_2(t)$ with the parameters $\alpha_0 = 0$, $\alpha_1 = -0.1$ and $\sigma = 1$.

t	$X(t)$	$\Phi_1(t)$	$\Phi_2(t)$	$\mid \Phi_1(t) - X(t) \mid$	$\mid \Phi_2(t) - X(t) \mid$
0.01	-0.0687	-0.0687	-0.0687	0	0
0.02	-0.1200	-0.1201	-0.1200	0.0001	0
0.03	-0.2200	-0.2202	-0.2200	0.0002	0.0000
0.04	-0.1839	-0.1843	-0.1839	0.0004	0.0000
0.05	-0.2270	-0.2276	-0.2270	0.0006	0.0000
0.06	-0.1945	-0.1953	-0.1945	0.0008	0.0000
0.07	-0.1122	-0.1133	-0.1122	0.0010	0.0000
0.08	-0.2993	-0.3005	-0.2993	0.0011	0.0000
0.09	-0.4628	-0.4642	-0.4628	0.0014	0.0000
0.1	-0.3329	-0.3348	-0.3329	0.0019	0.0000
0.2	-0.7238	-0.7309	-0.7237	0.0071	0.0000
0.3	-1.3394	-1.3562	-1.3393	0.0168	0.0002
0.4	-1.4366	-1.4678	-1.4362	0.0312	0.0004
0.5	-1.4988	-1.5432	-1.4980	0.0444	0.0008
0.6	-1.2293	-1.2878	-1.2281	0.0584	0.0013
0.7	-1.3498	-1.4212	-1.3479	0.0714	0.0019
0.8	-1.2432	-1.3276	-1.2405	0.0844	0.0027
0.9	-0.9772	-1.0726	-0.9736	0.0955	0.0036
1	-0.8250	-0.9292	-0.8204	0.1042	0.0046

Table 2: Absolut error of Φ_1 and Φ_2, with $\alpha_0 = 0$, $\alpha_1 = -0.1$, $\sigma = 1$, $h = 0.01$.

6. Conclusion

In this paper we propose an extension of the ADM for solving linear SDE. The numerical results for different values of parameters confirm the theoretical results obtained and illustrate the powerful of this method. We use Matlab for the computations associated with the section of numerical experiments in this

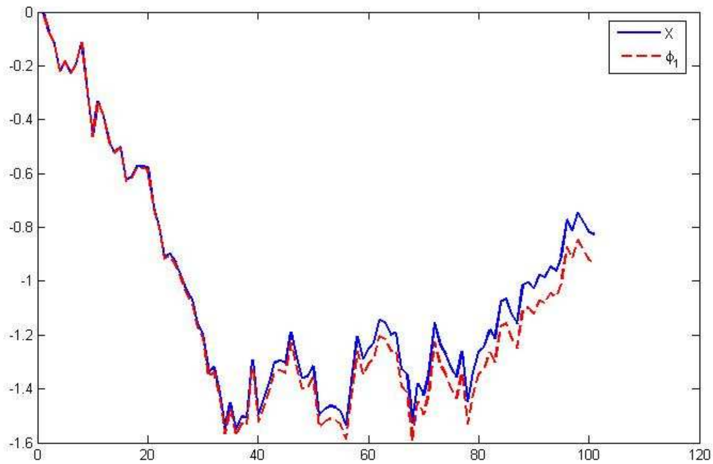

Figure 1: Exact solution X and approximate solution Φ_1 for $\alpha_0 = 0$, $\alpha_1 = -0.1$, $\sigma = 1$, $h = 0.01$.

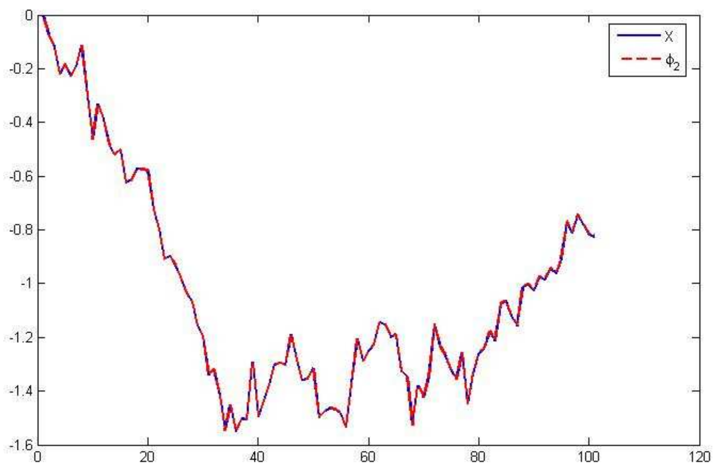

Figure 2: Exact solution X and approximate solution Φ_2 for $\alpha_0 = 0$, $\alpha_1 = -0.1$, $\sigma = 1$, $h = 0.01$.

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