Advance of perihelion

Kin-Ho Lo[a)] and Kenneth Young[b)]
Department of Physics, The Chinese University of Hong Kong, Hong Kong, China

Benjamin Y. P. Lee[c)]
Keble College, University of Oxford, Oxford, United Kingdom

(Received 24 July 2012; accepted 21 June 2013)

The advance of perihelion, in particular for Mercury, is regarded as a classical test of general relativity, but a number of other (in some cases much larger) contributions to this phenomenon are seldom discussed in detail in textbooks. This paper presents a unified framework for evaluating the advance of perihelion due to (a) general relativity, (b) the solar quadrupole moment, and (c) planetary perturbations, the last in a ring model where the mass of each perturbing planet is "smeared out" into a coplanar circular orbit. The exact solution of the ring model agrees to within 4% with the usually quoted figure. Time-dependent contributions beyond the ring model contain some surprising features: they are not small, and some with long periods could mimic a secular advance.
[http://dx.doi.org/10.1119/1.4813067]

I. INTRODUCTION

This paper is concerned with the advance of perihelion of a planet, say Mercury. General relativity (GR) famously predicts a rate of advance $\Omega \approx 43$ arcsec per century—in agreement with observations—as cited in elementary textbooks[1–4] and popular accounts,[5–12] and as derived in GR texts.[13–19] The best current value[20] for the GR effect is 42.98 ± 0.04 arcsec/century. However, the so-called "observed" value actually refers to the *remaining* discrepancy after other contributions are subtracted, a point not always made explicit.[1]

Apart from non-inertial effects[13,14,17] of $\approx$5025 arcsec/century, planetary perturbations[13,14,17] amount to $\approx$532 arcsec/century, but are seldom analyzed in an accessible manner, except for a ring model[21,22] to which reference will be made below. This article fills the pedagogical gap by developing a unified and accessible framework that applies to (a) GR, (b) the solar quadrupole moment,[13,14,17,23–25] and also (c) the ring model of planetary perturbations. Other possibilities are not discussed, such as modifications of the inverse-square law[26] or a hypothetical inner planet (Vulcan).[27] The entire discussion is placed within the context of a small perturbation upon an inverse-square force, excluding situations such as motion near a black hole.[28]

The analysis presented here is principally of pedagogical and historical interest. Modern solar system tests of GR[20] would not be restricted to the advance of perihelion, nor would Ω be expressed as a sum of additive contributions from different causes. Rather, numerical solutions of the N-body system in a parameterized post-Newtonian (PPN) formalism[9,13,14,17,18] are fitted to data including accurate modern measurements on the moon and on spacecraft. The Mercury data no longer constitute the principal constraints on the PPN parameters.

Section II reviews the Kepler problem, with an attractive central force per unit mass GM/r^2 in standard notation, leading to elliptic orbits with

$$r^{-1} \equiv u = \sigma^{-1}(1 + \varepsilon \cos\phi), \tag{1}$$

where ϕ is the heliocentric longitude in the orbital plane (x-axis along the perihelion), $\varepsilon > 0$ is the eccentricity, and σ is the semi-latus rectum, related to the semi-major axis a by $\sigma = a(1-\varepsilon^2)$. The orbit in Eq. (1) is closed, so that $\Omega = 0$. The mass of the planet relative to the Sun is ignored throughout, so that the Sun is regarded as stationary.

All three contributions to the advance of perihelion involve an additional attractive radial force per unit mass in the orbital plane of the form

$$(GM/r^2)\,\lambda f(r), \tag{2}$$

where λ is a formal small parameter and λf is dimensionless. Section II shows that such a perturbation leads to an advance that can be calculated exactly, and the leading term alone

$$\Omega \approx \frac{\pi}{T}\lambda f_1, \tag{3}$$

with $f_1 = \sigma^{-1}(df/du)$, provides a good estimate in practice $[(\varepsilon/2)^2 \approx 0.01$ for Mercury], and is exact for both GR and the quadrupole moment. This approximate result, in essentially this form, is given by several authors[29,30] and could have been guessed apart from the numerical prefactor—a constant value of f, equivalent to changing the solar mass M, does not cause an advance of perihelion; for small eccentricities only a small range of r is sampled, so only the first-order variation of $f(r)$ enters the final result.

The general formalism is applied to GR (Sec. III), the quadrupole moment (Sec. IV), and the ring model of planetary perturbations (Sec. V), thus giving a unified account of all three effects. Time-dependent contributions beyond the ring model (Sec. VI) contain some surprises: nearly resonant perturbations lead to long-period oscillations that could mimic a secular advance, with substantial amplitudes even when averaged over fairly long time windows. The amplitudes are large because the time-dependent effects contain dominant terms that cancel in the ring model. Some concluding remarks are given in Sec. VII.

II. GENERAL FORMALISM

A. Kepler problem

The familiar Kepler problem is first solved to set the stage. The radial equation of motion is, in standard notation

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{J^2}{r^3}, \tag{4}$$

where $J = r^2(d\phi/dt)$ is the conserved angular momentum per unit mass. Using $u = r^{-1}$ and eliminating t via $d/dt = Ju^2(d/d\phi)$ then leads to

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{J^2}, \tag{5}$$

where, crucially, the coefficient of u is unity. The solution to this equation is given by Eq. (1), where $\sigma = J^2/GM$. One constant of integration is eliminated and the sign of ε determined by a choice of $\phi = 0$. The orbit is closed, $r(\phi) = r(\phi + 2\pi)$, and the perihelion does not advance.

The time t and the angle ϕ are related by $t = T(\phi/2\pi) + \Delta(\varepsilon, \phi)$ (see Appendix A), where T is the period. The term Δ has the following properties: (a) it is 2π-periodic in ϕ; (b) in its Fourier series representation, the coefficients of $\sin m\phi$ and $\cos m\phi$ are $O(\varepsilon^m)$ (and in particular Δ vanishes for $\varepsilon = 0$); and (c) its average value is zero. Any function satisfying (a) and (b) will be said to be *weakly oscillatory* (WO).

