Advanced Applications of the Holonomic Systems Approach

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3 December 2009

1. The **Holonomic Systems Approach**: Implementation of the *Mathematica* package `HolonomicFunctions`
   - Noncommutative Gröbner bases in Ore algebras
   - Rational solutions of coupled linear systems of difference or differential equations
   - Closure properties for ∂-finite functions
   - Summation/integration algorithms due to Zeilberger, Takayama, and Chyzak
   - http://www.risc.uni-linz.ac.at/research/combinat/software/

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2. Three **Advanced Applications**
   - Proof of Ira Gessel's lattice path conjecture
   - Relations between basis functions in FEM
   - Computer proof of Stembridge's TSPP theorem

Introduction

Basic idea: describe functions/sequences via

- linear relations (PDEs, multivariate recurrences, mixed difference-differential equations) and
- finitely many initial values.

All possible manipulations (addition, multiplication, substitutions, summation, integration) are performed on this level.

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(Informal) Definition: A function $f(x_1, \ldots, x_d)$ is called ∂-finite w.r.t. the operators $\partial_1, \ldots, \partial_d$ if all its "derivatives" $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}(f)$ span a finite-dimensional $\mathbb{K}(x_1, \ldots, x_d)$-vector space.

Remark: The notion ∂-finiteness is closely related with that of holonomic systems.

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Example: The Legendre polynomials $P_n(x)$ are ∂-finite with respect to the operators S_n and D_x.

Standard Application 1

Abramowitz/Stegun (10.1.41)

$$\left.\frac{\partial j_\nu(x)}{\partial \nu}\right|_{\nu=0} = \frac{\mathrm{Ci}(2x)\sin(x) - \mathrm{Si}(2x)\cos(x)}{x}$$

What we need

- Closure properties
- a database of functions whose ∂-finite description is known:

hypergeometric expressions, hyperexponential expressions, algebraic expressions, AiryAi, AiryAiPrime, AiryBi, AiryBiPrime, AngerJ, AppellF1, ArcCos, ArcCosh, ArcCot, ArcCoth, ArcCsc, ArcCsch, ArcSec, ArcSech, ArcSin, ArcSinh, ArcTan, ArcTanh, ArithmeticGeometricMean, BellB, BernoulliB, BesselI, BesselJ, BesselK, BesselY, Beta, BetaRegularized, Binomial, CatalanNumber, ChebyshevT, ChebyshevU, Cos, Cosh, CoshIntegral, CosIntegral, EllipticE, EllipticF, EllipticK, EllipticPi, EllipticTheta, EllipticThetaPrime, Erf, Erfc, Erfi, EulerE, Exp, ExpIntegralE, ExpIntegralEi, Factorial, Factorial2, Fibonacci, FresnelC, FresnelS, Gamma, GammaRegularized, GegenbauerC, HankelH1, HankelH2, HarmonicNumber, HermiteH, Hypergeometric0F1, Hypergeometric0F1Regularized, ...

What we need

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...
Hypergeometric1F1, Hypergeometric1F1Regularized, Hypergeometric2F1, Hypergeometric2F1Regularized, HypergeometricPFQ, HypergeometricPFQRegularized, HypergeometricU, JacobiP, KelvinBei, KelvinBer, KelvinKei, KelvinKer, LaguerreL, LegendreP, LegendreQ, LerchPhi, Log, LogGamma, LucasL, Multinomial, NevilleThetaC, ParabolicCylinderD, Pochhammer, PolyGamma, PolyLog, QBinomial, QFactorial, QPochhammer, Root, Sin, Sinh, SinhIntegral, SinIntegral, SphericalBesselJ, SphericalBesselY, SphericalHankelH1, SphericalHankelH2, Sqrt, StirlingS1, StirlingS2, StruveH, StruveL, Subfactorial, WeberE, WhittakerM, WhittakerW, Zeta.

