
ADVANCED TOPICS IN FINITE ELEMENT ANALYSIS OF STRUCTURES

With *Mathematica*[®] and MATLAB[®] Computations

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JOHN WILEY & SONS, INC.

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