B. Time-independent perturbations

With an extra time-independent radial force as given in Eq. (2), the radial equation becomes

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}[1 + \lambda f(r)] + \frac{J^2}{r^3}. \tag{6}$$

Changing variables to u and eliminating t as before, one gets

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{J^2} + \frac{GM}{J^2}\lambda f(u). \tag{7}$$

Next expand in powers of λ: $u(\phi) = u_0(\phi) + \lambda u_1(\phi) + O(\lambda^2)$. Using the zeroth-order solution $u_0 = \sigma^{-1}(1 + \varepsilon\cos\phi)$, we get

$$\begin{aligned}\frac{d^2u_1}{d\phi^2} + u_1 &= \frac{GM}{J^2} f(u_0(\phi)) \\ &= \sigma^{-1} f(\sigma^{-1} + \sigma^{-1}\varepsilon\cos\phi) \\ &= \sigma^{-1}\sum_{k=0}^{\infty} f_k \varepsilon^k \cos^k\phi \\ &\equiv \sigma^{-1} g(\phi) = \sigma^{-1}\sum_m g_m \cos m\phi,\end{aligned} \tag{8}$$

where f has been expanded in a Taylor series about $u = \sigma^{-1}$, with

$$f_k = \frac{\sigma^{-k}}{k!}\frac{d^kf}{du^k}\bigg|_{u=\sigma^{-1}}, \tag{9}$$

and the even periodic function $g(\phi)$ has been expressed in a Fourier series. Incidentally, $g(\phi)$ is WO, and $g_m = O(\varepsilon^m)$.

The solution $u_1(\phi)$ is the sum of a homogeneous solution and inhomogeneous solutions due to each g_m. The homogeneous solution just takes u to another Kepler orbit, with no advance of perihelion. When $m \neq 1$, the inhomogeneous solution proportional to $\cos m\phi$ does not contribute to the advance of perihelion.[31] This leaves the *resonant* term $m = 1$, and the corresponding inhomogeneous solution satisfies $d^2u_1/d\phi^2 + u_1 = \sigma^{-1}g_1\cos\phi$, with solution[32] $u_1 = (1/2)\sigma^{-1}g_1\phi\sin\phi$. Putting u_0 and u_1 together gives

$$\begin{aligned}u &\approx u_0 + \lambda u_1 \\ &= \sigma^{-1}(1 + \varepsilon\cos\phi) + \frac{1}{2}\lambda\sigma^{-1}g_1\phi\sin\phi \\ &\approx \sigma^{-1}[1 + \varepsilon\cos((1 - (\lambda g_1/2\varepsilon))\phi)].\end{aligned} \tag{10}$$

The next perihelion occurs at $[1 - (\lambda g_1/2\varepsilon)]\phi_P = 2\pi$ or

$$\phi_P \approx 2\pi\left(1 + \frac{\lambda g_1}{2\varepsilon}\right), \tag{11}$$

so that the perihelion advance per cycle is $\Delta\phi_P = \pi\lambda g_1/\varepsilon$, giving a rate of advance[33]

$$\Omega = \frac{\Delta\phi_P}{T} = \frac{\pi}{T}\frac{\lambda g_1}{\varepsilon}. \tag{12}$$

It remains to determine g_1. Multiply Eq. (8) by $\cos\phi$ and integrate over $[0, 2\pi]$; then using

$$\int_0^{2\pi}\cos^{2k+2}\phi\, d\phi = \frac{\pi}{2^{2k+1}}\frac{(2k+2)!}{[(k+1)!]^2}, \tag{13}$$

we obtain

$$g_1 = \sum_{k=0}^{\infty} f_{2k+1} C_{k+1}^{2k+2}(\varepsilon/2)^{2k+1}, \tag{14}$$

where $C_m^n = n!/[m!(n-m)!]$. Substituting g_1 back into Eq. (12) yields the key result

$$\Omega = \frac{\pi}{2T}\lambda\sum_{k=0}^{\infty} C_{k+1}^{2k+2} f_{2k+1}(\varepsilon/2)^{2k}. \tag{15}$$

Several features of Eq. (15) can be understood from general considerations:

- As discussed, only the variation of $f(u)$ around some mean value (described by the coefficients f_k, $k > 0$) would contribute.
- For $\varepsilon \ll 1$, u has a small excursion around σ^{-1}, so f can be approximated by its linear variation f_1, and more generally higher coefficients f_k are associated with higher powers of ε.
- The problem is invariant under $\varepsilon \to -\varepsilon$, $\phi \to \phi + \pi$, so only even powers of ε appear.
- The series is rapidly convergent, e.g., $(\varepsilon/2)^2 \approx 0.01$ for Mercury; in fact, we shall see that for both GR and the quadrupole moment, all terms except f_1 are exactly zero.

III. GENERAL RELATIVITY

In GR, the equations of motion are[13–19] $r^2(d\phi/d\tau) = J =$ constant and

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{J^2}{r^3} + \frac{3GMJ^2}{c^2r^4}, \tag{16}$$

where τ is the proper time. But since the physical interpretation of τ does not enter the calculation, it can be regarded as t and the formalism in Sec. II can be applied. Comparing with Eq. (6), one has $\lambda f = -(3J^2/c^2r^2)$; since this expression is quadratic in u, the series (15) has only the first term

$$\lambda f_1 = -\frac{6J^2}{c^2\sigma^2} = -\frac{6GM}{a(1-\varepsilon^2)c^2}. \tag{17}$$

For Mercury, $\lambda f_1 = 1.60 \times 10^{-7}$ and hence $\Omega = (\pi/T)\lambda f_1 = 42.9$ arcsec/century, in agreement with the "observed" value. The result is correct to all orders in ε, a fact not always emphasized. In fact, some derivations unnecessarily use the approximation $\varepsilon \ll 1$.[16]

IV. QUADRUPOLE MOMENT

If the Sun is slightly oblate, then the gravitational potential is given by[13,14,17]

$$-\frac{GM}{r}\left[1 - J_2\left(\frac{r_s}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right)\right], \tag{18}$$

where r_s is the mean radius of the Sun and J_2 is the conventional quadrupole parameter. Estimates of J_2 range from "high" values[25] of order 10^{-5}, to "medium" values[34] of order 10^{-6}, with more recent consensus converging to "low" values[17,35] of order 10^{-7}. Our purpose here is not to examine the evidence for the different estimates, but simply to relate Ω to J_2, whatever the latter might be. A time-dependent quadrupole moment J_2 has also been suggested,[25] possibly varying with the sunspot cycle.

Using Eq. (18) in the equatorial plane and differentiating to obtain the force gives $\lambda f(r) = (3/2)J_2 r_s^2 u^2$. Again, the series in Eq. (15) has only one term

$$\lambda f_1 = 3J_2\left(\frac{r_s}{\sigma}\right)^2. \tag{19}$$

Putting this into the formalism of Sec. II then gives, in the case of Mercury,

$$\Omega = (J_2/10^{-7}) \times 0.0127\,\text{arcsec/century}. \tag{20}$$

For currently accepted "low" values of J_2, this contribution is very small compared to that of GR, but a derivation such as that presented here is still needed to establish this fact.