Standard Application 1

```
lhs = Annihilator[Derivative[1,0][SphericalBesselJ][0,x],
                  Der[x]]
```

$$\{x^3 D_x^4 + 8x^2 D_x^3 + (2x^3 + 14x) D_x^2 + (8x^2 + 4) D_x + (x^3 + 6x)\}$$

Standard Application 1

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```
rhs = Annihilator[
  1/x*(CosIntegral[2*x]*Sin[x]-SinIntegral[2*x]*Cos[x]),
  Der[x]]
```

$$\{(12x^5 + 5x^3) D_x^6 + (144x^4 + 70x^2) D_x^5 + (132x^5 + 475x^3 + 260x) D_x^4 + (1056x^4 + 796x^2 + 240) D_x^3 + (228x^5 + 1991x^3 + 1288x) D_x^2 + (912x^4 + 1110x^2 + 560) D_x + (108x^5 + 753x^3 + 516x)\}$$

Standard Application 1

One ODE is a multiple of the other one:

```
OreReduce[rhs, lhs]
```

$$\{0\}$$

Therefore 6 initial values had to be checked.

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$$\{0\}$$

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Alternatively:

```
Together[
 ApplyOreOperator[lhs,
  1/x*(CosIntegral[2*x]*Sin[x]-SinIntegral[2*x]*Cos[x])]]
```

$$\{0\}$$

Hence, only 4 initial values have to be compared.

Standard Application 2

A hypergeometric double sum

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{n+r+s}\binom{n}{r}\binom{n+r}{r}\binom{n}{s}\binom{n+s}{s}\binom{2n-(r+s)}{n} =$$

$$\sum_{k=0}^{\infty}\binom{n}{k}^4$$

Standard Application 2

```
Annihilator[(-1)^(n+r+s)*Binomial[n,r]*Binomial[n,s]*
  Binomial[n+r,r]*Binomial[n+s,s]*Binomial[2*n-(r+s),n],
  {S[r], S[s], S[n]}]
```

$$\{(n+1)(n-r+1)(n-s+1)(n-r-s+1)S_n \\ +(n+r+1)(n+s+1)(2n-r-s+1)(2n-r-s+2), \\ (s+1)^2(2n-r-s)S_s+(n-s)(n+s+1)(n-r-s), \\ -(r+1)^2(2n-r-s)S_r-(n-r)(n+r+1)(n-r-s)\}$$

```
Takayama[%, {r, s}]
```

$$\{(n+2)^3S_n^2-2(2n+3)\left(3n^2+9n+7\right)S_n-4(n+1)(4n+3)(4n+5)\}$$

```
Annihilator[Sum[Binomial[n, k]^4, {k, 0, n}], S[n]]
```

$$\{(n+2)^3S_n^2-2(2n+3)\left(3n^2+9n+7\right)S_n-4(n+1)(4n+3)(4n+5)\}$$

Standard Application 3

Andrews/Askey/Roy (6.8.10)

For $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$, for $l+m+n$ even and the sum of any two of l, m, n is not less than the third:

$$\int_{-1}^{1} C_l^{(\lambda)}(x)\, C_m^{(\lambda)}(x)\, C_n^{(\lambda)}(x) \left(1-x^2\right)^{\lambda-1/2}\, \mathrm{d}x =$$

$$\frac{\pi\, 2^{1-2\lambda}\Gamma\left(2\lambda+\frac{1}{2}(l+m+n)\right)}{\Gamma(\lambda)^2\left(\frac{1}{2}(l+m+n)+\lambda\right)} \times \frac{(\lambda)_{(m+n-l)/2}(\lambda)_{(l+n-m)/2}(\lambda)_{(l+m-n)/2}}{\left(\frac{1}{2}(m+n-l)\right)!\left(\frac{1}{2}(l+n-m)\right)!\left(\frac{1}{2}(l+m-n)\right)!(\lambda)_{(l+m+n)/2}}$$

Standard Application 3

An annihilating ideal for the integral is obtained via creative telescoping using Chyzak's algorithm. The same ideal is obtained for the right-hand side:

```
rhs = Annihilator[Pochhammer[..., {S[l], S[m], S[n]}]
```

$$
\begin{aligned}
\{&(l+m-n+1)(l-m+n+2\lambda-1)S_m \\
&\quad -(l-m+n+1)(l+m-n+2\lambda-1)S_n, \\
&(l+m-n+1)(l-m-n-2\lambda+1)S_l \\
&\quad -(l-m-n-1)(l+m-n+2\lambda-1)S_n, \\
&(l-m-n-2)(l-m+n+2)(l+m-n+2\lambda-2) \\
&\qquad \times(l+m+n+2\lambda+2)S_n^2 \\
&\quad -(l+m-n)(l-m-n-2\lambda)(l-m+n+2\lambda)(l+m+n+4\lambda)\}
\end{aligned}
$$

There seem to be lots of singularities!

Standard Application 3

```
AnnihilatorSingularities[rhs, {0, 0, 0},
  Assumptions -> lambda > 1/2 &&
                 Element[(l+m+n)/2, Integers] &&
                 l+m >= n && l+n >= m && m+n >= l]
```

$$\{\{\{l \to 0, m \to n\}, \lambda > \tfrac{1}{2} \wedge n \geq 0\}, \{\{l \to n, m \to 0\}, \lambda > \tfrac{1}{2} \wedge n \geq 0\},$$
$$\{\{l \to 0, m \to 0, n \to 0\}, \lambda > \tfrac{1}{2}\}, \{\{l \to 0, m \to 1, n \to 1\}, \lambda > \tfrac{1}{2}\},$$
$$\{\{l \to 1, m \to 0, n \to 1\}, \lambda > \tfrac{1}{2}\}, \{\{l \to 2, m \to 0, n \to 2\}, \lambda > \tfrac{1}{2}\}\}$$

The isolated singular points are checked immediately.

For the first two cases, we apply creative telescoping recursively.

Standard Application 4

Gradshteyn/Ryzhik (6.512.1)

$$\int_0^\infty J_m(ax)J_n(bx)\,dx =$$

$$\frac{a^{-n-1}b^n\Gamma\left(\frac{1}{2}(m+n+1)\right)}{\Gamma(n+1)\Gamma\left(\frac{1}{2}(m-n+1)\right)}$$

$$\times {}_2F_1\left(\tfrac{1}{2}(m+n+1), \tfrac{1}{2}(-m+n+1), n+1, \tfrac{b^2}{a^2}\right)$$

Standard Application 4

```
CreativeTelescoping[BesselJ[m, a*x]*BesselJ[n, b*x],
                    Der[x], {Der[a],Der[b],S[m],S[n]}]
```

$$\{\{aD_a + bD_b + 1,$$
$$(b^2m^2 - b^2n^2 - 2b^2n - b^2)S_n^2 + (2a^2bn + 2a^2b - 2b^3n - 2b^3)D_b$$
$$-2a^2n^2 - 2a^2n + b^2m^2 + b^2n^2 - b^2,$$
$$(abm + abn + ab)S_mS_n + (a^2b - b^3)D_b - a^2n - b^2m - b^2,$$
$$(a^2m^2 + 2a^2m - a^2n^2 + a^2)S_m^2 + (-2a^2bm - 2a^2b + 2b^3m + 2b^3)D_b$$
$$-a^2m^2 - 2a^2m - a^2n^2 - a^2 + 2b^2m^2 + 4b^2m + 2b^2,$$
$$(a^2b - b^3)D_bS_n + (abn - abm)S_m + (a^2n + a^2 - b^2m - b^2)S_n,$$
$$(a^2b - b^3)D_bS_m + (b^2m - a^2n)S_m + (abm - abn)S_n,$$
$$(a^2b^2 - b^4)D_b^2 + (a^2b - 3b^3)D_b - a^2n^2 + b^2m^2 - b^2\},$$

$$\{-x, 2(abnx + abx)S_mS_n - 2\left(bmn + bm + bn^2 + 2bn + b\right)S_n$$
$$+ 2\left(b^2nx + b^2x\right), abxS_mS_n + b^2x,$$
$$- 2(abmx + abx)S_mS_n + 2\left(am^2 + amn + 2am + an + a\right)S_m$$
$$- 2\left(b^2mx + b^2x\right), -abxS_m + b^2xS_n, b^2xS_m - abxS_n,$$
$$ab^2x^2S_m - b^3x^2S_n - b^2mx + b^2nx + b^2x\}\}$$

Standard Application 4