V. PLANETARY PERTURBATIONS

A. Ring model

In the ring model,[21,22] the mass M_i of each planet[36] $i = 2, 3, \ldots, 8$ is "smeared out" into its orbit, assumed to be a circle of radius R_i centered at the Sun and coplanar with the orbit of Mercury. Following Price and Rush,[21] R_i is taken to be the semi-major axis a_i. Other choices are equally plausible, but the difference is negligible. For example, if R_i is taken to be the semi-latus rectum σ_i or $(a_ib_i)^{1/2} = \sigma_i(1-\varepsilon_i^2)^{-3/4}$ (ε_i is the eccentricity of the orbit of each perturbing planet), the final result would increase by only 0.27% or 0.07%, respectively. [Although these ambiguities of order ε_i^2 are small, the same cannot be said of terms of order ε^2, e.g., in Eq. (15).]

The potential due to this ring at radius r in the orbital plane is

$$\lambda\Phi^{(i)}(r) = -G\int \frac{M_i}{2\pi R_i}\frac{1}{|\mathbf{r}-\mathbf{r}'|}R_i\,d\varphi, \tag{21}$$

where in polar coordinates $\mathbf{r}' = (R_i, \theta = \pi/2, \varphi)$, and a formal small parameter λ is again inserted. Without loss of generality, assume $\mathbf{r}$ is on the x-axis so that

$$\begin{aligned}\lambda\Phi^{(i)}(r) &= -\frac{GM_i}{2\pi R_i}\int_0^{2\pi}\frac{R_i\,d\varphi}{\sqrt{r^2 + R_i^2 - 2rR_i\cos\varphi}}\\ &= -\frac{2GM_i}{\pi R_i}K(\xi_i),\end{aligned} \tag{22}$$

where

$$\xi_i = \frac{r}{R_i} \tag{23}$$

and $K(\xi_i)$ is the complete elliptical integral of the first kind.[37] Expanding $K(\xi_i)$ in powers of ξ_i gives

$$\lambda\Phi^{(i)}(r) = -\frac{GM_i}{R_i}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\xi_i^{2n}, \tag{24}$$

which can also be obtained by expanding the integrand in Eq. (22) and integrating term by term, without involving the elliptic integral. Here $\xi_i < 1$; for perturbation by a hypothetical inner planet (Vulcan),[27] the expansion should be carried out in ξ_i^{-1}. The radial dependence of each term in Eq. (24) is $(1/R_i)\xi_i^{2n} = r^{2n}/R_i^{2n+1}$, so by comparison with the usual spherical harmonic expansion of $|\mathbf{r}-\mathbf{r}'|^{-1}$ in angular momentum indices (ℓ, m), it is recognized that $\ell = 2n$.

Thus each planet contributes to the force in Eq. (2) an amount

$$\lambda F^{(i)}(r) = \frac{GM_i}{R_i^2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n)\,\xi_i^{2n-1}, \tag{25}$$

and to $\lambda f(r)$ an amount

$$\begin{aligned}\lambda f^{(i)}(r) &= -\frac{M_i}{M}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n)\,\xi_i^{2n+1}\\ &= -\frac{M_i}{2M}\left(\xi_i^3 + \frac{9}{8}\xi_i^5 + \frac{75}{64}\xi_i^7 + \cdots\right).\end{aligned} \tag{26}$$

Defining the Taylor coefficients $f_k^{(i)}$ as in Eq. (9) and using Eqs. (15) and (26), one obtains a solution for Ω, exact to first order in the perturbation (i.e., in λ or M_i/M), as a triple sum

$$\begin{aligned}\Omega = \frac{\pi}{2T}\sum_i\frac{M_i}{M}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{2k+2}{[(k+1)!]^2}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\\ \times\frac{(2k+2n+1)!}{(2n-1)!}\,\xi_i^{2n+1}\left(\frac{\varepsilon}{2}\right)^{2k}.\end{aligned} \tag{27}$$

Because all functions such as $f^i(r)$ are expanded about $r = \sigma$, the quantity ξ_i defined in Eq. (23) should henceforth be evaluated at $\xi_i = \sigma/R_i$.

To make sense of the large number of terms, one can assign a weight to each:

$$W = \frac{M_i}{M} \xi^{\alpha} \varepsilon^{\beta}, \tag{28}$$

where, referring to Eq. (27), $\alpha = 2n+1 = \ell+1$, $\beta = 2k$. The largest weights ($W_0 \approx 3 \times 10^{-7}$) are associated with the leading terms ($\alpha = 3$, $\beta = 0$) for Venus and Jupiter; each contributes $\sim$100 arcsec/century to Ω. Truncating at $W < 10^{-3} W_0$, only 29 terms survive (respectively, 12, 7, 3, 4, 2, 1, 0 terms for the 7 planets).

The resultant contributions to Ω are shown in column (b) of Table I, with a total $\Omega = 554$ arcsec/century, 4% higher than the accepted value. The contributions of Uranus and Neptune are negligible. Column (a) of Table I is due to Price and Rush[21] and is explained below.

B. Discussion

Price and Rush[21] solved the ring model with two additional and unnecessary approximations. First, they assume ξ_i is small, resulting in a closed but inexact expression for the perturbing force

$$F^{(i)}(r) = \frac{GM_i}{2R_i} \frac{\xi_i}{1-\xi_i^2} = \frac{GM_i}{2R_i}(\xi_i + \xi_i^3 + \cdots), \tag{29}$$

agreeing with Eq. (25) only to first order in ξ_i; the leading relative error is $\xi_i^2/8 \sim 3\%$ in the case of Venus. Second, their calculation is accurate only to lowest order in ε.[38] For any perturbing force $F(r)$, the result in the second unnumbered equation below their Eq. (20) agrees exactly with the f_1 term in our Eq. (15). In fact, the analog to Eq. (27) in the calculation of Price and Rush[21] could be obtained from Eq. (27) by keeping only the ε^0 terms and replacing

$$2\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 (2n) \to 1, \tag{30}$$

to account for the difference between Eqs. (25) and (29).

Under the assumption[21] of a nearly circular Mercury orbit, it is somewhat arbitrary whether one takes the radius to be the semi-major axis a ($\xi_i = a/R_i$) or the semi-latus rectum σ ($\xi_i = \sigma/R_i$). The difference is $\sim$12% in ξ_i^3 and $\sim$20% in ξ_i^5.[39] Column (a) shows the results reported in Price and Rush,[21] evaluated using $\xi_i = a/R_i$ and giving $\Omega = 531.1$ arcsec/century. (Actually $\Omega = 531.9$ arcsec/century is quoted;[21] the small difference appears to be due to rounding.) Using $\xi_i = \sigma/R_i$ gives $\Omega = 451.5$ arcsec/century, while keeping all powers of ε^2 in Eq. (15) gives $\Omega = 531.8$ arcsec/century. Finally, if the correct force expression in Eq. (25) is used, our results in column (b) are obtained—these are *exact* for the ring model to first order in λ or M_i/M. The excellent agreement found by Price and Rush[21] is fortuitous, due to the cancelation of several errors: the error in the potential and the failure to keep higher powers of ε.

The same ring model was also solved by Stewart,[22] using the Laplace–Runge–Lenz (LRL) vector. The evaluation is carried out to all orders in ε, so it does not matter that the formalism is referenced to a rather than σ.[39] Our result essentially agrees with Stewart's (555.8 arcsec/century). Within the ring model, the two approaches are complementary. The LRL vector is more elegant but less accessible to students and also requires the numerical evaluation of an integral; our approach is more elementary and the result in Eq. (27) is in closed form (though as an infinite summation).

The ring model itself suffers from three further errors. First, the eccentricities of the orbits of the perturbing planets lead to corrections of $O(\varepsilon_i)$.[40] Second, these orbits are tilted from the Mercury orbit by some angle χ_i. And third, the perturbing planets are moving in their orbits. Stewart[22] has shown that the first two effects reduce Ω by 4.4%, giving excellent agreement with the usually quoted figure of 532 arcsec/century. That calculation is unavoidably complicated because of the geometry, and will not be further discussed here.

In anticipation of the time-dependent corrections (Sec. VI), it is useful to explain the leading indices $\alpha = 3$, $\beta = 0$ for the time-independent case in Eq. (28). First, $\alpha = \ell + 1$. The $\ell = 0$, $\alpha = 1$ monopole term is equivalent to changing the solar mass, which does not cause an advance of perihelion. The $\ell = 1$, $\alpha = 2$ dipole contribution vanishes by the symmetry of the ring; thus the leading term is $\ell = 2$, $\alpha = 3$. In the time-dependent treatment, when the perturbing planet is at a particular position on the ring, symmetry is destroyed and the dipole term will contribute with $\alpha = 2$.

To understand the powers of ε, first note that the effect of the perturbing force goes as λf_1, i.e., variations of λf, which can be sampled only if there is a nonzero ε; thus $g_1 = O(\varepsilon)$. This perturbation in modifying the second term in $u_0(\phi) = \sigma^{-1}(1 + \varepsilon \cos\phi)$ is therefore of order g_1/ε [see Eq. (10)], so Ω goes as ε^0. Again, the situation is different in the time-dependent case. When the perturbing planet is at a particular position on the ring, symmetry is broken and Mercury would experience a nonzero effect even if its orbit were circular. Therefore the force expression (analogous to g_1) starts at ε^0 and Ω starts at ε^{-1}, i.e., $\beta = -1$.

Thus, $\alpha = 2$ and $\beta = -1$ terms will appear in the time-dependent case because symmetry is broken. In particular, because the leading term goes as ξ_i^2 rather than ξ_i^3, Uranus and Neptune will turn out to be important, even though they are negligible in the ring model.

Table I. Precession caused by the ring masses of the other planets on Mercury, in arcsec/century. Column (a) is the result of Price and Rush;[21] column (b) is our result.

Planet	a_i(AU)	ξ_i	M_i/M	(a)	(b)
Venus	0.72	0.5126	2.448×10^{-6}	268.8	292.6
Earth	1	0.3707	3.003×10^{-6}	92.3	94.7
Mars	1.5	0.2433	3.227×10^{-7}	2.4	2.4
Jupiter	5.2	0.07123	9.545×10^{-4}	159.9	156.7
Saturn	9.2	0.03868	2.858×10^{-4}	7.6	7.5
Uranus	19.2	0.01931	4.364×10^{-5}	0.14	0.14
Neptune	30.1	0.01234	5.150×10^{-5}	0.04	0.04
Total				531.1	554.0

VI. TIME-DEPENDENT CONTRIBUTIONS

A. Derivation

This section sketches the salient features of the time-dependent contributions beyond the ring model, with details given in Appendix B. The perturbation is due to a planet[41] of mass M_i at position $\mathbf{r}_i(t) = (R_i, \theta_i = \pi/2, \phi_i(t))$, with $\phi_i(t) = \omega_i t + \phi_i^0$. The potential is

$\lambda\Phi(r,\phi,t) = -GM_i|\mathbf{r}-\mathbf{r}_i(t)|^{-1}$. For $R_i > r$, an expansion in terms of spherical harmonics[42] gives, for $\cos\theta = \cos\theta_i = 0$,

$$\lambda\Phi(r,\phi,t) = -GM_i \sum_{\ell,m} D_{\ell,m} \frac{r^\ell}{R_i^{\ell+1}} e^{im[\phi-\phi_i(t)]}, \tag{31}$$

where $D_{\ell,m} = [(\ell-m)!/(\ell+m)!]|P_\ell^m(0)|^2$. The $m=0$ terms in Eq. (31) reproduce the ring model, so the time-dependent contributions can be evaluated by summing the $m>0$ terms and adding the complex conjugate, to be denoted as $\sum'$.

A key parameter is the ratio of orbital periods

$$\rho_i = \frac{\omega_i}{\omega} = \frac{T}{T_i} \tag{32}$$

between Mercury (T) and the perturbing planet (T_i). The phase in the exponential in Eq. (31) is

$$m[\phi - \phi_i(t)] = m(1-\rho_i)\phi - m\phi_i^0. \tag{33}$$

In addition, various periodic functions such as $(1+\varepsilon\cos\phi)^{-\ell}$ in r^ℓ all combine into WO functions that can be decomposed into harmonics $e^{-in\phi}$, so that apart from a constant, the overall phase goes as

$$[m(1-\rho_i) - n]\phi \equiv \nu\phi \equiv (1+\mu)\phi. \tag{34}$$

If ρ_i is (nearly) rational, then for certain integer pairs (m,n), μ is (nearly) zero and the perturbing potential goes (nearly) as $e^{i\phi}$, which will cause a (nearly) secular response. Assuming ρ_i to be not exactly rational, this leads to two important and complementary conclusions. On one hand, theoretically, these time-dependent terms do not contribute to the secular rate of advance—on an infinite time scale, the average value is zero. On the other hand, over finite time windows nearly resonant terms cannot in practice be distinguished from truly secular terms.