```
gb = OreGroebnerBasis[
  Annihilator[BesselJ[m, a*x]*BesselJ[n, b*x],
    {Der[x], Der[a], Der[b], S[m], S[n]}],
  OreAlgebra[x, Der[x], Der[a], Der[b], S[m], S[n]],
  MonomialOrder -> EliminationOrder[1]
];
LeadingPowerProduct /@ gb
```

$$\{D_a S_n, D_a S_m, D_a^2, D_x S_n, D_x S_m, D_x D_b,$$
$$D_x D_a, S_m^2 S_n, D_b S_n^2, D_b S_m S_n, D_b S_m^2, x\}$$

Standard Application 5

Gradshteyn/Ryzhik (4.539)

$$\int_0^1 \frac{(-\log(t))^{s-1}\arctan(at)}{t}\,\mathrm{d}t = a2^{-s-1}\Gamma(s)\Phi\left(-a^2, s+1, \tfrac{1}{2}\right)$$

where $\Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}$ is the Lerch transcendent.

Standard Application 5

Both sides contain non-holonomic expressions (see the ISSAC'09 paper by Chyzak/Kauers/Salvy).

```
Annihilator[
  Integrate[(-Log[t])^(s-1)/t*ArcTan[a*t], {t, 0, 1}],
  {Der[a], S[s]}, Assumptions -> s>0]
```

Annihilator::nondf: The expression (-Log[t])^(-1 + s) is not recognized to be ∂-finite. The result might not generate a zero-dimensional ideal.

$$\{aD_aS_s - s\}$$

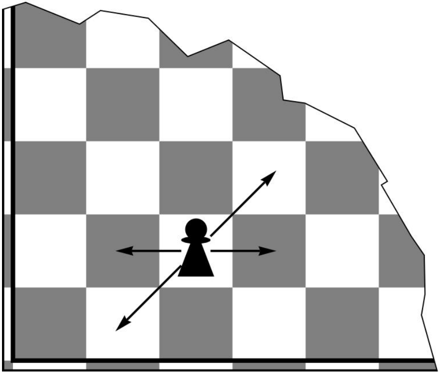
```
Annihilator[2^(-s-1)*Gamma[s]*a*LerchPhi[-a^2, s+1, 1/2]]
```

$$\{aD_aS_s - s\}$$

Advanced Application 1

Proof of Gessel's conjecture

(joint work with M. Kauers and D. Zeilberger)



Advanced Application 2

Relations for speeding up FEM

Problem setting

Joachim Schöberl (RWTH Aachen): Simulate the propagation of electromagnetic waves using the Maxwell equations

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively. Define basis functions (in 2D) in order to approximate the solution:

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x} - 1\right)$$

Problem: need to represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the original basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself)

The Gröbner approach

The numerists need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

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that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

- consider the operators D_x, S_i, and S_j
- basis functions $\varphi_{i,j}(x,y)$ are ∂-finite with respect to them
- compute generators of an annihilating left ideal for $\varphi_{i,j}(x,y)$
- represent them in the algebra $\mathbb{Q}(i,j)[x,y][D_x;1,D_x][S_i;S_i,0][S_j;S_j,0]$
- compute a Gröbner basis in order to eliminate x and y

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- consider the operators D_x, S_i, and S_j
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- represent them in the algebra $\mathbb{Q}(i,j)[x,y][D_x;1,D_x][S_i;S_i,0][S_j;S_j,0]$
- compute a Gröbner basis in order to eliminate x and y