Each term in the four-fold summation (over planets i, angular momenta ℓ, m, and the harmonic components n of the WO functions) is again associated with a weight W as in Eq. (28), with $\alpha = \ell+1 \geq m+1$ and $\beta = n-1$. We impose three conditions: (a) the weight should not be too small, e.g., $W \geq 10^{-3}\,W_0$; (b) the time-dependence is close to resonance, e.g., $|\mu| < 5\times 10^{-3}$ (the resultant sinusoidal variations have periods longer than 50 years); and (c) the sum $\ell+m$ should be even because $D_{\ell,m}$ vanishes otherwise. Then only two terms remain, due to Uranus and Neptune (see Table II), each giving a time-dependent rate of advance

$$\Omega = -\tilde{\Omega}_i \cos\phi_i, \quad \tilde{\Omega}_i = \frac{\pi}{T}\frac{M_i}{M}\xi_i^2\varepsilon^{-1}, \tag{35}$$

where ϕ_i is the azimuthal position of planet i at the time (measured from the perihelion of Mercury).[43] The sinusoidal terms have periods (namely, the orbital periods T_i of the perturbing planets) 84 years and 165 years, respectively, and large amplitudes $\tilde{\Omega}_i = 21.2$ and 10.3 arcsec/century. Note that μ is involved (schematically) in the combination $\sin(\mu\phi)/\mu$ [cf. Eq. (B9)], which for small μ becomes a linear term from which μ disappears.

Table II. The relevant contributions to Ω, with non-negligible weights and long periods possibly indistinguishable from a secular advance.

Planet i	ℓ	m	n	W	μ
Uranus	1	1	0	8×10^{-8}	–0.00286
Neptune	1	1	0	4×10^{-8}	–0.00146

Incidentally, because the two surviving terms have $m=1$, $n=0$, Eq. (34) shows that $\mu = -\rho_i$, so the condition $|\mu| \ll 1$ translates to $\rho_i = \omega_i/\omega \ll 1$. Therefore these terms can be evaluated by a much simpler quasi-static calculation, in which between two perihelions of Mercury the perturbing planet is assumed to be at a *fixed* position ϕ_i.

B. Comparison of terms

The leading time-dependent terms ($\alpha=2$, $\beta=-1$) are larger than those with $\alpha=3$, $\beta=0$ in the ring model, because symmetry is broken. Nevertheless, most terms have larger values of $|\mu|$; their effects can be identified and removed from data covering say at least 50 years.[44] For Uranus and Neptune, the next important term would have $n=1$, with an extra power of ε. In addition, to have a small μ, m would then have to be 2 and therefore $\ell \geq 2$, $\alpha \geq 3$, giving an extra power of ξ_i. Altogether, the next term is reduced by a factor of $\varepsilon\xi_i \sim 4\times 10^{-3}$ (the numerical value referring to Uranus) and can be ignored. This is the reason why we may take $\varepsilon = 0$ in the calculation in Appendix B.

C. Time averaging

The large oscillatory terms can be suppressed if data are averaged over a time Δt, so that the factors $\cos\phi_i$ are replaced by their averages, and the effective amplitudes become $\tilde{\Omega}_i \to (\sin\gamma_i/\gamma_i)\tilde{\Omega}_i$, where $\gamma_i = \pi\Delta t/T_i$. Using a window of 100 years, the two leading terms in Table II have amplitudes –3.2 and 5.1 arcsec/century, still by no means negligible. Of course, it is possible to remove such sinusoidal terms more effectively by fitting the data.

Therefore the advance of perihelion caused by planetary perturbations (imagining other effects to be turned off) during a century is *not* the figure of 532 arcsec often quoted, but depends, to the tune of a few arcsec, on which period one is talking about.

VII. CONCLUSION

A formalism has been developed for evaluating the precession of a Keplerian orbit (e.g., Mercury) due to an axially symmetric static perturbation. This formalism, solved to first order in the perturbation (but with no other approximations), provides a unified platform for discussing general relativity (GR), the solar quadrupole moment, and the perturbation of the other planets in a ring model. This solution for the ring model improves upon the work of Price and Rush;[21] the result agrees with that of Stewart[22] using the LRL vector, and with the usually quoted value to about 4%, a difference well accounted for by the eccentricities and departures from coplanarity of the orbits of the perturbing planets.[22]

In a more general model, the perturbing planets move in coplanar circular orbits. An angular momentum analysis reveals that the $m=0$ terms exactly recover the static ring model, so that the other terms provide the time-dependent corrections. Provided the ratios of orbital periods are not rational, all these terms lead to rates of advance that are sinusoidal in time. In theory, these are distinguishable from a secular advance, but in practice, terms with periods comparable to the observational window (say 50 to 100 years) must be considered. Only two such terms have significant amplitudes and they can be identified with the quasi-static perturbations due to Uranus and Neptune. Even when averaged over a century, the effects are, surprisingly, still at the level of a few arcsec/century.

In the planetary problem, all terms contribute as $(M_i/M)(r/R_i)^\alpha \varepsilon^\beta$, where M is the mass of the Sun, the perturbing planet i has mass M_i and orbital radius R_i, and the orbit of Mercury has radius r and eccentricity ε. The leading time-dependent contributions have values of α and β that are each one unit less than the corresponding values for the ring model; thus Uranus and Neptune have significant time-dependent effects even though they make negligible contributions in the ring model. The difference in the indices comes about because the ring model has axial symmetry, whereas a moving perturbing mass does not.

The approximation of circular orbits lends itself naturally to a frequency-domain analysis, in which secular terms and long-period oscillations are conceptually distinguished. In the time domain (including analysis of observational data), the two components could be confused unless the time window is at least several centuries. This potential pitfall does not afflict modern treatments based on fitting numerical solutions to observational data.

The formalism and results in this paper help to close the pedagogical gap in treatments of the advance of perihelion as a test of GR. In terms of pedagogy, the material can be presented at three levels: (a) the axially symmetric static model and keeping only f_1 in Eq. (15); (b) the same model keeping all f_k; and (c) including time-dependent contributions. The first and possibly the second of these, at least, should be accessible to students of GR.

The general result keeping only f_1 is, apart from notation, the same as that given by Adkins and McDonnell[29] in their Sec. II. For larger eccentricities, their result in Sec. III is expressed as an integral, whereas ours is expressed as a power series in $(\varepsilon/2)^2$. Chashchina and Silagadze[46] have also derived the precession as an integral, by considering the Hamilton vector that is perpendicular to the Runge–Lenz vector. Our result may be more convenient for numerical evaluation, especially since ε is small. Schmidt[30] has given an expression for the advance of perihelion, valid for arbitrary radial force fields F but small ε; converted to our notation, his result reads

$$\Omega = \frac{2\pi}{T}\left[(3 + rF'/F)^{-1/2} - 1\right]. \tag{36}$$

To compare to our result, let F be an inverse-square field perturbed by the additional force in Eq. (2), and evaluate Eq. (36) to first order in λ; a straightforward computation recovers Eq. (3). Beyond the common domain of validity (λ and ε both small), our result is valid for small λ and any ε, whereas Schmidt's[30] is valid for small ε and any λ (if λ is not regarded as a formal small parameter, then with the additional force in Eq. (2), the total force F is entirely arbitrary); each is therefore useful in a different situation. For the planetary problems under discussion here, $\lambda \sim 10^{-7}$ whereas $\varepsilon \sim 0.1$, so our formalism is the relevant one.

ACKNOWLEDGMENTS

We are grateful to K.-F. Li, L. M. Lin, and B. Sheen for advice on astronomical coordinates, and to C. M. Will for advice on the PPN formalism and for pointing us to the literature.[20] P. Goldreich and K.-F. Li have kindly commented on the manuscript.

APPENDIX A: RELATING TIME AND ANGLE FOR A KEPLER ORBIT

In order to express t in terms of ϕ, use $r^2(d\phi/dt) = J$ to get

$$\begin{aligned}\frac{dt}{d\phi} &= \frac{\sigma^2}{J}\frac{1}{(1+\varepsilon\cos\phi)^2}\\ &= \frac{\sigma^2}{J}\left[P_0(\varepsilon) + \sum_m{}' P_m(\varepsilon)\cos m\phi\right],\end{aligned} \tag{A1}$$

where the periodic function is expressed in a Fourier series, and $\sum'$ means the exclusion of the $m = 0$ term. Straightforward computation then gives[37] $P_0(\varepsilon) = (1-\varepsilon^2)^{-3/2}$, while $P_m = O(\varepsilon^m)$. Integrating Eq. (A1) then gives $t = T(\phi/2\pi) + \Delta(\varepsilon, \phi)$, in which the linear term has been related to the period, now recognized to be $T = 2\pi(\sigma^2/J)(1-\varepsilon^2)^{-3/2}$. The periodic term is

$$\Delta(\varepsilon, \phi) = \frac{\sigma^2}{J}\sum_m{}' \frac{P_m(\varepsilon)}{m}\sin m\phi, \tag{A2}$$

which has the three properties described in Sec. II A.

APPENDIX B: TIME-DEPENDENT MODEL

Start with the Lagrangian (per unit mass) for the motion of Mercury

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \left[-\frac{GM}{r} + \lambda\Phi(r,\phi,t)\right], \tag{B1}$$

with $\lambda\Phi$ given in Sec. VI A. This leads to the equations of motion

$$\begin{aligned}\frac{dJ}{d\phi} &= -\lambda\frac{r^2}{J}\frac{\partial\Phi(r,\phi,t)}{\partial\phi} \equiv \lambda S_J,\\ \frac{d^2u}{d\phi^2} + u - \frac{GM}{J^2} &= \lambda\frac{r^2}{J^2}\left(\frac{\partial\Phi}{\partial r} + \frac{\partial\Phi}{\partial\phi}\frac{du}{d\phi}\right) \equiv \lambda S_u.\end{aligned} \tag{B2}$$

Now write $J = J_0 + \lambda J_1 + \cdots$, $u = u_0 + \lambda u_1 + \cdots$, where J_0 and u_0 describe the zeroth-order solution of Sec. II, in particular, with $J_0 =$ constant. Then the first-order solution satisfies

$$\frac{dJ_1}{d\phi} = S_J(\phi), \quad \frac{d^2u_1}{d\phi^2} + u_1 = S_u(\phi), \tag{B3}$$

where the source terms on the right-hand sides are to be evaluated using J_0 and u_0.

The somewhat messy computation is exhibited only in the $\varepsilon \to 0$ limit; higher-order terms in ε will turn out not to be important (see Sec. VI B). The second term in S_u can be neglected because $du_0/d\phi = -\varepsilon\sin\phi = O(\varepsilon)$. Then evaluating the first term in S_u for the zeroth-order solution gives

$$S_u = -\frac{M_i}{M}\sigma^{-1}\sum_{\ell,m}{}' D_{\ell,m}\xi_i^{\ell+1}e^{im(1-\rho_i)\phi}e^{-im\phi_i^0}, \tag{B4}$$

where we have used Eq. (33) and put $r(\phi) = \sigma$ for $\varepsilon = 0$.

For $\varepsilon \neq 0$, extra periodic functions appear because of the factors of $(1+\varepsilon\cos\phi)^{-1}$ in $r(\phi)$, from the term $\Delta(\phi)$ when expressing t in terms of ϕ, and from the second term in S_u. All these lead to WO functions, whose Fourier series contains terms with $e^{-in\phi}$ and amplitude $O(\varepsilon^n)$. As indicated in Sec. VI B, these terms with $n > 0$ can be ignored.

Next evaluate the effect of one such typical term with

$$\frac{d^2u_1}{d\phi^2} + u_1 = -\frac{M_i}{M}\sigma^{-1}(se^{-im\phi_i^0})e^{i\nu\phi} + \text{c.c.}, \tag{B5}$$

where s stands for a typical coefficient in Eq. (B4): $s = D_{\ell,m}\xi_i^{\ell+1}$; note that s is real if only $n=0$ terms are kept. The solution to Eq. (B5) is

$$u_1(\phi) = \frac{M_i}{M}\sigma^{-1}\frac{2s}{\nu^2-1}\cos(\nu\phi - m\phi_i^0) \tag{B6}$$

and

$$u = \sigma^{-1}\left[(1+\varepsilon\cos\phi) + \lambda\frac{M_i}{M}\frac{2s}{\nu^2-1}\cos(\nu\phi - m\phi_i^0)\right]. \tag{B7}$$