- takes very long, interrupt as soon as a desired operator is found
- result is quite big (2 pages of output)
- because of "extension/contraction" we can not be sure that we obtain the smallest operator

The ansatz approach

The numerists need a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) = \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

- work in the algebra $\mathbb{Q}(i,j,x,y)[D_x;1,D_x][S_i;S_i,0][S_j;S_j,0]$
- compute a Gröbner basis G of a ∂-finite annihilating ideal for $\varphi_{i,j}(x,y)$
- choose index sets A and B
- reduce the above ansatz with G
- do coefficient comparison with respect to x and y
- solve the resulting linear system for $a_{k,l}, b_{m,n} \in \mathbb{Q}(i,j)$
- can find the "smallest" relation
- certain optimizations (e.g., using homomorphic images) reduce the computation time to a few seconds

Optimizations (1)

Of course,

$$\mathrm{nf}\left(\sum_k a_k \boldsymbol{\partial}^{\boldsymbol{\alpha}_k}\right) = \sum_k a_k \,\mathrm{nf}\left(\boldsymbol{\partial}^{\boldsymbol{\alpha}_k}\right)$$

- reduce each monomial $\boldsymbol{\partial}^{\boldsymbol{\alpha}_k}$ separately
- use previously computed normal forms

Optimizations (2)

Idea: Can we use homomorphic images for finding a good ansatz?

- surely we can compute in $\mathbb{Z}_p(i,j,x,y)[D_x;1,D_x][S_i;S_i,0][S_j;S_j,0]$
- this does not help much
- better: try to reduce polynomial arithmetic
- have to keep x and y symbolically (coefficient comparison)
- what about i and j? If we plug in values for them, we loose noncommutativity!

Recall: normal form computation

Let $\mathbb{O}$ be the operator algebra.

Input: $p \in \mathbb{O}$, a Gröbner basis $G = \{g_1, \ldots, g_n\} \subseteq \mathbb{O}$
Output: normal form of p modulo ${}_{\mathbb{O}}\langle G \rangle$

while exists $1 \leq k \leq n$ such that $\mathrm{lm}(g_k) \,|\, \mathrm{lm}(p)$
    $g := (\mathrm{lm}(p)/\mathrm{lm}(g_k)) \cdot g_k$
    $p := p - (\mathrm{lc}(p)/\mathrm{lc}(g)) \cdot g$
end while

Modular normal form computation

Input: $p \in \mathbb{O}$, a Gröbner basis $G = \{g_1, \ldots, g_n\} \subseteq \mathbb{O}$
Output: normal form of p modulo ${}_{\mathbb{O}}\langle G \rangle$

while exists $1 \leq k \leq n$ such that $\operatorname{lm}(g_k) \mid \operatorname{lm}(p)$
 $g := h((\operatorname{lm}(p)/\operatorname{lm}(g_k)) \cdot g_k)$
 $p := p - (\operatorname{lc}(p)/\operatorname{lc}(g)) \cdot g$
end while

where h is an insertion homomorphism, in our example

$$\begin{aligned} h: \mathbb{Q}(i,j,x,y) &\to \mathbb{Q}(x,y) \\ f(i,j,x,y) &\mapsto f(i_0,j_0,x,y), \quad \text{for } i_0, j_0 \in \mathbb{Z} \end{aligned}$$