Henceforth assume $\nu \approx 1$, $|\mu| \ll 1$, since otherwise the second term is observationally distinguishable from a slowly precessing ellipse and can be assumed to be filtered out from the data. Setting $du/d\phi = 0$ to find the perihelion angle ϕ_P, we get

$$\varepsilon\sin\phi_P + \lambda\frac{M_i}{M}\frac{s}{\mu}\sin(\nu\phi_P - m\phi_i^0) = 0. \tag{B8}$$

Now put $\phi_P = 2N\pi + \lambda\phi_{P1}(N) + \cdots$ and evaluate Eq. (B8) to first order in λ to obtain

$$\phi_{P1}(N) = -\frac{M_i}{M}\frac{s}{\varepsilon\mu}\sin(2N\pi\mu - m\phi_i^0). \tag{B9}$$

The perihelion shifts in each cycle by[45]

$$\Delta\phi_P = \frac{d\phi_{P1}}{dN} = -\frac{M_i}{M}\frac{2\pi s}{\varepsilon}\cos\psi, \tag{B10}$$

where $\psi = 2N\pi\mu - m\phi_i^0$ is readily recognized to be $-m\phi_i$ evaluated at the time $t = t_N$ when Mercury has executed N cycles. Thus, each such term contributes to Ω a time-varying term given by

$$\Omega = -\frac{\pi}{T\varepsilon}\frac{M_i}{M}2s\cos m\phi_i. \tag{B11}$$

The actual solution consists of many such terms, in a fourfold sum involving i, ℓ, m, n. Fortunately, most terms can be dropped on account of the combination of several requirements:

Weight. Each term contributes to Ω with a weight W given by Eq. (28), now with $\alpha = \ell + 1$ and $\beta = n - 1$. [The latter is shifted by one unit on account of the prefactor $(1/\varepsilon)$ in Eq. (B11).] If we require $W \geq 10^{-3}W_0$, then 140 sets of (i, ℓ, n) values remain—roughly four times as many as in the corresponding time-independent case, because α need not be odd and β need not be even.

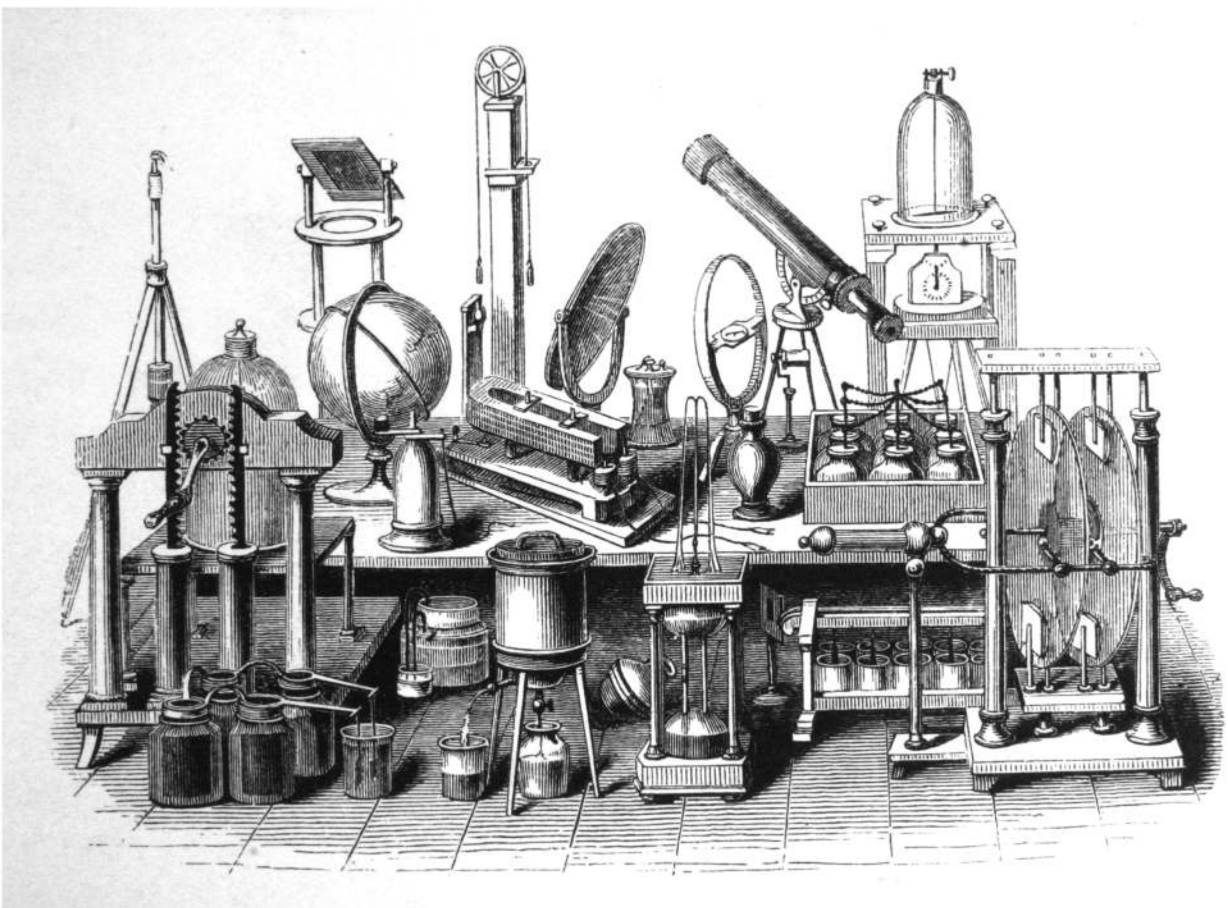
Oscillatory period. Assume (somewhat arbitrarily) that all observations of Ω are made over periods >50 years, or >200 orbital periods for Mercury. In this case only $|\mu| < 5\times10^{-3}$ need be considered, and only four sets of (i, ℓ, n) values remain; moreover, for each of these, at most one value of m will make $|\mu|$ small.

Other conditions. Only those terms with $\ell + m$ even will contribute.

Only two terms remain (Table II). Both have $n = 0$, and this result has been anticipated and used to simplify many intermediate expressions. The rest of the calculation is shown in Sec. VI.