Result

With this method, we find in a few seconds relations like

$$
\begin{aligned}
&(2i+j+5)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)\\
&+2(2i+1)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)\\
&-(j+3)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)\\
&+(j+1)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)\\
&-2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)\\
&+(2i+j+5)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)=\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)\\
&-2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}
$$

$\longrightarrow$ these formulae already caused a speed-up of 20 percent (!) in the numerical simulations.

3D case

We would like to do the same thing in 3D.

- now the basis functions

$$\varphi(i,j,k,x,y,z) := \begin{array}{l} P_i\left(\frac{2z}{(1-x)(1-y)}-1\right)(1-x)^i(1-y)^i \\ P_j^{(2i+1,0)}\left(\frac{2y}{1-x}-1\right)(1-x)^j \\ P_k^{(2i+2j+2,0)}(2x-1) \end{array}$$

  contain 6 variables
- computations become too big and too slow

A first result for 3D

One of the supports looks as follows:

$$\{S_j S_k^4, S_j^2 S_k^3, S_j^3 S_k^2, S_j^4 S_k, D_x S_j S_k^3, D_x S_j^2 S_k^2, D_x S_j^3 S_k, D_x S_j^4, S_j S_k^5,$$
$$S_j^2 S_k^4, S_j^3 S_k^3, S_j^4 S_k^2, S_i S_k^5, S_i S_j S_k^4, S_i S_j^2 S_k^3, S_i S_j^3 S_k^2, D_x S_j S_k^4, D_x S_j^2 S_k^3,$$
$$D_x S_j^3 S_k^2, D_x S_j^4 S_k, D_x S_i S_k^4, D_x S_i S_j S_k^3, D_x S_i S_j^2 S_k^2, D_x S_i S_j^3 S_k, S_j S_k^6,$$
$$S_j^2 S_k^5, S_j^3 S_k^4, S_j^4 S_k^3, S_i S_k^6, S_i S_j S_k^5, S_i S_j^2 S_k^4, S_i S_j^3 S_k^3, D_x S_j S_k^5, D_x S_j^2 S_k^4,$$
$$D_x S_j^3 S_k^3, D_x S_j^4 S_k^2, D_x S_i S_k^5, D_x S_i S_j S_k^4, D_x S_i S_j^2 S_k^3, D_x S_i S_j^3 S_k^2, S_j S_k^7,$$
$$S_j^2 S_k^6, S_j^3 S_k^5, S_j^4 S_k^4, S_i S_k^7, S_i S_j S_k^6, S_i S_j^2 S_k^5, S_i S_j^3 S_k^4, D_x S_j S_k^6, D_x S_j^2 S_k^5,$$
$$D_x S_j^3 S_k^4, D_x S_j^4 S_k^3, D_x S_i S_k^6, D_x S_i S_j S_k^5, D_x S_i S_j^2 S_k^4, D_x S_i S_j^3 S_k^3, S_j S_k^8,$$
$$S_j^2 S_k^7, S_j^3 S_k^6, S_j^4 S_k^5, D_x S_j S_k^7, D_x S_j^2 S_k^6, D_x S_j^3 S_k^5, D_x S_j^4 S_k^4, D_x S_i S_k^7,$$
$$D_x S_i S_j S_k^6, D_x S_i S_j^2 S_k^5, D_x S_i S_j^3 S_k^4, D_x S_j S_k^8, D_x S_j^2 S_k^7, D_x S_j^3 S_k^6,$$
$$D_x S_j^4 S_k^5, D_x S_i S_k^8, D_x S_i S_j S_k^7, D_x S_i S_j^2 S_k^6, D_x S_i S_j^3 S_k^5, D_x S_j S_k^9,$$
$$D_x S_j^2 S_k^8, D_x S_j^3 S_k^7, D_x S_j^4 S_k^6\}$$

Joachim Schöberl was impressed but not too happy about these results...

Divide ...

Idea: Write $\varphi = u \cdot v \cdot w$ with

$$\begin{aligned} u &= P_i\left(\tfrac{2z}{(1-x)(1-y)} - 1\right)(1-x)^i(1-y)^i \\ v &= P_j^{(2i+1,0)}\left(\tfrac{2y}{1-x} - 1\right)(1-x)^j \\ w &= P_k^{(2i+2j+2,0)}(2x-1) \end{aligned}$$

and use the product rule

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}vw + u\frac{\mathrm{d}v}{\mathrm{d}x}w + uv\frac{\mathrm{d}w}{\mathrm{d}x}$$

We now want to find a relation between e.g. uvw and $\frac{\mathrm{d}u}{\mathrm{d}x}vw$.

Task: find relation between uvw and $\frac{\mathrm{d}u}{\mathrm{d}x}vw$

How does this fit into our framework?

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$$\mathrm{op}_1 \bullet f = \mathrm{op}_2 \bullet g.$$

Trivial solution: $\mathrm{op}_1 \in \mathrm{Ann}\, f$ and $\mathrm{op}_2 \in \mathrm{Ann}\, g$. But since f and g are closely related we expect that there exists something “better”.

. . . and conquer

The natural way to express a relation like

$$\mathrm{op}_1 \bullet f = \mathrm{op}_2 \bullet g$$

is by using operator vectors in $M = \mathbb{O} \times \mathbb{O}$ which we let act on $\mathcal{F} \times \mathcal{F}$ by

$$P \bullet F = (P_1, P_2) \bullet (f, g) := P_1 \bullet f + P_2 \bullet g, \quad \text{where } P \in M, F \in \mathcal{F} \times \mathcal{F}$$

But how to compute a Gröbner basis for the ideal of relations between f and g, i.e. the annihilator $\mathrm{Ann}_M(f, g)$?

Closure properties

Let $f = uvw$ and $g = \frac{\mathrm{d}u}{\mathrm{d}x}vw$.

We start with u and $u' = \frac{\mathrm{d}u}{\mathrm{d}x}$:
$\operatorname{Ann}_M(u, u') =$

$$_{\mathbb{O}}\Big\langle \big\{(p,0) | p \in \operatorname{Ann}_{\mathbb{O}} u\big\} \cup \big\{(0,p) | p \in \operatorname{Ann}_{\mathbb{O}} u'\big\} \cup \big\{(D_x, -1)\big\} \Big\rangle$$

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After computing a Gröbner basis of the above, we can perform the closure property "multiplication by vw" in a very similar fashion as usual (using an FGLM-like approach).

Result

Finally we can use the ansatz technique as before in order to find an $\{x, y, z\}$-free operator:

$$-2(1+2i)(2+j)(3+2i+j)(7+2i+2j)(5+i+j+k)$$
$$(7+i+j+k)(8+i+j+k)(8+2i+2j+k)(9+2i+2j+k)$$
$$(11+2i+2j+2k)(15+2i+2j+2k)f(i,j+1,k+3)+$$

$$\vdots$$

$\langle$ 31 similar terms $\rangle$

$$\vdots$$

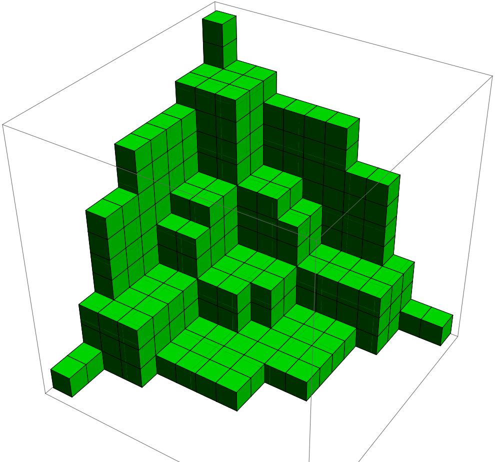
$$-2(4+2i+j)(5+2i+j)(5+2i+2j)(5+i+j+k)$$
$$(6+i+j+k)(8+i+j+k)(10+2i+2j+k)$$
$$(11+2i+2j+k)(11+2i+2j+2k)(15+2i+2j+2k)$$
$$g(i+1,j+2,k+3)=0$$

where $f = uvw$ and $g = \frac{\mathrm{d}u}{\mathrm{d}x}vw$.

Advanced Application 3

Stembridge's TSPP Theorem

(motivated by a $300 prize from D. Zeilberger)



Totally Symmetric Plane Partitions (TSPP)

Theorem: (John Stembridge, 1995)