[a]Electronic mail: lucienlo@pas.rochester.edu; Present address: Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA.
[b]Electronic mail: kyoung@cuhk.edu.hk
[c]Electronic mail: leebyp@gmail.com
[1]R. Resnick, D. Halliday, and K. S. Krane, *Physics*, 4th ed. (Wiley, New York, 1992), Vol. 1.
[2]R. Resnick, D. Halliday, and K. S. Krane, *Physics*, 5th ed. (Wiley, New York, 2002), Vol. 1.
[3]R. Wolfson and J. M. Pasachoff, *Physics for Scientists and Engineers*, 2nd ed. (HarperCollins, New York, 1995).
[4]J. R. Taylor, C. D. Zafiratos, and M. A. Dubson, *Modern Physics for Scientists and Engineers*, 2nd ed. (Pearson Prentice-Hall, Upper Saddle River, NJ, 2004).
[5]J. A. Coleman, *Relativity for the Layman: A Simplified Account of the History, Theory, and Proofs of Relativity* (Macmillan, New York, 1959).
[6]A. S. Eddington, *Space, Time, and Gravitation: An Outline of the General Relativity Theory* (Harper, New York, 1959).
[7]L. Brillouin, *Relativity Reexamined* (Academic Press, New York, 1970).
[8]L. Fang and Y. Chu, *From Newton's Laws to Einstein's Theory of Relativity*, translated by H. Huang (Science Press, Beijing, 1987).
[9]C. M. Will, *Was Einstein Right?: Putting General Relativity to the Test* (Oxford U.P., Oxford, U.K., 1988).
[10]R. Wolfson, *Simply Einstein: Relativity Demystified* (Norton, New York, 2003).
[11]R. Stannard, *Relativity: A Very Short Introduction* (Oxford U.P., Oxford, U.K., 2008).
[12]W. Isaacson, *Einstein: His Life and Universe* (Simon and Schuster, New York, 2007).
[13]S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
[14]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[15]B. F. Schutz, *A First Course in General Relativity* (Cambridge U.P., Cambridge, U.K., 1985).
[16]T. L. Chow, *General Relativity and Cosmology: A First Course* (Wuerz Publishing, Winnipeg, 1994).
[17]J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addsion-Wesley, San Francisco, 2003).
[18]M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge U.P., Cambridge, U.K., 2006).
[19]T. P. Cheng, *Relativity, Gravitation and Cosmology: A Basic Introduction* (Oxford U.P., Oxford, U.K., 2010).
[20]C. M. Will, "The confrontation between general relativity and experiment," Living Rev. Relat. **9**, 3 (2006), <http://www.livingreviews.org/lrr-2006-3>.
[21]M. P. Price and W. F. Rush, "Nonrelativistic contribution to Mercury's perihelion precession," Am. J. Phys. **47**, 531–534 (1979).
[22]M. G. Stewart, "Precession of the perihelion of Mercury's orbit," Am. J. Phys. **73**, 730–734 (2005).
[23]R. H. Dicke and H. M. Goldenberg, "Solar oblateness and general relativity," Phys. Rev. Lett. **18**, 313–316 (1967).
[24]R. H. Dicke, "The solar oblateness and the gravitational quadrupole moment," Ap. J. **159**, 1–24 (1970).
[25]R. H. Dicke, J. R. Kuhn, and K. G. Libbrecht, "Oblateness of the Sun in 1983 and relativity," Nature **316**, 687–690 (1985).
[26]N. T. Roseveare, *Mercury's Perihelion: From Le Verrier to Einstein* (Clarendon Press, Oxford, U.K., 1982).

[27]R. Baum and W. Sheehaan, *In Search of Planet Vulcan: The Ghost in Newton's Clockwork Universe* (Plenum, New York, 1997).
[28]H.-J. Schmidt, "Perihelion advance for orbits with large eccentricities in the Schwarzschild black hole," Phys. Rev. D **83**, 124010 (2011).
[29]G. S. Adkins and J. McDonnell, "Orbital precession due to central-force perturbations," Phys. Rev. D **75**, 082001 (2007).
[30]H.-J. Schmidt, "Perihelion precession for modified Newtonian gravity," Phys. Rev. D **78**, 023512 (2008).
[31]This is analogous to a simple harmonic oscillator with natural frequency 1 and responding at the driving frequency m.
[32]This is analogous to a simple harmonic oscillator driven at resonance, and responding with secular terms.
[33]It will be seen below that $g_1 = O(\varepsilon)$ as $\varepsilon \to 0$, so g_1/ε is finite.
[34]L. Campbell, *et al.*, "The Sun's quadrupole moment and perihelion precession of Mercury," Nature **305**, 508–510 (1983).
[35]T. M. Brown *et al.*, "Inferring the sun's internal angular velocity from observed p-mode frequency splittings," Ap. J. **343**, 526–546 (1989), cited by Hartle (Ref. 17).
[36]Whether Pluto is regarded as a planet is irrelevant.
[37]I. S. Gradyshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York and London, 1965).
[38]This is evidenced by the *first*-order expansion in $X = r - a$ in arriving at their Eq. (10).
[39]In our formalism and in that of Stewart,[22] it does not matter whether the expansion is carried out around $u_1 = a^{-1}$ or $u_2 = \sigma^{-1}$, since all terms are kept.
[40]When only one planet is discussed, there is the freedom to change $\phi \to \phi + \pi$, $\varepsilon \to -\varepsilon$, so only ε^2 enters but not ε. But when $\phi = 0$ is already defined by the orbit of one planet, this freedom is no longer available to change the sign of ε_i for another planet, so corrections will enter to $O(\varepsilon_i^1)$.
[41]A sum over different planets i is understood.
[42]See, e.g., J. D. Jackosn, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 92, Eq. (3.38) and p. 102, Eq. (3.70).
[43]Heliocentric longitudes on 1 January 1960 are found from <http://cohoweb.gsfc.nasa.gov/helios/planet.html>: 77.56° for the perihelion of Mercury, and 138.15°, 217.02° respectively for the longitudes of Uranus and Neptune. Within the approximation of circular orbits for the perturbing planets, the latter two are calculated at other times by assuming a uniform rate of change according to the known periods.
[44]This situation is not axiomatic. One can imagine a hypothetical system where $\rho_i = 0.500001$ for Venus.
[45]Replacing the finite difference by a derivative in N is justified only for those terms with $|\mu| \ll 1$.
[46]O. I. Chashchina and Z. K. Silagadze, "Remark on orbital precession due to central-force perturbations," Phys. Rev. D **77**, 107502 (2008).



Frontispiece

The illustration on the title page to J. Muller's *Principles of Physics and Meteorology* (London, 1847) contains images of at least twenty three pieces of apparatus, including Nörrenberg's apparatus, vacuum pump, tangent galvanometer, Atwood's machine, barometer, battery of Leyden jars, powder bomb, electroplating bath, plunge battery, magneto-electric machine, telescope and prism. How many can you identify? (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)