The number of TSPPs whose 3D Ferrers diagram is bounded inside the cube $[0,n]^3$ is given by the product-formula

$$\prod_{1 \le i \le j \le k \le n} \frac{i+j+k-1}{i+j+k-2}.$$

Soichi Okada proved that the TSPP formula is true if

$$\det\left(a(i,j)\right)_{1 \le i,j \le n} = \prod_{1 \le i \le j \le k \le n} \left(\frac{i+j+k-1}{i+j+k-2}\right)^2,$$

where

$$a(i,j) = \binom{i+j-2}{i-1} + \binom{i+j-1}{i} + 2\delta(i,j) - \delta(i,j+1).$$

Translation to holonomic framework

Doron Zeilberger proposed a method for proving that

$$\det(a(i,j))_{1\le i,j\le n} = \mathrm{Nice}(n),$$

for some explicit expressions $a(i,j)$ and $\mathrm{Nice}(n)$, and for all $n \in \mathbb{N}$:

Find another discrete function $B(n,j)$ such that the following identities hold:

$$\sum_{j=1}^{n} B(n,j)a(i,j) = 0, \qquad i,n \in \mathbb{N}, i < n$$

$$B(n,n) = 1, \qquad n \in \mathbb{N},$$

$$\sum_{j=1}^{n} B(n,j)a(n,j) = \frac{\mathrm{Nice}(n)}{\mathrm{Nice}(n-1)}, \qquad n \in \mathbb{N}.$$

Then the determinant evaluation follows as a consequence.

How to find $B(n, j)$

- we do not know a closed form for $B(n, j)$, but
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∂-finite description: We succeeded in computing a Gröbner basis of these recurrences.
The Gröbner basis consists of 5 operators; their leading monomials $S_j^4, S_j^3 S_n, S_j^2 S_n^2, S_j S_n^3, S_n^4$ form a staircase of regular shape.

Several summation approaches

$$\sum_{j=1}^{n} B(n,j)a(n,j) = \frac{\mathrm{Nice}(n)}{\mathrm{Nice}(n-1)}$$

There are several methods for treating such holonomic sums; we unsuccessfully tried

- elimination (Zeilberger's slow algorithm),
- Takayama's algorithm,
- Chyzak's algorithm

(but could not accomplish the necessary computations).

The right ansatz

Chyzak's algorithm makes an ansatz of the form

$$\sum_{i=0}^{d} p_i(n) S_n^i \quad + \quad (S_j - 1) \sum_{S_j^l S_n^m \in U} q_{l,m}(n,j) S_j^l S_n^m$$

for $p_i \in \mathbb{Q}(n)$ and $q_{l,m} \in \mathbb{Q}(n,j)$. Uncoupling is needed!

Finally, we succeeded by using a "polynomial ansatz" for a creative telescoping operator:

$$\sum_{i=0}^{d} p_i(n) S_n^i \quad + \quad (S_j - 1) \sum_{k,l,m} q_{k,l,m}(n) j^k S_j^l S_n^m$$

Nevertheless, the computations were very much involved; some of the output relations consume up to 700 MB of memory.

Outlook

This technique can be extended in a straight-forward manner to the q-case (which is an open problem for more than 25 years!).

From the computational point of view, this is still a big challenge!
$\longrightarrow$ START project

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Thank you for your attention!